A possible interpretation of the Barbero–Immirzi parameter and its consequences

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- Motivations: The Barbero–Immirzi (BI) parameter is a free constant that appears in the spectra of area and volume operators of LQG. Many authors have contributed to shed light on the origin of this ambiguity.[a] Here we present an extension of the idea originally proposed by Gambini, Obregon and Pullin.
- Idea: The BI parameter is a quantization ambiguity analogous to the θ-angle of Yang–Mills gauge theories.
- Further extension: We present a motivation to promote the BI parameter to be a field rather than a constant, digressing on the dynamics determined by the coupling of the BI field with the Nieh–Yan topological density. By the way, an extension of the Holst action to spacetimes with torsion is also discussed.

[a] Rovelli & Thiemann, (1998), R. Gambini et al., (1999), Perez & Rovelli (2005), Freidel et al. (2005), Chou et al. (2005), Randono (2005).

Outline

- General remarks: brief review of the role of the BI parameter in classical and quantum gravity.

- Einstein-Cartan theory: description of the interaction between spinor matter fields and gravity. Torsion and its geometrical features.

- Generalization of the Holst action: brief description of the Nieh–Yan topological density and its relation to the Pontryagin densities. Generalization of the Holst theory to spacetimes with torsion. Comparison between the BI parameter and the vacuum angle of Yang–Mills gauge theories.

Outline

- Promoting the BI parameter to be a field: motivation to consider the hypothesis that the BI parameter is actually a field rather than a constant. Proposal on the nature of the BI parameter. Dynamics and comparison with previously proposed models.

- Interaction with fermions and Peccei–Quinn mechanism: introduction of fermion fields and dynamical consequences related to the presence of the BI field. Description of the PQ mechanism for determining the value of the BI parameter.

- Canonical theory: constraints of the new theory with the BI field. Proposal for the Ashtekar–Barbero formulation.

- Concluding remarks and discussion: summary and discussion of the described results, proposals for future research.

General Remarks: the BI parameter

The BI parameter is a free constant appearing in the spectra of length, area and volume operators of LQG, e.g., the area spectrum is:

$$A_\gamma = 8\pi\gamma\ell_{Pl}^2 \sum_k \sqrt{j_k\,(j_k+1)}\,. \tag{1}$$

Rovelli and Thiemann in 1998 noted that the canonical transformation $U(\gamma)\,:\,(A,E)\,\rightarrow\,(A',E')$, where $A'^i_\alpha = \gamma A^i_\alpha + (1-\gamma)\,\Gamma^i_\alpha$ and $E'^\beta_k = \frac{1}{\gamma}E^\beta_k$, does not correspond to a unitary transformation in the quantum theory.

A rescaling of the canonical variables usually generates different, but unitary equivalent quantum representations.

So, what happens here?

General Remarks: the BI parameter

One possibility is that the BI parameter has a topological origin.

If, in fact, the BI parameter has a topological origin, then different choices of γ leads to *inequivalent* representations of the quantum canonical commutations relations.

Gambini, Obregon and Pullin suggested that it exists an interesting analogy between BI parameter and the θ-ambiguity appearing in Yang–Mills gauge theories. Both do not affect the classical theory, but produce striking effects in the quantum regime.

But in pure gravity this analogy is not completely convincing.

The BI parameter appears as a multiplicative constant in front of the Holst modification, which cannot be ascribable among topological densities.

General Remarks: the Holst action

The Holst action is:

$$S_{\mathrm{Hol}}\left[e,\omega\right] = S_{\mathrm{HP}}\left[e,\omega\right] + S_{\mathrm{Mod}}\left[e,\omega\right] = -\frac{1}{16\pi G}\int e_a \wedge e_b \wedge \left(\star R^{ab} + \beta R^{ab}\right) \,, \tag{2}$$

where for later convenience we defined $\beta = -\frac{1}{\gamma}$.

The signature is $+,-,-,-$ and we set $\hbar = c = 1$; $[G] = M^{-2} = L^2$.

The Holst modification vanishes on (half-)shell, namely on the solution of the homogeneous second Cartan structure equation. Specifically, we have that:

$$de^a + \omega^a{}_b \wedge e^b = 0 \implies R^a{}_b \wedge e^b = 0\,, \tag{3}$$

namely if the connection 1-form ω^{ab} satisfies the structure equation, its associated curvature 2-form satisfies the cyclic Bianchi identity. So that

The term S_{Mod} does not affect the classical equations of motion of GR.

General Remarks: the Holst action

Since the Holst modification is not a topological term, the same conclusion cannot be drawn in spacetimes with torsion.

The Bianchi cyclic identity is modified by the presence of torsion, i.e.

$$de^a + \omega^a_{\ b} \wedge e^b = T^a \implies R^a_{\ b} \wedge e^b = d^{(\omega)} T^a \,, \tag{4}$$

The case of fermion fields coupled to gravity is particularly instructive. It is well known that the presence of fermions coupled to first order Palatini gravity generates torsion in spacetime, thus revealing some interesting features about the Holst theory and suggesting a possible generalization.

Einstein–Cartan theory: generalities

We can describe a system of spin-$1/2$ fields coupled to gravity via the Einstein–Cartan action:

$$S_{EC}\left[e,\omega,\psi,\overline{\psi}\right] = -\frac{1}{16\pi G}\int e_a \wedge e_b \wedge \star R^{ab} + \frac{i}{2}\int \star\, e_a \wedge \left(\overline{\psi}\gamma^a \mathcal{D}\psi - \overline{\mathcal{D}\psi}\gamma^a\psi + \frac{i}{2} m e^a \overline{\psi}\psi\right), \quad (5)$$

where the covariant derivatives are defined as:

$$\mathcal{D}\psi = d\psi - \frac{i}{4}\,\omega^{ab}\Sigma_{ab}\psi \qquad \text{and} \qquad \overline{\mathcal{D}\psi} = d\overline{\psi} + \frac{i}{4}\,\overline{\psi}\Sigma_{ab}\omega^{ab}. \quad (6)$$

Spinor fields generate torsion: the variation with respect to ω^{ab} gives

$$d^{(\omega)}e^a = T^a = -4\pi G \star \left(e^a \wedge e_b J^b_{(A)}\right) \quad \text{where} \quad J^d_{(A)} = \overline{\psi}\gamma^d\gamma^5\psi\,. \quad (7)$$

The unique solution is: $\omega^{ab}\left(e,\psi,\overline{\psi}\right) = \underset{\circ}{\omega}{}^{ab}\left(e\right) + 2\pi G\, \epsilon^{ab}{}_{cd} e^c J^d_{(A)}$.

Einstein–Cartan theory: generalities

It is worth noting that the Cartan structure equation does not contain any dynamical information, it is an algebraic relation that makes it possible to *uniquely* express the Lorentz valued connection ω^{ab} as a function of the other fields, namely the gravitational and the spinor fields.

By pulling back the EC action on the solution of the structure equation, we obtain the following effective action:

$$
\begin{aligned}
S_{\text{eff}}\left[e,\psi,\overline{\psi}\right] = & -\frac{1}{16\pi G}\int \frac{1}{2}\epsilon_{abcd}\, e^a \wedge e^b \wedge \underset{\circ}{R}{}^{cd} \\
& + \frac{i}{2}\int \star\, e_a \wedge \left(\overline{\psi}\gamma^a \underset{\circ}{\mathcal{D}}\psi - \overline{\underset{\circ}{\mathcal{D}}\psi}\gamma^a\psi + \frac{i}{2} m e^a \overline{\psi}\psi\right) \\
& + \frac{3}{2}\pi G \int dV\, \eta_{ab} J^a_{(A)} J^b_{(A)} \,,
\end{aligned}
$$

known as EC effective action

Einstein–Cartan theory: Holst case

For pure gravity the Holst action is dynamically equivalent to HP action: in the presence of minimally coupled fermions the situation changes:

$$S_{ECH}\left[e,\omega,\psi,\overline{\psi}\right] = -\frac{1}{16\pi G}\int e_a \wedge e_b \wedge \left(\star R^{ab} + \beta R^{ab}\right) + \frac{i}{2}\int \star\, e_a \wedge \left(\overline{\psi}\gamma^a \mathcal{D}\psi - \overline{\mathcal{D}\psi}\gamma^a\psi + \frac{i}{2} m e^a \overline{\psi}\psi\right), \quad (8)$$

The structure equation becomes:

$$d^{(\omega)}e^a = \tau^a = -\frac{4\pi G}{1+\beta^2}\star\left(e^a \wedge e_b J^b_{(A)}\right)\boxed{-4\pi G\frac{\beta}{1+\beta^2}e^a \wedge e_b J^b_{(A)}}. \quad (9)$$

The new torsion tensor depends explicitly on the BI parameter. It differs from the one calculated before because of the presence of an additional trace component which vanishes as soon as $\beta = 0$. Strangely enough the trace component is an internal axial vector rather than a vector.

Einstein–Cartan theory: Holst case

The solution of the new Cartan structure equation can be easily calculated:

$$\omega^{ab}\left(e,\psi,\overline{\psi}\right) = \underset{\circ}{\omega}{}^{ab}\left(e\right) + \frac{2\pi G}{1+\beta^2}\,\epsilon^{ab}{}_{cd}e^{c}J^{d}_{(A)} - 8\pi G\frac{\beta}{1+\beta^2}e^{[a}J^{b]}_{(A)}\,. \qquad (10)$$

By pulling the action back on the solution above, we obtain:

$$\begin{aligned} S_{\text{eff}}\left[e,\psi,\overline{\psi}\right] = &-\frac{1}{16\pi G}\int\frac{1}{2}\epsilon_{abcd}\,e^{a}\wedge e^{b}\wedge \underset{\circ}{R}{}^{cd} \\ &+\frac{i}{2}\int\star\, e_a\wedge\left(\overline{\psi}\gamma^{a}\underset{\circ}{\mathcal{D}}\psi - \overline{\underset{\circ}{\mathcal{D}}\psi}\gamma^{a}\psi + \frac{i}{2}me^{a}\overline{\psi}\psi\right) \\ &+\frac{3}{2}\pi G\frac{1}{1+\beta^2}\int dV\,\eta_{ab}J^{a}_{(A)}J^{b}_{(A)}\,, \end{aligned} \qquad (11)$$

The effective action deviates from the EC one and the BI parameter acquires a classical physical meaning.

Einstein–Cartan theory: extension

To construct an action which reduces to the Ashtekar–Romano–Tate one for $\beta = \pm i$, we have to modify also the fermionic sector:[a]

$$S\left[e,\omega,\psi,\overline{\psi}\right] = -\frac{1}{16\pi G}\int e_a \wedge e_b \wedge \left(\star R^{ab} + \beta R^{ab}\right) + \frac{i}{2}\int \star\, e_a \wedge \left[\overline{\psi}\gamma^a S^5_{(\beta)}\mathcal{D}\psi - \overline{\mathcal{D}\psi}S^5_{(\beta)}\gamma^a\psi\right],$$

where $S^5_{(\beta)} = 1 + i\beta\gamma^5$.

<u>THIS ACTION IS DYNAMICALLY EQUIVALENT TO THAT OF THE EINSTEIN-CARTAN THEORY</u>

Note that for $\beta = i$ only the left handed fermions interact with the self dual Ashtekar connections. The above action can be considered a natural generalization of the ART one for arbitrary values of the BI parameter.

[a] SM, *Phys. Rev.* **D73**, 084016, (2006) [gr-qc/0601013] (see also SM, gr-qc/0610026).

Einstein–Cartan theory: extension

To understand the reason why the introduced modifications do not affect the classical theory, consider:

$$\mathcal{I} = \int \left[\frac{\beta}{16\pi G} R^{ab} \wedge e_a \wedge e_b + \frac{\beta}{2} \star e_a \wedge \left(\overline{\psi} \gamma^a \gamma^5 \mathcal{D}\psi - \overline{\mathcal{D}\psi} \gamma^5 \gamma^a \psi \right) \right] . \quad (12)$$

By using the solution of the Cartan structure equation obtained by varying the non-minimal action, we can rewrite the fermionic term, so that we obtain

$$\begin{aligned} \mathcal{I} &= \frac{\beta}{16\pi G} \int \left[R^{ab} \wedge e_a \wedge e_b - T^a \wedge T_a \right] \\ &= -\frac{\beta}{16\pi G} \int d\left(e_a \wedge T^a \right) = -S_{NY}\left[e, \omega \right] , \end{aligned} \quad (13)$$

The integrand above is known as Nieh–Yan topological density.

It represents a generalization of the Holst term to spacetimes with torsion.

Holst theory: generalization

Consequently: we propose the following generalized action for spacetimes with torsion

$$\begin{aligned} S_{\text{new}}\left[e,\omega,\psi,\overline{\psi}\right] = &-\frac{1}{16\pi G}\int e_a\wedge e_b\wedge\star R^{ab} + S_{\text{matter}}\left[e,\omega,\psi,\overline{\psi}\right] \\ &-\frac{\beta}{16\pi G}\int\left(e_a\wedge e_b\wedge R^{ab} - T^a\wedge T_a\right)\,, \end{aligned} \tag{14}$$

REMARKS:

- the proposed action describes the same classical dynamics of the Einstein(-Cartan) action;
- the BI parameter disappears from the effective classical action;
- it generalizes the Holst approach to Riemann–Cartan spacetimes;
- the new action suggests a possible physical interpretation of the BI parameter in analogy with the θ-angle of Yang-Mills gauge theories.

From EC to AB formulation of Gravity

The action now contains a true topological term, which provides insight into the nature of the BI parameter, completing the picture proposed by Gambini, Obregon and Pullin. Specifically. . .

. . . the Ashtekar–Barbero constraints of GR can be obtained by rescaling the heuristic state functional of the Einstein–Cartan theory by the exponential of the Nieh–Yan functional.[a]

WORK IN PROGRESS. . . More rigorously, one should work with the well defined states of LQG, we expect that the rescaling of the states by the exponential of the Nieh–Yan functional multiplied by the constant $\tilde{\beta}$ generates a shift of the BI parameter $\beta \to \beta + \tilde{\beta}$.

The rescaling is motivated by a particular large gauge transformation related to the temporal gauge fixing, specifically consisting in modifying the Wigner boost parameter which fixes the gauge.

[a] SM, PRD 77, (2008), 024036

End of first part

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Second part

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Motivation from particle physics

Let us consider the following action:

$$S_{\mathrm{new}}\left[e,\omega,\psi,\overline{\psi},q,\overline{q}\right] = S_{\mathrm{HP}}\left[e,\omega\right] + S_{\mathrm{matter}}\left[e,\omega,\psi,\overline{\psi},q,\overline{q}\right] - \frac{\beta}{16\pi G}\int \left(e_a \wedge e_b \wedge R^{ab} - T^a \wedge T_a\right) , \tag{15}$$

where the matter term describes leptons as well as quarks.

It is well known that the quarks mass matrices M resulting from the spontaneous breaking of the $SU(2)\times U(1)$ symmetry are neither diagonal nor Hermitian. We can diagonalize M by chiral rotating the quark fields:

$$q_R \longrightarrow q'_R = e^{\frac{i}{2n_f}\mathrm{Arg}\,\det M} q_R\,, \tag{16a}$$

$$q_L \longrightarrow q'_L = e^{-\frac{i}{2n_f}\mathrm{Arg}\,\det M} q_L\,. \tag{16b}$$

Such a chiral transformation introduces a divergent term in the effective action.

Motivation from particle physics

We recall that, in spacetime with torsion, the jacobian of a chiral rotation of the fermionic measure in the Euclidean path-integral contains a divergent term,[a] i.e.

$$\delta q \delta \overline{q} \xrightarrow{U(1)_A} \delta q \delta \overline{q} e^{\frac{i}{8\pi^2} \int \alpha \left[R_{ab} \wedge R^{ab} + 2\Lambda^2 \left(T_a \wedge T^a - e_a \wedge e_b \wedge R^{ab} \right) \right]},$$

Λ being the regulator and α the parameter of the transformation.

The resulting action is

$$S\left[e,\omega,\psi,\overline{\psi},q,\overline{q}\right] = S_{\mathrm{HP}}\left[e,\omega\right] + S_{\mathrm{D}}\left[e,\omega,\psi,\overline{\psi},q,\overline{q}\right] + \frac{1}{8\pi^2}\,\alpha \int R_{ab} \wedge R^{ab}$$
$$+ \frac{1}{16\pi G}\left(\beta + \frac{4G}{\pi}\,\alpha\Lambda^2\right) \int \left(T^a \wedge T_a - e_a \wedge e_b \wedge R^{ab}\right).$$

As soon as we try to remove the regulator we obtain a divergence.

[a] Chandía & Zanelli, PRD 55, (1997), 7580.

Motivation from particle physics

Let us introduce in the fundamental action the field $\beta(x)$, which interacts with the gravitational field through the NY term, i.e.

$$S_{\rm Tot}\left[e,\omega,\psi,\overline{\psi},q,\overline{q},\beta\right] = S_{\rm HP}\left[e,\omega\right] + S_{\rm D}\left[e,\omega,\psi,\overline{\psi},q,\overline{q}\right] + \frac{1}{16\pi G}\int \beta(x)\left(T^a\wedge T_a - e_a\wedge e_b\wedge R^{ab}\right)\ . \qquad (17)$$

Dynamically, the new action (17) is no longer equivalent to the EC one.

The new action above is invariant under a rescaling of $\beta(x)$, so that, without affecting the dynamical content of the theory, we can incorporate the divergence in the definition of a new field $\beta'\left(x\right) = \beta\left(x\right) + \frac{4G}{\pi}\,\alpha\Lambda^2$.

In this new framework the BI parameter, β_0, can be naturally associated to the expectation value of the field $\beta(x)$, namely $\beta_0 = \langle\beta(x)\rangle$.

The BI field: effective dynamics

Let us study the effective dynamics of the BI field in the pure gravitational case:

$$S_{\rm grav}=-\frac{1}{16\pi G}\int e_a\wedge e_b\wedge\star R^{ab}-\frac{1}{16\pi G}\int\beta\left(e_a\wedge e_b\wedge R^{ab}-T^a\wedge T_a\right)\,.$$

By varying the action with respect to the connection 1-form ω^{ab} and manipulating it, we obtain the following structure equation:

$$de^a+\omega^a{}_b\wedge e^b=T^a=\frac{1}{2}\epsilon^{ab}{}_{cd}\partial_b\beta\, e^c\wedge e^d\,.$$

Pulling back the action on the solution of the structure equation, we have:[a]

$$S_{\rm eff}=-\frac{1}{16\pi G}\int e_a\wedge e_b\wedge\star R^{ab}+\frac{1}{2}\int\star d\tilde\beta\wedge d\tilde\beta\,, \qquad (18)$$

where we defined $\tilde\beta(x)=\frac{1}{4}\sqrt{\frac{3}{\pi G}}\beta(x)$. $\left[\tilde\beta(x)\right]=M=L^{-1}$.

[a] SM, arXiv:0902.2764. Calcagni & SM, arXiv:0902.0957

The BI field: remarks

- When the parameter β is promoted to be a field, the NY term in the action ceases to be topological, generating a modification with respect to the EC theory.

- To preserve the usual transformation properties of the irreducible components of torsion, the $\tilde{\beta}(x)$ field has to be a pseudo-scalar. Interestingly enough, the pseudo-scalar nature of $\tilde{\beta}(x)$ is not assumed *a priori*, but is a geometrical consequence of the theory.

- The BI field decouples from gravity and behaves exactly as a (pseudo) scalar field. In the previous models [a] based on the usual Holst action, $\phi = \sinh \beta$ plays the role of a scalar field rather than β itself.

[a] Taveras & Yunes, PRD 78, (2008). Torres-Gomez & Krasnov, arXiv:0811.1998. Castellani et al., (1991).

The BI field: coupling with fermions

Let me consider the following action:

$$
\begin{aligned}
S_{\mathrm{Tot}} = & -\frac{1}{16\pi G}\int e_a\wedge e_b\wedge\star R^{ab} \\
& -\frac{1}{16\pi G}\int \beta(x)\left(e_a\wedge e_b\wedge R^{ab}-T^a\wedge T_a\right) \\
& +\frac{i}{2}\int \star e_a\wedge\left(\overline{\psi}\gamma^a D\psi-\overline{D\psi}\gamma^a\psi+\frac{i}{2}me^a\overline{\psi}\psi\right),
\end{aligned}
\tag{19}
$$

Dynamically, the action (19) is not equivalent to the EC one, in fact the presence of the new field $\beta(x)$ modifies the expression of the torsion 2-form, i.e.

$$
de^a+\omega^a_{\ b}\wedge e^b = T^a = \epsilon^a_{\ bcd}\left(\frac{1}{2}\eta^{bf}\partial_f\beta-2\pi G J^b_{(A)}\right)e^c\wedge e^d\,,
$$

where $J_{(A)}=e_aJ^a_{(A)}=e_a\sum_{f=1}^{n}\overline{\psi}_f\gamma^a\gamma^5\psi_f$ is the fermionic axial current.

The BI field: coupling with fermions

By using the expression we found for torsion an interesting form of the effective action can be easily calculated, namely

$$S_{\text{eff}} = S_{\text{HP}}\left[e, \underset{\circ}{\omega}\right] + S_{\text{D}}\left[e, \underset{\circ}{\omega}, \psi, \overline{\psi}\right] + \frac{3}{2}\pi G \int dV \, \eta_{ab} J^a_{(A)} J^b_{(A)} + \frac{1}{2}\int \star d\tilde{\beta}(x) \wedge d\tilde{\beta}(x) - \sqrt{3\pi G}\int \star J_{(A)} \wedge d\tilde{\beta}(x) \,. \qquad (20)$$

An interaction between $\tilde{\beta}(x)$ and the fermionic axial current appears. Remembering that the chiral anomaly prevents the axial current to be conserved, i.e. $d\star J_{(A)} = -\frac{1}{192\pi^2} R_{ab} \wedge R^{ab}$, as noted previously, the $\tilde{\beta}(x)$ turns out to couple to the gravitational Pontryagin density, namely

$$S_{\text{eff}} = S_{\text{HP}}\left[e, \underset{\circ}{\omega}\right] + S_{\text{D}}\left[e, \underset{\circ}{\omega}, \psi, \overline{\psi}\right] + \frac{3}{2}\pi G \int dV \, \eta_{ab} J^a_{(A)} J^b_{(A)} + \frac{1}{2}\int \star d\tilde{\beta}(x) \wedge d\tilde{\beta}(x) + \frac{\sqrt{3\pi G}}{192\pi^2}\int \tilde{\beta}(x) \underset{\circ}{R}_{ab} \wedge \underset{\circ}{R}^{ab} \,. \qquad (21)$$

The BI field: coupling with fermions

The effective action can also contain P and CP violating terms.

Possible sources of discrete symmetries violations are a θ term correlated to the non-trivial global structure of the local gauge group. Moreover the possible presence of quarks introduces another term proportional to $\alpha = \mathrm{Arg}\det M$. We include these terms via the parameter $\tilde{\theta} = \theta + \frac{1}{8\pi^2}\,\alpha$. So, finally:

$$S_{\text{eff}} = S_{\text{HP}}\left[e, \underset{\circ}{\omega}\right] + S_{\text{D}}\left[e, \underset{\circ}{\omega}, \psi, \overline{\psi}\right] + \frac{3}{2}\,\pi G \int dV\, \eta_{ab} J^{a}_{(A)} J^{b}_{(A)}$$

$$+ \frac{1}{2}\int \star d\tilde{\beta}(x) \wedge d\tilde{\beta}(x) + \frac{1}{8\pi^2}\int \left(\tilde{\theta} + \frac{\sqrt{3\pi G}}{24}\tilde{\beta}(x)\right) \underset{\circ}{R}_{ab} \wedge \underset{\circ}{R}^{ab}\,. \qquad (22)$$

Focus the attention on the last term in the action. It is the only CP violating term. Studying the effects connected with this term,[a] it could be possible to put some restrictive limits on the expectation value of $\tilde{\beta}(x)$.

[a] see, e.g., Alexander & Yunes (2008), Yunes & Spergel (2008), Alexander, Finn & Yunes (2007)

The BI field: Peccei–Quinn mechanism

We can relate the value of the BI parameter to that of the vacuum angle $\tilde{\theta}$. In particular,

$$S_{\mathrm{CP}} = \frac{1}{8\pi^2} \int \left(\tilde{\theta} + \frac{\sqrt{3\pi G}}{24} \tilde{\beta}(x) \right) \underset{\circ}{R}_{ab} \wedge \underset{\circ}{R}^{ab} \,, \tag{23}$$

being the only CP violating term, then the effective potential will be even in $\tilde{\theta} + \frac{\sqrt{3\pi G}}{24}\tilde{\beta}(x)$, so it will have a stationary point in $\tilde{\theta} + \frac{\sqrt{3\pi G}}{24}\tilde{\beta}(x) = 0$, preserving the P and CP symmetries.[a] According to this mechanism, we have $\beta_0 = -32\tilde{\theta}$.

The BI ambiguity is correlated to the vacuum angle of gravity.

Side remark: only the physical field $\tilde{\beta}_{\mathrm{phys}} = \tilde{\beta} - \left\langle \tilde{\beta} \right\rangle$ interacts with gravity via the Pontryagin density.

[a] Peccei & Quinn, PRL 38, 1440, (1977); PRD 16, 1791, (1977).

The BI field: Canonical Theory

The gravitational theory with the BI field has been canonically formulated in a recent paper.[a] Once fixed the temporal gauge and solved the second class constraints related to torsion, the evolution of the system in the phase space turns out to be limited by the following set of first class constraints:

$$\mathcal{R}_i := \epsilon_{ij}{}^{k} K^j_\alpha E^\alpha_k \approx 0\,, \tag{24}$$

$$\mathcal{H}_\alpha := 2E^\gamma_i D_{[\alpha} K^i_{\gamma]} + \Pi \partial_\alpha \tilde{\beta} \approx 0\,, \tag{25}$$

$$\mathcal{H} := -\frac{8\pi G}{2e} E^\alpha_i E^\gamma_j \left(\epsilon^{ij}{}_k R^k_{\alpha\gamma} + 2K^i_{[\alpha} K^j_{\gamma]} \right) + \frac{1}{2e}\Pi^2 - \frac{1}{2} e\, \partial_\alpha \tilde{\beta} \partial^\alpha \tilde{\beta} \approx 0\,. \tag{26}$$

where $\tilde{\beta} = \frac{1}{4}\sqrt{\frac{3}{\pi G}}\beta$, so that $\left[\tilde{\beta}\right] = L^{-1} = M$ and the symplectic structure being:

$$\left\{ K^i_\alpha(t, \underline{x}), E^\beta_j(t, \underline{x}') \right\} = \delta^\beta_\alpha \delta^i_j \delta(\underline{x}, \underline{x}')\,, \quad \left\{ \tilde{\beta}(t, \underline{x}), \Pi(t, \underline{x}') \right\} = \delta(\underline{x}, \underline{x}')\,, \tag{27}$$

[a] Calcagni & SM, arXiv:0902.0957

Ashtekar–Barbero variables

One is immediately tempted to define the new variables as:

$$A_\alpha^i(t,\underline{x}) = -\frac{1}{\beta(t,\underline{x})}K_\alpha^i(t,\underline{x}) + \Gamma_\alpha^i(t,\underline{x})\,, \qquad P_i^\alpha(t,\underline{x}) = -\beta(t,\underline{x})E_i^\alpha(t,\underline{x})\,,$$

but this definition generates a complicated new canonical algebra.

The above definition breaks the rescaling symmetry of the theory

So, according to the interpretation of the BI parameter previously proposed, we suggest to define the new variables as:

$$A_\alpha^i(t,\underline{x}) = -\frac{1}{\beta_0}K_\alpha^i(t,\underline{x}) + \Gamma_\alpha^i(t,\underline{x})\,. \tag{28}$$

The expectation value β_0 is completely arbitrary being associated to a free scalar field. But in the non-perturbative quantum theory the rescaling symmetry cannot remain unbroken, generating a non-trivial effective potential which can dynamically fix the value of $\beta_0 = -\frac{1}{\gamma}$.

Ashtekar–Barbero constraints

By introducing the new variables into the constraints, we can rewrite them in the Ashtekar–Barbero form, with the presence of the scalar field $\tilde{\beta}$, i.e.

$$\mathcal{R}_i \rightarrow \mathcal{G}_i := \partial_\alpha E^\alpha_i + \epsilon_{ij}{}^k A^j_\alpha E^\alpha_k \approx 0 \,, \tag{29}$$

$$\mathcal{H}_\alpha \rightarrow \mathcal{H}'_\alpha := E^\gamma_i F^i_{\alpha\gamma} + \Pi \partial_\alpha \tilde{\beta} \approx 0 \,, \tag{30}$$

$$\begin{aligned}\mathcal{H} \rightarrow \mathcal{H}' := & -\frac{8\pi G}{2e} E^\alpha_i E^\gamma_j \left(\epsilon^{ij}{}_k F^k_{\alpha\gamma} + 2 \left(\frac{1+\beta_0^2}{\beta_0^2} \right) K^i_{[\alpha} K^j_{\gamma]} \right) \\ & + \frac{1}{2e}\Pi^2 - \frac{1}{2} e \, \partial_\alpha \tilde{\beta} \partial^\alpha \tilde{\beta} \approx 0 \,, \end{aligned} \tag{31}$$

where $F^i_{\alpha\gamma}$ is the curvature of the $SU(2)$ valued connection A^i_α.

We expect that the quantum theory breaks the $U(1)_A$ rescaling symmetry of the starting action.

The net result is the possible presence of an additional interaction through the Pontryagin class $P = \int F^i \wedge F_i$ (work in progress!).

Conclusions

- The BI parameter is a free constant appearing in the kinematical observables of LQG. Its value can be determined by studying the black holes modes, but its nature is still debated.
- Here, motivated by a previous suggestion of Gambini, Obregon and Pullin, we initially proposed a topological interpretation of the BI parameter in analogy with the θ-angle of QCD.
- To complete the analogy between the θ-angle of QCD and the BI parameter of LQG, we extended the Holst approach to torsional spacetimes, where a true topological term containing the Holst modification can be introduced.
- We gave a general argument that the BI parameter has to be promoted to be a field. The effective dynamics revealed that the BI field behaves as a pseudo-scalar field.

Conclusions

- Moreover, the introduction of fermions in this framework seems to be particularly interesting and promising for new developments of the theory.

- We have showed how the analogous of the Peccei–Quinn mechanism can provide a dynamical determination of the BI parameter, which turns out to be related to the vacuum angle of the gravitational theory.

- In order to study the outcomes in a rigorous way, we have to introduce these new concepts in the well posed formalism of LQG. Through an analogous mechanism, the BI parameter could be dynamically related to other topological quantities. In particular, by studying the interaction with fermions in the canonical framework, new interactions could appear, providing interesting effects.