The imaginary part of the GR action and the large-spin 4-simplex amplitude

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Summary

Classical GR

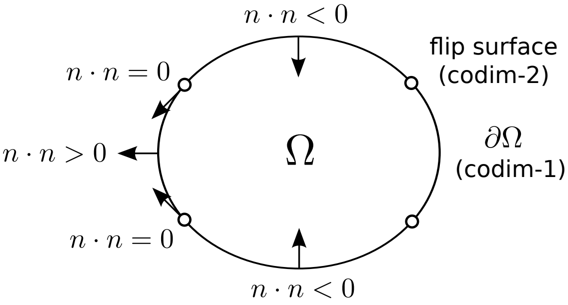
- First-order and second-order actions agree on-shell.
- The action has a nonvanishing imaginary part:

$$\mathrm{Im}\, S = \frac{1}{16G} \sum_{\text{flips}} A_{\text{flip}} = \frac{1}{4} \sum_{\text{flips}} \sigma_{\text{flip}}$$

The large-spin limit of LQG

- The large-spin vertex amplitude reproduces the classical action iff analytically continued to $\gamma = \pm i$.
- A first-ever agreement in LQG between BH entropy derived from an effective action and from a microscopic state counting.
- The limit should be thought of as a high-energy ("transplanckian") regime.

Generic spacetime regions Ω with boundary $\partial\Omega$



- $\partial\Omega$ doesn't have to be at infinity!

Normal vector n^μ

- Sign of n^μ: chosen so that $n_\mu v^\mu > 0$ for outgoing v^μ.
- Sign of $n \cdot n$: has to flip between spacelike and timelike patches.
  $\Rightarrow$ $n \cdot n$ cannot be constant everywhere!

The second-order GR action

$$\boxed{S = \frac{1}{16\pi G}\left(\int_\Omega \sqrt{-g}\, R\, d^4x + 2\int_{\partial\Omega}\sqrt{\frac{-h}{n\cdot n}}\, K\, d^3x\right)}$$

- The independent variable is $g_{\mu\nu}$. All the rest derived from it.
- Bulk term: Einstein-Hilbert.
- Boundary term: York-Gibbons-Hawking. [York, 1972; Gibbons & Hawking, 1977]
  - Defining property: $\delta S = 0$ for $\delta g_{\mu\nu}|_{\partial\Omega} = 0$ (no assumptions on $\partial_\mu g_{\nu\rho}$).

Boundary term notation

- h_{ab} - intrinsic metric of boundary $\partial\Omega$; $h = \det h_{ab}$.
- n^μ - boundary normal; $K_a^b = \nabla_a n^b$ - extrinsic curvature; $K = K_a^a$.
- Valid for arbitrary norm and signature of n^μ.
  - Rescaling n^μ is a gauge freedom.

Alternative - a densitized normal

For a clean variational principle, would be nicer to have a normal that is a unique function of $\partial\Omega$ and $g_{\mu\nu}$ everywhere.

- Can't use a unit normal: not well-defined at signature flips.
- Can use a densitized version: $N_\mu = \dfrac{1}{3!}\sqrt{-g}\,\epsilon_{\mu abc}\epsilon^{abc}$.
- Related to undensitized n^μ as: $N_\mu = \sqrt{\dfrac{-h}{n\cdot n}}\,n_\mu$.

- However, covariant derivatives of N_μ are awkward.
- Will be easier in the first-order formalism.

The first-order Palatini action

$$\boxed{S = \frac{1}{16\pi G}\left(\int_\Omega \Sigma^{IJ}\wedge F_{IJ} - \int_{\partial\Omega}\Sigma^{IJ}\wedge\left(A_{IJ} - \frac{2N_I dN_J}{N\cdot N}\right)\right)}$$

The independent variables

- Vielbein e^I_μ and connection A^{IJ}_μ.
- In form notation, just e^I and A^{IJ}.

Useful derived quantities

- $\Sigma^{IJ} = \frac{1}{2}\epsilon^{IJKL}e_K\wedge e_L = \star(e\wedge e)^{IJ}$.
- Curvature: $F_{IJ} = dA_{IJ} + A_I{}^K\wedge A_{KJ}$.
- Densitized normal: $N_I = \frac{1}{3!}\epsilon^{abc}\epsilon_{IJKL}e^J_a e^K_b e^L_c$.

The first piece of the boundary term

$$\boxed{S_{\partial\Omega,\text{old}} = -\frac{1}{16\pi G}\int_{\partial\Omega}\Sigma^{IJ}\wedge A_{IJ}}$$

Why have this term?

- Cancels the A_{IJ} variation from the bulk term.
- Thus, enforces the variational principle: $\delta S = 0$ for $\delta e^I\big|_{\partial\Omega} = 0$ (no assumptions on A_{IJ}).

Not gauge invariant

- Invariant if the direction of N_I is constant along the boundary.
- Cannot hold everywhere on a closed boundary, because of signature flips!

The new, second piece of the boundary term

$$\boxed{S_{\partial\Omega,\text{new}} = \frac{1}{8\pi G}\int_{\partial\Omega} \Sigma^{IJ} \wedge \frac{N_I dN_J}{N \cdot N} = \frac{1}{8\pi G}\int_{\partial\Omega} \Sigma^{IJ} \wedge \frac{n_I dn_J}{n \cdot n}}$$

In second expression, introduced an undensitized normal n_I (free to rescale).

Why have this term?

- For gauge invariance - cancels the gauge transformation of the $\Sigma^{IJ} \wedge A_{IJ}$ term.

Properties of the $N_I dN_J$ term

- Non-polynomial, due to the $N \cdot N$ in the denominator.
- Depends only on the boundary value of e^I.
  $\Rightarrow$ Doesn't affect the variational principle.

On-shell equality of the Einstein-Hilbert and Palatini actions

Bulk and boundary terms are equal separately

- $\sqrt{-g}\, R\, d^4x = \Sigma^{IJ} \wedge F_{IJ}$
- $\sqrt{\frac{-h}{n \cdot n}}\, K\, d^3x = -\frac{1}{2}\,\Sigma^{IJ} \wedge \left(A_{IJ} - \frac{2 n_I d n_J}{n \cdot n} \right)$
- The boundary terms are equal thanks to the new $n_I dn_J$ piece.

Requires "half" of the equations of motion

- $T_I \equiv de_I + A_I{}^J e_J = 0.$
  (from variation of the first-order action w.r.t. A_{IJ})

The first-order action with Holst term

$$\boxed{\begin{aligned} S = \frac{1}{16\pi G}\left(\int_\Omega \Sigma^{IJ} \wedge F_{IJ} - \int_{\partial\Omega} \Sigma^{IJ} \wedge \left(A_{IJ} - \frac{2N_I dN_J}{N \cdot N}\right)\right) \\ + \frac{1}{16\pi G\gamma}\left(\int_\Omega e^I \wedge e^J \wedge F_{IJ} - \int_{\partial\Omega} e^I \wedge (e^J \wedge A_{IJ} - de_I)\right) \end{aligned}}$$

The $e^I \wedge e^J \wedge A_{IJ}$ piece of the boundary term

- Cancels the A_{IJ} variation from the $e^I \wedge e^J \wedge F_{IJ}$ bulk term.
- Thus, enforces the variational principle: $\delta S = 0$ for $\delta e^I\big|_{\partial\Omega} = 0$.

The $e^I \wedge de_I$ piece of the boundary term

- Combines with the $e^I \wedge e^J \wedge A_{IJ}$ term to give $e^I \wedge T_I$.
- Thus, restores gauge invariance.
- Unlike the $N_I dN_J$ piece, is polynomial.

Holst action - properties

Both bulk and boundary pieces of the Holst term vanish on-shell

- Again, assume "half" the equations of motion $T^I = 0$.
- Bulk piece then vanishes due to Bianchi identity $e^J \wedge F_{IJ} = 0$.
- Boundary piece vanishes trivially.

The Holst term is not topological

- The boundary piece alone *is* topological.
  - $d(e^I \wedge T_I)$ is the Nieh-Yan density.
- However, only the correct bulk+boundary combination gives a good variational principle.

Holst action - alternative expression

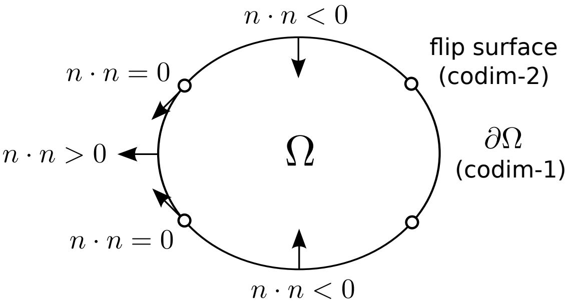
$$\boxed{\begin{aligned} S = \frac{1}{16\pi G}\Bigg(&\int_\Omega \left(\star + \frac{1}{\gamma}\right)(e\wedge e)^{IJ}\wedge F_{IJ} \\ &- \int_{\partial\Omega}\left(\star + \frac{1}{\gamma}\right)(e\wedge e)^{IJ}\wedge\left(A_{IJ} - \Gamma^{\mathrm{H}}_{IJ}\right)\Bigg) \end{aligned}}$$

- Γ^{H}_{IJ} is Peldan's hybrid connection (a non-polynomial functional of e^I). [Peldan,1994]
- Manifest Hodge-duality structure, gauge invariance.
- Credit to Andreas Thurn.

Summary on GR actions

- We now have the correct boundary terms for the Palatini and Holst actions.
- The correct Holst term vanishes on-shell, but is not topological.
- The Einstein-Hilbert, Palatini and Holst actions all agree on-shell.
  - Using only "half" the equations of motion $T_I = 0$.
  - The bulk and boundary terms agree separately.

The imaginary part of the GR action



In GR:

$$\operatorname{Im} S = \frac{1}{16G} \sum_{\text{flips}} A_{\text{flip}}$$

(A - area)

In more general theories:

$$\operatorname{Im} S = \frac{1}{4} \sum_{\text{flips}} \sigma_{\text{flip}}$$

(σ - Wald entropy functional)

What do you mean, an imaginary part?

This sounds ridiculous

Lorentzian actions are supposed to be real.

- Unitary amplitudes e^{iS}.

Euclidean actions are imaginary.

- Exponentially damped amplitudes $e^{-\operatorname{Im} S}$.
- Tunneling, thermal behavior.

But in gravity...

- Classical Lorentzian solutions can have thermal properties.
  - Black holes, other horizons.
- Unitarity is not at all understood for general spacetime regions.
  - Where it *is* understood (S-matrix), $\operatorname{Im} S$ vanishes.

Deriving Im S

Arises from the action's boundary term, integrated over the infinitesimal neighborhood $\delta\mathcal{F}$ of a flip surface $\mathcal{F}$:

$$\boxed{\operatorname{Im} S = \frac{1}{8\pi G} \sum_{\text{flips}} \operatorname{Im} \int_{\delta\mathcal{F}} \sqrt{\frac{-h}{n \cdot n}}\, K\, d^3x}$$

In first-order formulation - arises from the $n_I dn_J$ piece of the boundary term:

$$\boxed{\operatorname{Im} S = \frac{1}{8\pi G} \sum_{\text{flips}} \operatorname{Im} \int_{\delta\mathcal{F}} \Sigma^{IJ} \wedge \frac{n_I dn_J}{n \cdot n}}$$

Simplest example

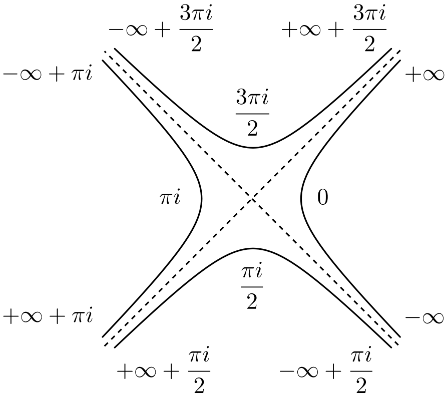
- 1+1d: $\Sigma^{IJ} \to \epsilon^{IJ}$
- Flat space, trivial gauge: $A_{IJ} = 0$

$$\boxed{S = \frac{1}{8\pi G}\int \sqrt{\frac{-h}{n\cdot n}}\,K\,dx = \frac{1}{8\pi G}\int \epsilon^{IJ}\frac{n_I dn_J}{n\cdot n} = \frac{1}{8\pi G}\int d\alpha}$$

- In Euclidean: $\int d\alpha = 2\pi$
- In Lorentzian: $\int d\alpha = 2\pi i$ [Sorkin, 1974]

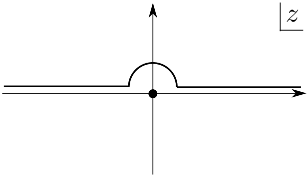
Angles in the Lorentzian plane

When n_I flips signature (crosses a null direction), its angle jumps by $\pi i/2$.



Derivation of the $\pi i/2$ jump:

- Write your vector as: $n^\mu \sim L^\mu + z\ell^\mu$.
- Boost angle is: $d\alpha = \dfrac{dz}{2z}$.
- At signature flip, z crosses through zero.
- Must deform the integration contour to avoid the pole!



$$\operatorname{Im}\alpha = \frac{1}{2i}\oint\frac{dz}{2z} = \frac{\pi}{2}$$

Back to GR in 1+1d

In flat space:

- The action for a region in flat space is: $S = \dfrac{1}{8\pi G} \cdot 2\pi i = \dfrac{i}{4G}$.
- Arises from the 4 poles in the $\dfrac{n_I dn_J}{n \cdot n}$ boundary term at signature flips (null n_I).
- Sign chosen so that $\operatorname{Im} S > 0$.
  $\Rightarrow$ Amplitudes $e^{iS} \sim e^{-\operatorname{Im} S}$ not exploding.

In curved space (in presence of matter):

- Only the real part changes. Still have: $\operatorname{Im} S = \dfrac{1}{4G}$.
- The $\dfrac{n_I dn_J}{n \cdot n}$ integral with its poles remains unchanged.
- The bulk term and the A_{IJ} boundary term have no poles.

Upgrade to 3+1d (or any other dimension)

$$\boxed{\operatorname{Im} S = \frac{1}{8\pi G}\sum_{\text{flips}}\frac{\pi}{2}\cdot A_{\text{flip}} = \frac{1}{16G}\sum_{\text{flips}} A_{\text{flip}}}$$

- $\operatorname{Im} S$ is due to poles in the $\Sigma^{IJ} \wedge \dfrac{n_I dn_J}{n \cdot n}$ boundary term.
- Poles now appear at 2d signature-flip surfaces (null n_I).

On a flip surface, only one component contributes to the numerator

- Form indices on Σ^{IJ} - along the flip surface.
- Form index on dn_J - along the null normal/tangent.
- IJ indices - in the 1+1d plane transverse to the flip surface.

The calculation reduces to 1+1d

- The component of Σ^{IJ} provides an area-density factor.

Classical action of a flat 4-simplex

$$\boxed{S = \sum_{\substack{\text{2d}\\\text{corners}}} \frac{A\theta}{8\pi G}}$$

$$\boxed{\operatorname{Im} S = \sum_{\substack{\text{thin}\\\text{corners}}} \frac{A}{8G}}$$

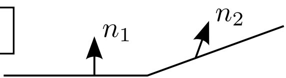
4-simplex with spacelike 3d faces in Minkowski space

- S is purely a boundary term, concentrated at 2d corners (triangles).
- Dihedral angles θ - discrete version of extrinsic curvature.

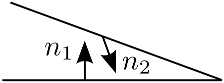
Angles θ are complex

- Every "thin" corner hides two signature flips.

$\boxed{\operatorname{Im}\theta = 0}$



$\boxed{\operatorname{Im}\theta = \pi}$



EPRL spinfoam amplitude for a large-j 4-simplex

[Barrett, Dowdall, Fairbairn, Hellmann & Pereira, 2010]

$$\boxed{\mathcal{A} = N_+ e^{iS_+} + N_- e^{iS_-}}$$

$$\boxed{S_\pm = \sum_{\text{corners}} j(\mp\gamma \operatorname{Re}\theta - \operatorname{Im}\theta)}$$

$\pm$ denotes the two stationary points of the "path integral".

- Correspond to two parity-related configurations of the 4-simplex.
- Can isolate one of the two by smearing over spins j. [Bianchi & Ding, 2011]

Expression for $S_\pm$ adapted from Barrett et.al. by comparing the angle conventions.

- $\operatorname{Re}\theta$ is $-\Theta$ from Barrett et.al.
- $\operatorname{Im}\theta$ is $\Pi = 0, \pi$ from Barrett et.al.

From spins to areas

$$\boxed{A = 8\pi G \sqrt{\gamma^2}\, j} \qquad \boxed{S_\pm = \sum_{\text{corners}} \frac{\text{sign}(\gamma) A}{8\pi G} \left(\mp \operatorname{Re}\theta - \frac{1}{\gamma} \operatorname{Im}\theta \right)}$$

In the area spectrum, used $\sqrt{\gamma^2} = \text{sign}(\gamma)\,\gamma$

- Rather than γ - to cover both signs of γ.
- Rather than $|\gamma|$ - to be as complex-analytic as possible.

Two halves of the complex γ plane

- $\sqrt{\gamma^2}$ and $\text{sign}(\gamma)$ can be analytically continued from the real line.
- However, only in two disjoint domains of the complex plane.
- Positive and negative real γ lie in different domains.
- Away from the real line, the domain boundary is arbitrary.

Comparing to the classical action

$$\boxed{S_\pm = \sum_{\text{corners}} \frac{\text{sign}(\gamma)A}{8\pi G}\left(\mp \operatorname{Re}\theta - \frac{1}{\gamma}\operatorname{Im}\theta\right)}$$

Old interpretation

- For real γ, the second term is an integer multiple of $\pi \Rightarrow$ throw away.
- The first term reproduces the classical action, up to sign.

New interpretation

- The first term reproduces only the real part of the classical action.
- The imaginary part is reproduced by the second term, iff analytically continue $\gamma \to \text{sign}(\gamma)i$.
  - Sign chosen so that $\operatorname{Im} S_\pm > 0$.
  - $\gamma > 0$ goes to $\gamma = i$; $\gamma < 0$ goes to $\gamma = -i$.
  - This is allowed - we had two separate analytic domains to begin with.

Let's send γ to $\mathrm{sign}(\gamma)i$

$$\boxed{S_\pm \;\to\; \mp\,\mathrm{sign}(\gamma)\,\mathrm{Re}\,S_{\mathrm{class}} + i\,\mathrm{Im}\,S_{\mathrm{class}}}$$

Properties of the analytically-continued amplitude

- $S_\pm$ reproduces the classical action, up to the sign of the real part.
- e^{iS_+} and e^{iS_-} are still complex-conjugate to each other.

What does $\gamma = \pm i$ mean?

- Back to the self-dual Ashtekar connection!
- However, don't know how to quantize with non-real γ.
- Use real γ as a regulator - remove at the end of the calculation.

Black hole state-counting in LQG

Does the entropy from state counting agree with the Bekenstein-Hawking formula?

- Bekenstein-Hawking is derived from a classical action.
- To compare, must express the state-counting entropy through parameters of an effective action in a classical limit.
- In LQG, there is no calculation of a continuum effective action to compare with.

The large-spin limit is a semiclassical regime where we have an action to compare with!

- Carry out the LQG state counting, with restriction to large spins and fixed number of punctures. [Frodden, Geiller, Noui & Perez, 2012]
- Bekenstein-Hawking reproduced, iff γ is analytically continued to $\pm i$.
- $\gamma = \pm i$ leads to a correct action *and* a matching entropy!

Large-spin limit vs. the continuum

Large-spin LQG with $\gamma \to \pm i$ reproduces key aspects of continuum GR.

- However, this happens non-trivially.

In the continuum:

- γ doesn't matter.
- $\operatorname{Im} S$ comes from the Palatini boundary term.

In the large-spin limit:

- γ matters.
- $\operatorname{Im} S$ comes from a Holst-type $1/\gamma$ term. Large quantum effect!

The two limits seem distinct.

- How should we understand this?

An RG flow between large spins and the continuum?

A tempting interpretation

γ^2 "flows" from arbitrary values in the continuum to the self-dual value at large spins.

A suggestive coincidence

- In a 1-loop perturbative calculation (in Euclidean, with fermions), γ^2 flows from arbitrary values in the IR to the self-dual value in the UV.
  [Benedetti & Speziale, 2011]
- Sounds familiar...
- The continuum is certainly "in the IR".
- Is the large-spin limit somehow "in the UV"?

The large-spin limit as a high-energy regime

UV and IR are misleading words

- The large-spin limit describes large *distances* $L \sim \sqrt{j}\,\ell_{\text{planck}}$.
  In quantum gravity, this can mean either large or small *energies* E.
- In QFT: small energies $E \sim \hbar/L = (\ell_{\text{planck}}/L)M_{\text{planck}}$
- In classical GR: large energies $E \sim L/G = (L/\ell_{\text{planck}})M_{\text{planck}}$

What's the appropriate physical picture?

- QFT picture:
  - Long-wavelength gravitons *perturbing* a background.
  - Action (of graviton) over distance L is $S/\hbar \sim 1$.
- Classical GR picture:
  - Large black holes *defining* a background.
  - Action (of classical metric) over distance L is $S/\hbar \sim (L/\ell_{\text{planck}})^2 \gg 1$.
- The large-spin limit is more like the classical GR picture.
- Should be thought of as high-energy: $E/M_{\text{planck}} \sim \sqrt{j} \gg 1$.

Two classical limits of LQG

Continuum classical GR		LQG with $j \sim 1$		LQG with $j \gg 1$ (discrete classical GR)
$E \ll M_{\mathrm{planck}}$	$\longleftarrow$	$E \sim M_{\mathrm{planck}}$	$\longrightarrow$	$E \gg M_{\mathrm{planck}}$
	Many spins (coarse-graining)		Large spins (no coarse-graining)	

- Continuum limit: many low-energy quanta.
- Large-spin limit: few high-energy quanta.

- All this refers to the large-spin limit *of the fundamental theory*.
- Large spins may also arise in a coarse-graining picture.
  This will involve some new, effective dynamics.

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- The large-spin vertex amplitude reproduces the classical action iff analytically continued to $\gamma = \pm i$.
- A first-ever agreement in LQG between BH entropy derived from an effective action and from a microscopic state counting.
- The limit should be thought of as a high-energy (“transplanckian”) regime.

New questions

Understand Im S!

- Proper relation to black hole entropy?
- Implications on unitarity?

The large-spin limit wants $\gamma = \pm i$.

- Must be a way to quantize with $\gamma = \pm i$ directly!
- Use lightlike structures?

New understanding of large-spin limit?

- Conceptual implications on "IR" divergences?