GROUP FIELD THEORY AND ALL THAT

A NEW NON-COMMUTATIVE REPRESENTATION OF GROUP FIELD THEORY: DUALITY OF SIMPLICIAL PATH INTEGRALS AND SPIN FOAM MODELS

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MAIN AIM(S) OF THIS TALK

- Show that:
  - there exist an exact duality between simplicial path integrals and spin foam models
  - this, in turn, is the covariant counterpart of a duality between a non-commutative triad (metric) representation and the spin network representation of quantum states of geometry
  - this duality can be seen explicitly and put to use at the group field theory level, as a new non-commutative representation of GFTs
  - exploiting this duality allows progress in:
    - understanding the geometry of spin network states and spin foam amplitudes and to encode it in GFT action
    - studying the symmetries of spin foam models (diffeos) also at GFT level
    - constructing GFT/spin foam models for 4d quantum gravity
    - ....
- Point out some open issue for further work
- Stimulate discussion......

AN ANALOGY: POINT PARTICLE ON A CIRCLE

- to understand meaning of duality between simplicial gravity path integral and spin foam representation, look at simpler case
- consider particle moving on circle, with position-dependent potential $\Rightarrow$ position $q(t) \in [0, 2\pi)$, $q(t) + 2\pi m \sim q(t)$ and with discretized time interval $\Gamma = \cup_{i=1}^{N} [t_{i-1}, t_i]$ ("spacetime")
- transition amplitude can be written in two equivalent forms:

$$\langle q, t | q_0, t_0 \rangle_\Gamma = \sum_{m=-\infty}^{+\infty} \prod_{j=1}^{N} \int_{-\infty}^{+\infty} dq_j \prod_{j=1}^{N+1} \int_{-\infty}^{+\infty} \frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar} \sum_{j=1}^{N+1} \left[p_j (q_j - q_{j-1} + 2\pi m) - \delta t \hat{H}(p_j, q_j) \right]} =$$

$$= \prod_{j=1}^{N} \int_{0}^{2\pi} dq_j \prod_{j=1}^{N+1} \sum_{n_j=-\infty}^{+\infty} \frac{1}{2\pi} e^{\frac{i}{\hbar} \sum_{j=1}^{N+1} \left[n_j (q_j - q_{j-1}) - \delta t \hat{H}(\hbar n_j, q_j) \right]}$$

- the first expression is the discretized "classical"path integral using classical continuum variables; the second is the (spin foam) re-writing of the same in terms of the quantum numbers labeling quantum states of the theory
- above example still a little deceptive because also (spin foam) expression in terms of quantum numbers has path integral form

ANOTHER ANALOGY: POINT PARTICLE ON A SPHERE

- for a particle on $S^3 \sim \mathrm{SU}(2)$ the path integral expression has same form in continuum approximation (with appropriate boundary conditions on q)
- the (spin foam) representation is given by the eigenfunctions expansion of the propagator (expansion in spherical harmonics $\sim$ spin network basis)

$$K(q,t;q_0,t_0)_\Gamma \approx_{\delta t\sim 0} \prod_{k=1}^{N} \int_{S^3} d\Omega_k(\theta_k) \prod_{k=1}^{N+1} \sum_{j_k} (2j_k+1)\, e^{-i\delta t(j_k^2-\frac{1}{4})+ij_k\Theta_k(\theta_k)} =$$

$$= \sum_{l=0}^{\infty} (2l+1) e^{-\frac{i(t-t_0)}{2m} l(l+1)}\, C_{2l}(q,q_0)$$

and does not have a path integral form for a classical particle action

- quantum discreteness of momenta ($l(l+1)$) in states encoded in boundary conditions for classical position variables in the classical path integral, still involving classical continuous momenta
- these two equivalent representations have complementary advantages/difficulties and are useful for answering different questions

A DUALITY OF DISCRETE PATH INTEGRALS AND SPIN FOAMS?

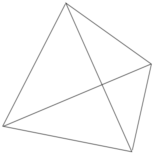
- one can then suggest that there exist a similar duality between spin foam models and gravity path integrals (in a simplicial context because spin foam models are constructed in a piecewise-flat context)
- if it is so, we then have two equivalent ways of expressing the dynamics of quantum geometry:
- advantages of discrete path integral formulation:
  - clearer geometric structure, semi-classical limit, relation with action and classical dynamics, symmetries, (calculation of observables?)
- advantages of spin foam formulation:
  - clearer algebraic properties, structure of quantum states, connection with canonical theory, (calculation of observables?)
- we already know such duality at level of partition function for 3d gravity
- we also know it is true in lattice gauge theory (Oeckl-Pfeiffer '00)

3D GRAVITY: SIMPLICIAL PATH INTEGRAL VS SPIN FOAM REPRESENTATION

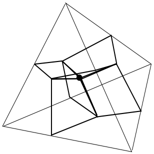
The theory we want to quantize in simplicial and then spin foam formalism is:

$$S(e,\omega) \;=\; \int_{\mathcal{M}} tr\left(e \wedge F(\omega)\right)$$

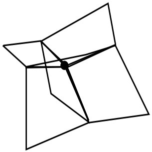
$e^I(x) \in su(2)$ triad 1-form - $\omega^J(x) \in su(2)$ connection 1-form with curvature $F^{IJ}(\omega)$
introduce simplicial complex and its topological dual



tetrahedron



tetra + dual



dual

discretize the continuum gravity variables on the discrete space:
$e(x) \longrightarrow E_e = E_f = \int_e e(x) = E^i J_i \in \mathfrak{su}(2)$ $\omega(x) \longrightarrow h_L = e^{\int_L \omega} \in \mathrm{SU}(2)$
$F(\omega) \longrightarrow G_f = G_e \;=\; \prod_{L\in\partial f} h_L = e^{F_f} \in \mathrm{SU}(2)$

3D GFT AMPLITUDES AS SIMPLICIAL GRAVITY PATH INTEGRALS

define a discrete action for the discrete variables, corresponding (in equations, symmetries, naive continuum limit) to the continuum 3d gravity action:

$$S(E,g) = \sum_{f\in\Gamma} tr\,(E_f\, G_f) = \sum_{e\in\Delta} tr\,(E_e\, G_e)$$

define path integral for discrete theory:

$$\mathcal{Z} = \prod_f \int_{\mathfrak{su}(2)} dE_f \prod_L \int_{\mathrm{SU}(2)} dh_L\, e^{i\,\sum_f\, tr\left(E_f\, G_f\right)}$$

the integral over the Lie algebra variables E can actually be computed:

$$\mathcal{Z} = \prod_f \int_{\mathfrak{su}(2)} dE_f \prod_L \int_{\mathrm{SU}(2)} dh_L\, e^{i\,\sum_f\, tr\left(E_f\, G_f\right)} = \prod_L \int_{\mathrm{SU}(2)} dh_L \prod_f \delta\left(G_f(h_L)\right)$$

we can derive the corresponding spin foam model from the simplicial path integral; this shows that a spin foam model can simply be seen as a convenient re-writing of a simplicial gravity path integral when this is expressed in connection (group) variables

3D SPIN FOAM QG

1) decompose δ in representations $(\Delta_j = 2j+1)$: $\delta\left(G_f\right) = \sum_{j_f} \Delta_{j_f} \chi^{j_f}\left(G_f\right)$

2) decompose the characters in repr. functions $\chi^{j_f}\left(\prod_L h_L\right) = \sum_{\{m\}} \prod_L D^{j_f}_{mm'}\left(h_L\right)$

$$Z = \left(\prod_f \sum_{j_f} \Delta_{j_f}\right) \prod_L \int dh_L D^{j_{f_1}}_{kl}\left(h_L\right) D^{j_{f_2}}_{st}\left(h_L\right) D^{j_{f_3}}_{mn}\left(h_L\right)$$

3) do the integrals over SU(2):

$$\int_{\mathrm{SU}(2)} dh_L D^{j_1}_{kl}\left(h_L\right) D^{j_2}_{st}\left(h_L\right) D^{j_3}_{mn}\left(h_L\right) = C^{j_1 j_2 j_3}_{ksm} C^{j_1 j_2 j_3}_{ltn}$$

4) contract (do the sum) the 3j-symbols
the final result is the Ponzano-Regge spin foam model:

$$Z(\Gamma) = \left(\prod_f \sum_{j_f}\right) \prod_f \Delta_{j_f} \prod_v \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}_v$$

SIMPLICIAL GEOMETRY FROM SPIN FOAMS? THE ASYMPTOTIC LIMIT OF THE VERTEX AMPLITUDE

- can we see the simplicial geometry (manifest in amplitudes written as simplicial gravity path integral) also in the spin foam representation? (heuristic)
- from LQG: j_e's label links of boundary spin networks, are eigenvalues of length operators, so characterize edge lengths at quantum level
- in the simplicial gravity path integral (that uses only classical variables), this role is played by the (modulus of the) discrete triad (Lie algebra) variables E_e
- the relation between them should appear in the semi-classical approximation (quantum numbers go to classical variables)
- if we take a semi-classical approximation of the simplicial gravity path integral (say, for single 3-simplex), for fixed (large) E_e, we expect to obtain something of the form

$$A(E) \sim (\ldots..) \left(e^{+\,i\sum_e tr(E_e G_e(E))} + e^{-\,i\sum_e tr(E_e G_e(E))} \right)$$

where $S_R = \sum_e tr\left(E_e G_e(E)\right)$ is Regge action, depending only on edge lengths

because semi-classical approximation amounts, for fixed E_e, to approximating amplitude with solutions of the equation enforcing metricity of the connection

THE ASYMPTOTIC LIMIT OF THE VERTEX AMPLITUDE

- now we look at the spin foam expression:

  one can show, for a single vertex (tetrahedron), that for large, fixed j's:

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}_{v*} \simeq \cos S_R(l_e) \simeq e^{iS_R} + e^{-iS_R}$$

  where $S_R(l_e = 2j+1)$ is the Regge action for simplicial gravity, with edge lengths given by $2j_e + 1$

- thus amplitudes in spin foam representation match expected form of semi-classical simplicial gravity path integral

  it means that the connection degrees of freedom have a good semi-classical geometry (metricity correctly imposed)

  this confirms the geometric meaning of the quantum variables j_e and

  suggests in itself (if we didn't have it already) a simplicial gravity formulation of same spin foam amplitudes

REALIZING THE DUALITY IN FULL....

Not enough:

- in 3d: need to go beyond partition function, no duality at the level of quantum states or observables, no clear duality for transition amplitudes
- no analogous (complete) path integral representation (derivation) for 4d models (EPR(L)), not beyond partition function for BC (measure unclear) or FK models (no obvious relation with standard BF theory) (Livine-Bonzom 08, Bonzom 09)
- no definition of mapping between the two representations - no duality

Convenient/promising setting to realize the duality is Group Field Theory:

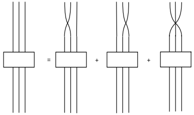
- GFTs are interpreted as 2nd quantized theories
  - of simplicial geometry and
  - of canonical LQG (QFT of spin networks)

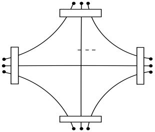
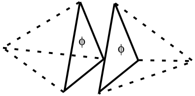
  (field ϕ represents "2nd quantized (D-1)-simplex or spin net vertex")
- FD are 2-complexes Γ topologically dual to simplicial complexes Δ
- FD amplitudes have interpretation of covariant implementation of dynamics of simplices/spin nets, thus expected to be related to path integrals for simplicial quantum gravity but can also be -always- represented as Spin Foam models
- other: new take on LQG dynamics, complete definition of SF models, topology change, use of QFT language and techniques, ...

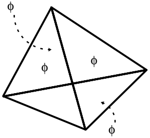
3D RIEMANNIAN QUANTUM GRAVITY AS A GFT - BOULATOV MODEL

- $G = SU(2)$, Real field: $\phi(g_1, g_2, g_3) = \phi(g_1 g, g_2 g, g_3 g) : SU(2)^{\times 3} \rightarrow \mathbb{R}$
- ϕ corresponding to LQG cylindrical function (connection representation)
- invariance imposed by projection: $\phi(g_1, g_2, g_3) = \int dh\, \Phi(g_1 h, g_2 h, g_3 h)$
- the action is:

$$S[\phi] = \frac{1}{2} \int dg_i\, \phi(g_1, g_2, g_3)^2 + \frac{\lambda}{4!} \int dg_j\, \phi(g_1, g_2, g_3)\phi(g_3, g_4, g_5)\phi(g_5, g_2, g_6)\phi(g_6, g_4, g_1)$$





- quantum theory defined by perturbative expansion of partition function:

$$Z = \int \mathcal{D}\phi\, e^{iS[\phi]} = \sum_{\Gamma} \frac{\lambda^{N_\Gamma}}{sym[\Gamma]} Z(\Gamma)$$

- Feynman diagrams dual to 3d simplicial complexes, with Feynman amplitudes:

$$Z(\Gamma) = \prod_{L \in \Gamma} \int dh_L \prod_{f} \delta\Big(\prod_{L \in \partial f} h_L\Big)$$

BOULATOV GFT MODEL - REPRESENTATION SPACE

- we can repeat the same calculation in representation space, expanding the field over $SU(2)^3$ in representations j

$$\phi(g_1, g_2, g_3) = \sum_{j_1, j_2, j_3} \phi^{j_1 j_2 j_3}_{m_1 m_2 m_3} D^{j_1}_{m_1 l_1}(g_1) D^{j_2}_{m_2 l_2}(g_2) D^{j_3}_{m_3 l_3}(g_3) C^{j_1 j_2 j_3}_{l_1 l_2 l_3}$$

- analogous to spin network representation of LQG wave function
- the end result for the Feynman amplitude is:

$$Z(\Gamma) = \left(\prod_f \sum_{j_f} \right) \prod_f (2j_f + 1) \prod_v \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}$$

- so, one finds that:

$$Z(\Gamma) = \prod_L \int dg_L \prod_f \delta(\prod_{L|f} g_L) = \left(\prod_f \sum_{j_f} \right) \prod_f (2j_f + 1) \prod_v \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}$$

- still, full simplicial path integral representation is missing: no triad (metric) variables in the formalism - only group elements or quantum numbers
  - difficult to read out simplicial geometry in quantum amplitudes or in GFT action
  - difficult to study semi-classical approximation
  - difficult to identify symmetries at GFT level (e.g. translation symmetry of BF)
  - in trying to move to 4d: difficult to impose simplicity (Plebanski) constraints

Metric representation for GFTs, simplicial path integrals and spin foam models

work in progress:

- A. Baratin, D. Oriti - non-commutative metric representation of GFTs
- A. Baratin, F. Girelli, D. Oriti - translation symmetry of BF theory at GFT level
- A. Baratin, B. Dittrich, D. Oriti, J. Tambornino - flux representation of kinematical LQG states
- A. Baratin, D. Oriti - GFT construction of 4d simplicial gravity path integrals and spin foam models

based on previous work and ideas on:

- GFT and simplicial gravity path integrals [Livine-Oriti, '02; Oriti, '05; Oriti, '06; Oriti-Tlas, '07; Oriti, '09; Oriti-Tlas, to appear]
- non-commutative Fourier transform [Majid,'00; Livine-Freidel, '05; Noui, '06, Joung-Mourad-Noui, '06, Freidel-Majid, '07]
- Lagrangian approach to spin foam models and discrete geometric conditions [Livine-Bonzom, '08; Conrady-Freidel, '08, Conrady-Freidel, 08, Oriti, '09; Bonzom, '09, Bonzom, '09]
- projected spin networks [Livine, '02; Alexandrov-Livine, '02; Alexandrov, '08]
- coupling of 4-simplices and extended GFT formalism [Livine-Oriti, '05]

PRELIMINARIES: NON-COMMUTATIVE FOURIER TRANSFORM

- consider the non-commutative $\mathfrak{su}(2)$ Lie algebra with basis $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$
- on such non-commutative space, one can define non-commutative plane waves $e_g(x)$, forming basis of functions on Lie algebra $\simeq$ functions on $\mathbb{R}^3$ endowed with a star product "*" reflecting the non-commutativity of algebra:
  - consider the group element $g \in \mathrm{SU}(2)$ and the Lie algebra element $x \in \mathfrak{su}(2)$, and define: $e_g(x) : \mathfrak{su}(2) \times \mathrm{SU}(2) \to \mathbb{C} : (x, g) \to e^{i\frac{1}{2}Tr(xg)}$ (fundamental representation)
  - this leads to a non-commutative Fourier transform: $C(\mathrm{SU}(2)) \to C_*(\mathbb{R}^3) \sim C(\mathfrak{su}(2))$

$$\phi(x) = \int_{\mathrm{SU}(2)} dg\, \phi(g) e_g(x) = \int_{\mathrm{SU}(2)} dg\, \phi(g)\, e^{i\frac{1}{2}Tr(xg)}$$

  - with star product defined on plane waves to respect the group multiplication, as:

$$(e_{g_1} * e_{g_2})(x) = e^{i\frac{1}{2}Tr(xg_1)} * e^{i\frac{1}{2}Tr(xg_2)} = e^{i\frac{1}{2}Tr(xg_1g_2)} = e_{g_1g_2}(x)$$

    and by linearity on generic functions
  - the same NC Fourier transform can be inverted (some issues with SU(2) vs SO(3)): $\phi(g) = \int d\vec{x} \left(\phi * e_{g^{-1}}\right)(x)$
- note: extension to $\mathrm{Spin}(4) \sim \mathrm{SU}(2) \times \mathrm{SU}(2)$ is straightforward

PRELIMINARIES: NON-COMMUTATIVE FOURIER TRANSFORM

- note: $C_*(\mathbb{R}^3)$ is space of functions on $\mathbb{R}^3$ with finite resolution, whose usual Fourier transform has bounded momenta
- one has delta functions: $\delta(g) = \int dx\, e_g(x) \qquad \delta_x(y) = \int dg\, e_{g^{-1}}(x)\, e_g(y)$
- important: $\int dy\ (\delta_x * \phi)(y) = \int dy\ (\phi * \delta_x)(y) = \phi(x) \qquad \delta(0) < \infty$
- $\phi(g^{-1}) \to \phi(-x)$
- crucial for later calculations:

$$(\phi \cdot \phi)(h) = \int_{SU(2)} dg\, \phi(g)\, \phi(h^{-1}g) \quad = \quad \int_{\mathbb{R}^3} dxdy\ (\phi *_x G(\cdot,\cdot) *_y \phi)(x,y)$$

$$\text{with} \quad G(x,y) = (\delta_{-x} * e_h)(y) \quad = \quad \left(e_h * \delta_{-h^{-1}xh}\right)(y)$$

- idea: use non-commutative Fourier transform to turn GFTs into non-commutative field theories on Lie algebras (B-variables)
- note: in BF, holonomy and B variables are canonically conjugate; in LQG, (smeared) triad (B) variables do not commute
- extension of NC Fourier transform to functions on $SU(2)^D$ is immediate:

$$\phi(x_1,..,x_D) = \int dg_1 \,..\, dg_D\, \phi(g_1,...,g_D)\, e_{g_1}(x_1)....e_{g_D}(x_D)$$

$$\phi(g_1,..,g_D) = \int dx_1 \,..\, dx_D \left(\phi *_{1,..,D} e_{g_1^{-1}}.....e_{g_D^{-1}}\right)(x_1,...,x_D)$$

A NEW REPRESENTATION FOR THE BOULATOV 3D GRAVITY MODEL

- $G = SU(2)$, real field: $\phi(g_1, g_2, g_3) = \int dh\, \Phi(hg_1, hg_2, hg_3)$
- action is

$$S[\phi] = \frac{1}{2}\int dg_i\, \phi(g_1,g_2,g_3)^2 + \frac{\lambda}{4!}\int dg_j\, \phi(g_1,g_2,g_3)\phi(g_3,g_4,g_5)\phi(g_5,g_2,g_6)\phi(g_6,g_4,g_1)$$

- after NC Fourier transform:

$$\begin{aligned}(C * \phi)(B_1,B_2,B_3) &= \text{“}\delta(B_1+B_2+B_3)\,\phi(B_1,B_2,B_3)\text{”} = \\ &= \int dg_1 \,..\, dg_D\, \phi(g_1,g_2,g_3)\, e_{g_1}(B_1)\, e_{g_2}(B_2)\, e_{g_3}(B_3) = \\ &= \int dg_1 ... dg_D \int dh \phi(h^{-1}g_1, h^{-1}g_2, h^{-1}g_3)\, e_{g_1}(B_1)\, e_{g_2}(B_2)\, e_{g_3}(B_3) = \\ &= \left(\int dh\,(e_h \otimes e_h \otimes e_h) * \phi\right)(B_1,B_2,B_3)\end{aligned}$$

- geometry: triangle (field) characterized by triads B_i associated to its edges
- note: the presence of the projector/constraint C= closure of edge triads makes it a 2nd order (metric) theory

A new representation for the Boulatov 3d gravity model

- action can be re-written in B basis using NC Fourier transform of the field ϕ
- kinetic part takes the form:

$$\mathcal{K}(\phi) = \prod_i \int_{\mathbb{R}^3} dB_i d\tilde{B}_i \int_{\mathrm{SU}(2)} dh \left(\phi *_{B_i} \prod_i G_h(\cdot,\cdot) *_{\tilde{B}_i} \phi \right)(B_i, \tilde{B}_i)$$

$G(B_i, B_i) = \left(e_h * \delta_{-h^{-1}B_i h}\right)(\tilde{B}_i)$ replace the δ functions of group picture

- interaction term can be nicely expressed graphically, and presents a function $G(B_i^\tau, B_i^{\tau'})$ in place of δ over the group, with combinatorial pattern of a simplex; in place of h above, one finds $h_{\tau\sigma} h^{-1}_{\tau'\sigma}$; meaning: B_i^τ and $B_i^{\tau'}$ associated to same edge i in two different triangles/fields τ and τ' are identified after parallel transport through center of tetrahedron σ by means of connection h
- we can compute the Feynman amplitudes $Z(\Gamma)$ (the -same- $Z(\Gamma)$ of the Boulatov model) in new representation, taking into account the $*$-product structure (fundamental non-commutativity)
- construction: (ordered) composition of 2-point functions $G(\cdot,\cdot)$ on Lie algebra, dual to usual convolution of delta functions on the group

Simplicial 3d gravity path integral from Boulatov GFT

- the result is (! little abuse of notation !):

$$Z(\Gamma) \quad = \quad \prod_f \int_{\mathbb{R}^3} dB_f \prod_L \int_{\mathrm{SU}(2)} dh_L \prod_f \left(e^{i\frac{1}{2}Tr(B_f H_f)} * \delta_{H_f^{-1} B_f H_f}(B_f) \right)$$

- 3d gravity simplicial path integral! - Lie algebra variables B_f correctly identified with the discrete triad associated to each edge
- additional constraint: $\delta_{H_f^{-1} B_f H_f}(B_f)$:
  - non-commutative analogue of constraint $\delta(B_f - H_f B_f H_f^{-1})$, which can be easily rewritten as delta function on group, and interpreted as constraint on connection
  - imposes that holonomy H_f lies in the plane orthogonal to the edge to which B_f is associated (characterizing Levi-Civita connection)
  - its meaning nicely identified in presence of boundary (e.g. single 3-simplex) with fixed B_f on boundary: it constrains the angles characterizing the connection elements h to be the dihedral angles corresponding to the B_f's (characterizing Levi-Civita connection)
- note: model contains more than 1st order BF theory
- same construction can be repeated for 4d BF theory, with same result (but no geometric interpretation of B or h variables)

Translation symmetry in 3d BF theory and in GFT

Continuum 3d BF theory possesses three type of symmetries

- translation symmetry: $\delta^T_\phi e \, = \, d_\omega \phi \qquad \delta_\phi \omega \, = \, 0 \qquad \phi = \mathfrak{su}(2)$-valued scalar
- local rotation symmetry: $\delta^R_\Lambda e \, = \, [e, \Lambda] \qquad \delta^R_\Lambda \omega \, = \, d_\omega \Lambda \qquad \Lambda \in \mathfrak{su}(2)$
- diffeomorphism symmetry:
  $\delta^D_\xi e \, = \, d(\iota_\xi e) + \iota_\xi(de) \qquad \delta^D_\xi \omega \, = \, d(\iota_\xi \omega) + \iota_\xi(d\omega) \qquad \xi$ vector field
- they are not independent:
  $\delta^D_\xi e \, = \, \delta^T_{\iota_\xi e} e \, + \, \delta^R_{\iota_\xi \omega} e \, + \, \iota_\xi(d_\omega e) \qquad \delta^D_\xi \omega \, = \, \delta^T_{\iota_\xi \omega} \omega \, + \, \delta^R_{\iota_\xi \omega} \omega \, + \, \iota_\xi(F(\omega))$
- on-shell (classically) diffeos obtained by combination of translation and rotation

In discrete 3d BF theory diffeos are broken (Dittrich-Bahr '09) but leave residual symmetry (Freidel-Louapre '02):

- discrete translation symmetry:
  $B_e \longrightarrow B_e \, + \, \phi_{v1} + [\Omega_{v1}(g_L), \phi_{v1}] \, - \, \phi_{v2} \, - \, [\Omega_{v2}(g_L), \phi_{v2}]$
- discrete rotation symmetry: $B_e(\sigma) \longrightarrow k_\sigma B_e(\sigma) k_\sigma^{-1} \qquad g_L \longrightarrow k_{\sigma 1} g_L k_{\sigma 2}^{-1}$

Rotation symmetry easily identified in Boulatov GFT in terms of group elements

Difficult to identify translation symmetry, in absence of a B-representation

TRANSLATION SYMMETRY IN BOULATOV MODEL

We now have a B-representation for the GFT action

$$S = \int [dB_i](\phi * \phi)(B_1, B_2, B_3) + $$
$$+ \frac{\lambda}{4!} \int dB] \text{“}\phi(B_1, B_2, B_3) * \phi(B_3, B_4, B_5) * \phi(B_5, B_2, B_6) * \phi(B_6, B_4, B_1)\text{”}$$

- We define first a simple transformation of the GFT field (for $\epsilon_e \in \mathfrak{su}(2)$):

  $$(T_{\epsilon_e}\phi)(B_1, B_2, B_3) = (T_{\epsilon_e} \triangleright \phi)(B_1, B_2, B_3) = \phi(B_1 + \epsilon_1, B_2 + \epsilon_2, B_3 + \epsilon_3)$$

  with the transformation parameters, one per edge e of the triangle corresponding to ϕ, restricted to satisfy $\sum_{e=1,2,3} \epsilon_e = 0$ because of closure constraint C
- Kinetic term invariant
  $\int [dB_e]\,(T_{\epsilon_e}\phi * T_{\tilde{\epsilon}_e}\phi)\,(B_1, B_2, B_3)\ \int [dB_e]\,(\phi * \phi)\,(B_1, B_2, B_3)$ iff $\epsilon_e = \tilde{\epsilon}_e$ (locality)
- Interaction term imposes further restriction on the transformation parameters:

  $$\epsilon_e = \tilde{\epsilon}_e = \epsilon_{v_e^1} - \epsilon_{v_e^2}$$

  where $v_e^{1,2}$ are the two vertices of the triangulation joined by the edge e

- the resulting transformation is generated by Lie algebra elements located at the vertices of the triangulation affecting simultaneously 3 field arguments (edges), thus common to 3 fields (triangles) in the interaction (tetrahedron) (everything is here expressed in the frame of the tetrahedron)
- the resulting transformation of the field is a symmetry of the GFT action
- it can be verified that it indeed corresponds to the translation symmetry at the level of the Feynman amplitudes (simplicial BF path integrals)
- note: what was a (discretized) local gauge symmetry in the Feynman amplitude (2nd quantized theory of discrete gravity), is a global symmetry at the level of the GFT (in line with the 3rd quantization interpretation)
- the presence of the above symmetry suggests a change of variables in each GFT field: $B_e \rightarrow B_v$, leading to a new "multi-local"form of the interaction, making the symmetry manifest:

$$\int [dB_v] \text{ ``}\phi(B_a, B_b, B_c) * \phi(B_a, B_b, B_d) * \phi(B_b, B_c, B_d) * \phi(B_d, B_b, B_a)\text{''}$$

- note: the closure constraint, satisfied by the fields in the GFT action, is discrete equivalent of $d_\omega B_e = 0$, thus this GFT symmetry is closer to diffeomorphisms than the 1st order BF translation symmetry

A FLUX REPRESENTATION FOR LQG KINEMATICAL STATES

This non-commutative metric representation of the GFT suggests a similar triad/flux representation for general LQG kinematical states in 4d

Indeed, smeared triad variables E_S, conjugate to the holonomies g_L of the $SU(2)$ connection, do not commute $\{E_S, E_{S'}\} \neq 0$, if the surfaces of smearing S, S' intersect

Using the NC tools, we can transform generic cylindrical functions $\Psi_\Gamma(g_1, ..., g_{L_\Gamma})$ based on graph Γ with L links, into functions of conjugate Lie algebra variables:

$$\Psi_\Gamma(E_1, .., E_{L_\Gamma}) = \int dg_1 .. dg_{L_\Gamma} \, \Psi(g_1, ..., g_{L_\Gamma}) \, e_{g_1}(E_1) .... e_{g_{L_\Gamma}}(E_{L_\Gamma})$$

- triad operators act by (non-commutative) multiplication: $\left(\hat{E}_L \triangleright \Psi_\Gamma\right)(E_i) = E_L * \Psi_\Gamma(E_i)$
- holonomy operators $\hat{e}_a(g) \equiv e_g(a)$ act as translations: $(\hat{e}_{a_i} \triangleright \Psi_\Gamma)(E_i) = \Psi_\Gamma(E_1, .., E_i + a_i, ..E_{L_\Gamma})$
- can define usual kinematical inner product between wave functions defined on two arbitrary graphs, in flux representation, by "Fourier transformation"
- cylindrical consistency, etc (in progress)

Correct (covariant) duality induced by simplicity constraints (see 4d construction....)

GFT MODELS FOR 4D QUANTUM GRAVITY

- 4d gravity is constrained BF theory: $\mathfrak{so}(4)$-Plebanski action

$$S(\omega, B, \phi) = \int_{\mathcal{M}} \left[B^{IJ} \wedge F_{IJ}(\omega) - \frac{1}{2} \phi_{IJKL} B^{KL} \wedge B^{IJ} \right]$$

from which: equations of motion:

$$\mathcal{D}_\omega B = dB + [\omega, B] = 0 \quad F^{IJ}(\omega) = \phi^{IJKL} B_{KL}$$

$$B^{IJ} \wedge B^{KL} = e\, \epsilon^{IJKL} \Leftrightarrow \quad \begin{array}{lll} I & B^{IJ} = \pm e^I \wedge e^J \\ II & B^{IJ} = \pm \frac{1}{2} \epsilon^{IJ}{}_{KL}\, e^K \wedge e^L \end{array}$$

- substituting constraints in Plebanski action on gets the Palatini action for gravity, in the “II ”sector:

$$S(\omega, e) = \int_{\mathcal{M}} \left[\frac{1}{2} \epsilon^{IJ}{}_{KL}\, e^K \wedge e^L \wedge F_{IJ}(\omega) \right]$$

GFT models for 4d quantum gravity

- strategy: start from GFT/spin foam models for 4d BF theory and apply on them suitable constraints, simplicial or quantum versions of the Plebanski constraints, to reduce them to gravitational models
- discrete BF theory in 4d:

$$B(x) \rightarrow B_t = B_f = \int_t B(x) = B_f^{IJ} T_{IJ} \in \mathfrak{so}(4)$$

$$\omega(x) \rightarrow g_L = e^{\int_L \omega} \in \mathrm{SO}(4) \qquad F(\omega) \rightarrow G_f = G_t = \prod_{L \in \partial f} g_L = e^{F_f} \in \mathrm{SO}(4)$$

$$S(B,g) = \sum_{f \in \Gamma} tr\,(B_f\, G_f) = \sum_{e \in \Delta} tr\,(B_t\, G_t)$$

- discrete simplicity constraints on B's [Barrett, Crane, De Pietri, Freidel, Reisenberger, Alexandrov,Krasnov,...]:
  $a) \forall\, tetra\, t \in \Delta \;\; \exists n \in S^3 \;/\; B_f^{IJ} n_J = 0 \;\; \forall B_f \;\; f \subset t$
  $b) \forall\, tetra\, t \in \Delta \;\; \exists n \in S^3 \;/\; (*B_f)^{IJ} n_J = 0 \;\; \forall B_f \;\; f \subset t$ (gravitational sector)

GFT MODELS FOR 4D QUANTUM GRAVITY

- start from GFT for 4d BF theory (analogous to Boulatov model in 3d): real field: $\phi(g_1,..,g_4) : G^{\times 4} \to \mathbb{R}$, symmetric under (local Lorentz): $\phi(g_1 g, g_2 g, ..., g_4 g) = \phi(g_1, ..., g_4)$

$$S[\phi] = \frac{1}{2}\int dg_i\, [\phi(g_1,g_2,g_3,g_4)]^2 + \frac{\lambda}{5!}\int dg_j\, [\phi(g_1,g_2,g_3,g_4)\phi(g_4,g_5,g_6,g_7) \phi(g_7,g_3,g_8,g_9)\phi(g_9,g_6,g_2,g_{10})\phi(g_{10},g_8,g_5,g_1)]$$

- of this (BF) model we know the quantum boundary states ($SO(4)$ spin networks, and the Feynman amplitudes:

$$Z(\Gamma) = \prod_f \int_{\mathfrak{so}(4)} dB_f \prod_L \int_{\mathrm{SO}(4)} dg_L\, e^{i\sum_f tr(B_f G_f)} = \left(\prod_{L\in\Gamma}\int dg_L\right)\prod_f \delta(\prod_{L\in\partial f} g_L) = \\ = \sum_{\{j_+,j_-\}}\prod_f (2j_+ + 1)(2j_- + 1)\prod_v \{15-j\}^v_+\{15-j\}^v_-$$

and also how the amplitudes in spin foam representation can be reconstructed from the structure of the boundary states

GFT models for 4d quantum gravity

- 4d gravity models: insert appropriate modifications in the kinetic and/or vertex term that implement Plebanski constraints of gravitational sector
- two possible strategies:
  1. define a GFT model with Feynman amplitudes being manifestly simplicial path integrals for BF with gravity constraints; then re-write as spin foam models
  2. find quantum version of Plebanski constraints, impose them on BF spin network states to get gravity spin network states, then construct spin foam/GFT amplitudes; then check they encode simplicial geometry
- first strategy difficult: we do not have a formulation of GFT reproducing a BF simplicial path integral with both B and g variables - (constraints should be imposed on B's!)
- second strategy leads to Barrett-Crane, Engle-Pereira-Rovelli(-Livine) spin foam model
- other (closer to the second) strategy:
  re-write BF spin foam in $\mathrm{SO}(4)$ coherent states, identify CS parameters as semi-classical counterpart of classical B's, impose gravity constraints on them; this leads, depending on how it is done, to EPR or Freidel-Krasnov models

4D QG FROM CONSTRAINING BF BOUNDARY STATES

- start from BF Spin(4) spin networks (labeled by pair (j^+, j^-)) and impose on them quantum version of gravity constraints
- constraints are:
  $\forall\, tetra\, t \in \Delta \;\; \exists n \in S^3 \; / \; (*B_f)^{IJ} n_J \; = \; 0 \;\; \forall B_f \;\; f \subset t$ (geometric sector)
  $\forall\, tetra\, t \in \Delta \;\; \exists n \in S^3 \; / \; (B_f)^{IJ} n_J \; = \; 0 \;\; \forall B_f \;\; f \subset t$ (topological sector)
  $\forall\, tetra\, t \in \Delta \;\; \exists n \in S^3 \; / \; (\gamma * B_f - B_f)^{IJ} n_J \; = \; 0 \;\; \forall B_f \;\; f \subset t$ (Immirzi γ)
- at quantum level, $B^{IJ} \rightarrow *T^{IJ}$ (or T^{IJ}) (or $*T^{IJ} + \frac{1}{\gamma} T^{IJ}$) generators of $\mathfrak{so}(4)$ algebra, act as operators on functions on the group/spin networks
- links of spin network vertices correspond to triangles, node of spin networks corresponds to tetrahedra; equations for B's give operator equations for T's and restrictions on representations and intertwiners labelling spin networks
- from such constrained spin networks construct amplitudes: take five gravity spin network vertices (corresponding to five tetrahedra in boundary of 4-simplex); “glue them together”by tracing out internal (vector) variables in each common representation space (corresponding to triangles common to two tetrahedra) $\longrightarrow$ amplitude for 4-simplex/spin foam vertex
  Barrett-Crane, Engle-Pereira-Rovelli(-Livine), Engle-Pereira, ...

- this methods gives: structure of boundary states, simplest possible vertex amplitude compatible with symmetries and with constraints
- takes nice care of constraint classes (using Master constraint) (Engle-Pereira)
- it does not allow to specify the other contributions to the spin foam amplitudes
- it gives:
  - geometric sector ($\gamma \to \infty$): Barrett-Crane vertex
  - topological sector ($\gamma \to 0$): EPR vertex
  - finite Immirzi parameter: EPRL vertex
- note: from this point of view, nothing wrong with the Barrett-Crane model
- other method: using coherent states (Livine-Speziale, Freidel-Krasnov):
  - use coherent states for $Spin(4)$: $|j^+, j^-, (n^+, n^-)\rangle$
  - coherent state parameters can be identified with classical bivector variables in mean values: $\langle j^+, j^-, (n^+, n^-)|(b^i_+ T^i_+, b^i_- T^i_-)|j^+, j^-, (n^+, n^-)\rangle = (j^+ n^i_+, j^- n^i_-)$ thus expected to play the same geometric role *in semi-classical limit*
  - idea: express BF spin foam in coherent states and impose the Plebanski constraints strongly on the coherent state parameters (treating them as classical variables)
- allows gluing performed by identifying "bivector variables"$(j^+, j^-, (n^+, n^-))$
- doubt: CS parameters are not really classical B variables.......
- it gives:
  - geometric sector: Freidel-Krasnov vertex
  - topological sector: EPR vertex
  - finite Immirzi parameter: FK vertex (coincides with EPRL for $\gamma < 1$)

4D QG FROM NC REPRESENTATION OF GFTS

- we can instead put to use the new non-commutative representation of GFTs
- transform GFT field for BF theory: $\phi(g_1, g_2, g_3, g_4)$, with $g_i \in \mathrm{Spin}(4)$ into a field of bivectors: $\phi(B_1, B_2, B_3, B_4)$ by NC Fourier transform
- 4d gravity: impose constraints on the bivector variables $B_f = (B_f^+, B_f^-)$
  easy to do, because they appear as classical variables in simplicial path integral: constraints to be imposed strongly by means of delta functions
- geometricity of the tetrahedron t ($\leftrightarrow \phi$) is imposed by:
  simplicity (geometric sector): $\exists k_t \in S^3 \sim \mathrm{SU}(2)$ (normal to tetrahedron t), s.t.
  $B_f^- = -k_t^{-1} B_f^+ k_t \quad \forall f \subset t$
  closure: $\sum_{f \subset t} B_f = 0$
- both can be imposed by means of projection operators:
  simplicity:
  $S_k \equiv \prod_{f \subset t} \delta_{-k_t^{-1} B_f^+ k_t}(B_f^-) = \prod_{f \subset t} \int_{\mathrm{SU}(2)} du_f \, e_{u_f}(k_t^{-1} B_f^+ k_t) \, e_{u_f}(B_f^-)$
  closure: $C \equiv$ "$\delta\left(\sum_{f \subset t} B_f\right)$" $\equiv \int_{\mathrm{Spin}(4)} dh \, e_h(B_1) \, e_h(B_2) \, e_h(B_3) \, e_h(B_4)$
- note: closure on the B is equivalent to invariance under left shift:
  $(C * \phi)(B_1, .., B_4) = \int dg_1 ... dg_4 \int dh \, \phi(hg_1, ..., hg_4) \, e_{g_1}(B_1) \dots e_{g_4}(B_4)$

4d QG from NC representation of GFTs

- we have now to decide how to impose these constraints in the GFT action
- simplest way: transform GFT field for BF theory: $\phi(g_1, g_2, g_3, g_4)$, with $g_i \in \mathrm{Spin}(4)$ into a field of bivectors: $\phi(B_1, B_2, B_3, B_4)$ by NC Fourier transform, and impose both constraints for arbitrary normal vectors k_t
- define field corresponding to geometric NC tetrahedron: $\Phi_{k_t}(B_1^+, B_2^+, B_3^+, B_4^+) = (S_{k_t} * C * \phi)(B_1, B_2, B_3, B_4)$
- consider “projected Ooguri action”:

$$S[\Phi] = \frac{1}{2}\int_{\mathfrak{su}(2)} dB_i^+ \, [\Phi_{k_t}(B_1^+, B_2^+, B_3^+, B_4^+)]^2 + \frac{\lambda}{5!}\int dB_j^+ \, [\Phi_{k_1}(B_1^+, B_2^+, B_3^+, B_4^+) \Phi_{k_2}(B_4^+, B_5^+, B_6^+, B_7^+)\Phi_{k_3}(B_7^+, B_3^+, B_8^+, B_9^+)\Phi_{k_4}(B_9^+, B_6^+, B_2^+, B_{10}^+)\Phi_{k_5}(B_{10}^+, B_8^+, B_5^+, B_1^+)$$

- note: dependence on the normal vectors k_t in each field/tetrahedron can be trivialized, by using the left-shift invariance (projector C)

4d QG from NC representation of GFTs

- amplitudes of the model give again nice simplicial gravity path integral:

$$Z(\Gamma) = \prod_f \int_{\mathfrak{so}(4)} dB_f \prod_t \int_{\mathrm{Spin}(4)} dh_L \prod_f (S_1 * S_2 * e_{H_e})(B_f) =$$

$$= \prod_f \int_{\mathfrak{so}(4)} dB_f \prod_t \int_{\mathrm{Spin}(4)} dh_L \prod_f (S_1 * S_2)(B_f) * e^{\frac{i}{2} Tr(B_f H_f)}$$

  where S_1 = simplicity constraints for bivector B_f, S_2 = constraints (on the connection) imposing consistency between simplicity constraints and parallel transport (gluing constraints $\sim$ discrete secondary second class constraints $\sim$ metricity of the connection)

- the same amplitudes can be easily expressed in the spin foam representation
- it gives Barrett-Crane model
- easy to see: because the dependence on the normal vectors k_t can be trivialized, the projected field $P_S\phi(g_i^+, g_i^-) = \prod \int_{SU(2)} du_i\, \phi((u_i g_i^+, u_i g_i^-))$ is a function of 4 copies of S^3, and the same S^3 is used in all fields in the action
- this action becomes indeed the usual GFT formulation of the BC model

4D QG FROM NC REPRESENTATION OF GFTS

- only problem (from point of view of simplicial geometry) is trivial dependence of the model on the normals of the tetrahedra: the normals to the same tetrahedron, as seen in two different 4-simplices, are uncorrelated
- need to identify them to define a unique geometry for the tetrahedron in both 4-simplices (induces additional correlation among 4-simplices)
- coupling of normals is not possible in standard formalism
- strategy (in progress):
  - go to extended field $\phi(B_1, B_2, B_3, B_4; k)$, with $k \in S^3$
  - use additional projector (to ensure gauge covariance): $(G * \phi)(B_1, B_2, B_3, B_4; k) = \int dh \phi(h \triangleright B_1, h \triangleright B_2, \triangleright B_3, h \triangleright B_4; h \triangleright k)$ with $h \in \mathrm{Spin}(4)$
  - can use h-invariance to fix $k = I = (1, 0, 0, 0)$; this fixes the S^3 component of the connection to be equal to k, but leaves free diagonal (with respect to k) part $\Rightarrow$ projected spin networks
  - geometric (NC) tetrahedron now: $(G * S_k * C * \phi)(B_1, B_2, B_3, B_4; k)$
  - k's are now independent variables, and can be coupled non-trivially, in particular can be identified in GFT kinetic term (same normal in both 4-simplices)
- note: issue of correct imposition of the constraints in the quantum theory is purely geometrical, dictated by purely classical considerations, as all geometric variables are present in GFT action and are under control

4d QG from NC representation of GFTs

Construction and analysis of new model in progress, but it seems that:

- it has amplitudes with simplicial gravity path integral form,
  with simplicity constraints
  as well as secondary constraints ensuring consistency between simplicity constraints and parallel transport,
  and dependence on normals k_t under control
- these constraints are those that allow to solve the connection for the bivectors (in commutative case) (Bonzom ’09)
- is based on projected spin networks
- in geometric sector, there are indications that the spin foam vertex amplitude is still the Barrett-Crane one, and that coupling of normals only affects other contributions to total amplitude
- thus, even from the point of view of simplicial geometry, there is nothing wrong with the Barrett-Crane vertex
- in topological sector, there are indications that the spin foam vertex amplitude is the EPR vertex (with boundary states given by pure $SU(2)$ spin networks)
- Immirzi parameter easily introduced by simple modification of projector S

PARTIAL CONCLUSIONS

- new GFT representation realizes explicit duality (unification) between simplicial quantum gravity and spin foam models/LQG
- it allows to keep under control the simplicial geometry of spin foam models directly at GFT level
- it allows to identify and study symmetries of the theory, e.g. diffeomorphisms
- it can provide a flux representation for LQG
- it provides a new (purely geometric) framework to construct spin foam models/simplicial path integrals for 4d quantum gravity
- $\ast$-product can be of help for relating LQG and GFT in definition of evolution operator
- it brings LQG/simplicial QG in closer contact with non-commutative geometry
- ....

Brief overview of open issues

- correct GFT model for 4d quantum gravity
  - compelling expression as simplicial gravity path integral in 4d
  - spin foam representation
  - space of spin network boundary states (with clear relation with LQG ones)
- a rigorous and physically transparent link between the canonical LQG framework and the GFT/spin foam one
  - 2nd quantization of spin network wave functions and Fock structure
  - derivation of the GFT path integral from LQG using coherent states and new B-representation
  - understanding how the dynamics of Hamiltonian/Master constraint is encoded in GFT action
- classical solutions of GFT equations
  - find more of them and understand their physical/geometric meaning
  - understand how they correspond to solutions of Hamiltonian/Master constraint
  - learn to control and use the GFT tree level expansion to this end
  - physical meaning of GFT coupling constant
- control and understanding of GFT Feynman expansion
  - manifold conditions
  - divergence of individual diagrams - perturbative renormalization (results in 3d)
  - divergence of total sum: Borel summability (results in 3d)
  - role of topology change, control over sum over topologies, physical consequences
  - physical meaning of coupling constant
  - a new notion of locality?

Brief overview of open issues

- GFT symmetries and simplicial gravity - diffeomorphisms
  - understand how diffeo symmetry is implemented/broken/modified in 4d gravity models and how to recover it in some regime
  - work at level of simplicial gravity path integral
  - develop a systematic analysis of these symmetries (and others) at GFT level, using QFT tools
- the continuum approximation and the link with General Relativity
  - physical interpretation of simplicial regime: few simplices/spin net vertices enough?
  - use results from simplicial gravity
  - develop coherent state techniques
  - study in detail continuum and semi-classical approximation of appropriate observables (e.g. correlations)
  - continuum approximation: regime of many GFT particles?
- statistical GFT: phase transitions, GR from GFT hydrodynamics?
  - what is the correct GFT vacuum (phase) for recovering continuum gravity as described by (modified) GR?
  - develop methods for extracting effective dynamics around different vacua
  - treat quantum space as a condensed matter system with microscopic (atomic) description given by a GFT
  - is continuum space a fluid of (very many) spin network/simplices?
  - is continuum GR a sort of hydrodynamics for them in such continuum/fluid regime?
  - physical meaning of GFT coupling constant

Thank you for your attention!