The unforgiving universe (semiclassicality in Loop Quantum Cosmology)

ILQG SEMINAR

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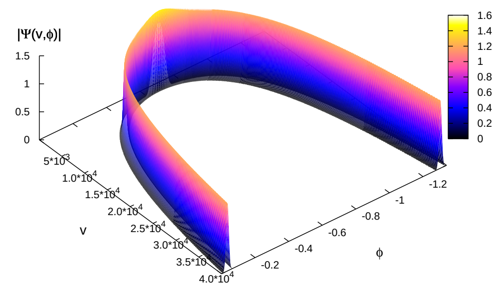
work by:

W. Kamiński, TP: arXiv:1001.2663

A. Ashtekar, TP *unpublished*

The problem

- Loop quantization of the isotropic/homogeneous cosmological models $\Rightarrow$ changes of the dynamics at near-Planck densities causing the Big Bounce.



  Observation: States sharply peaked throughout the evolution.

- Problem: In numerical simulations one has to select an example of the state for evolution:

  is the preservation of the semiclassicality robust?

- Addressing on the quantum level: Monte Carlo methods – probing the space of solutions via random samples (brute force approach).

  Need to run large number of time-costly simulations!

- Any Alternative approach?

Alternatives

- If the system solvable analytically: One can find the relation between dispersions at distant future and past.
  - Affirmative answer in sLQC: *A. Corichi, P. Singh, arXiv:0710.4543*
    States sufficiently sharply peaked initially remain so!
  - Problem: We need exact solvability of the system.
- Semiclassical dynamics: *M. Bojowald at al, arXiv:0911.4950, ...*
  - Choose canonical pair of variables X, P s.t. X, P and the momenta $G^{m,n} := \langle (\hat{X} - X)^m (\hat{P} - P)^n \rangle$ form closed algebra with evolution generator.
  - Capture the quantum dynamics as the EOMs for $G^{m,n}$.
  - Problem: Definiteness of the system of EOMs requires $|G^{m,n}| < \infty$: states decay faster than polynomially for both X, P. If X, P chosen naively, for many systems such states may not exist!
- The goal: Flexible and reliable method of comparing the distant future and past states.
  The means: The scattering picture.

Outline

An application to the simplest model:

- The flat FRW universe: specification.
- Geometrodynamical (WDW) vs LQC quantization.
- Method introduction for $\Lambda = 0$
  - WDW limit of LQC state.
  - The definition of the scattering picture.
  - An application: relation between dispersions.
- More complicated application: $\Lambda > 0$
  - WDW and LQC quantum system, deSitter limit.
  - The instantiations.
  - Appl: semiclassicality preservation between the cycles of evolution.
- Summary: Results and method properties.

An example model

Flat isotropic universe with massless scalar field.

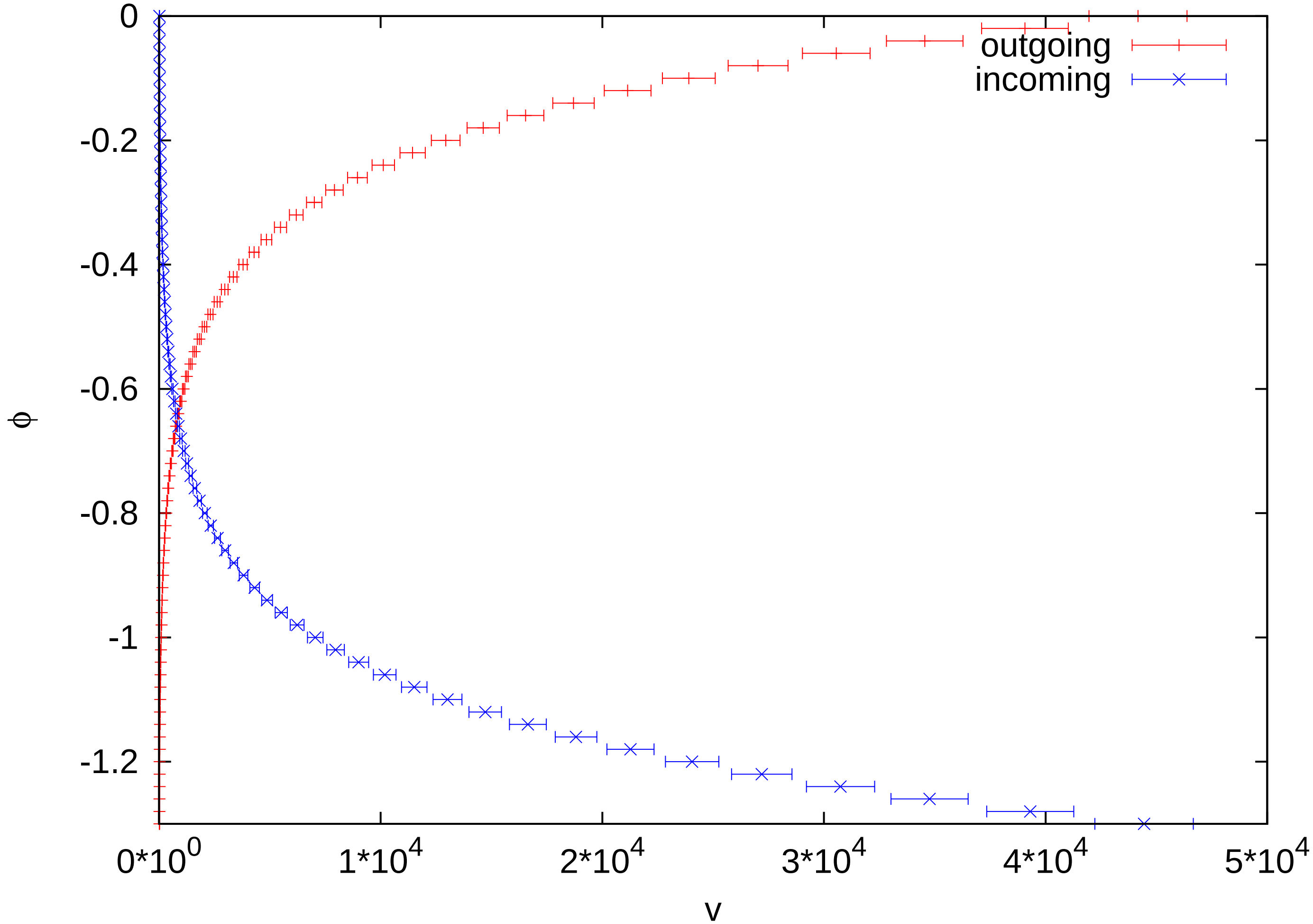
- Spacetime: manifold $M \times \mathbb{R}$ where M is topologically $\mathbb{R}^3$.
  $M \times \{t\}$ (where $t \in \mathbb{R}$) – homogeneous slices.
- Metric: $g = -dt^2 + a^2(t)^o q$
  - ${}^o q$ - flat fiducial metric ($dx^2 + dy^2 + dz^2$).
- The treatment:
  - The degrees of freedom (canonical variables):
    - Geometry: (v, b), v - prop. to the volume of chosen fiducial region.
    - Matter: (ϕ, p_ϕ), ϕ - scalar field value
  - The quantization:
    - Matter: standard Schrödinger representation
    - Geometry:
      - Wheeler-DeWitt: Schrödinger representation
      - LQC: methods of Loop Quantum Gravity
    - Nontrivial Hamiltonian constraint: Dirac program.
    - Scalar field used as an internal time

Wheeler-DeWitt quantization

All elements expressed in (c, p, ϕ, p_ϕ). Standard quantization.

- Kinematical Hibert space: ($v \propto \mathrm{sgn}(p)|p|^{3/2}$)
  - $\mathcal{H}^{\mathrm{kin}} = \mathcal{H}^{\mathrm{grav}} \otimes \mathcal{H}^{\phi}, \quad \mathcal{H}^{\mathrm{grav}} = L^2(\mathbb{R}, \mathrm{d}v), \quad \mathcal{H}^{\phi} = L^2(\mathbb{R}, \mathrm{d}\phi)$.
  - Basic operators: $(\hat{p},\ \hat{c} \propto i\partial_p,\ \hat{\phi},\ \hat{p}_\phi \propto i\partial_\phi)$.
  - Basis: eigenstates $(v|\hat{p} = v(v|, \quad (\phi|\hat{\phi} = \phi(\phi|$.
- Quantum constraint:
  $[\partial_\phi^2 \Psi](v, \phi) = -[\underline{\Theta}\Psi](v, \phi) := 12\pi G[(v\partial_v)^2 + v\partial_v + 1/4]\Psi(v, \phi)$.
- Physical states: $\Psi(v, \phi) = \int_{\mathbb{R}} \mathrm{d}k \underline{\tilde{\Psi}}(k) \underline{e}_k(v) e^{i\omega\phi}$,
  $\underline{\tilde{\Psi}} \in L^2(\mathbb{R}, \mathrm{d}k), \quad \omega = \sqrt{12\pi G}|k|, \quad \underline{e}_k(v) = (1/\sqrt{2\pi v}) e^{ik \ln(v)}$.
- Observables:
  - $\hat{p}_\phi : \underline{\tilde{\Psi}}(k) \mapsto \hbar\omega(k)\underline{\tilde{\Psi}}(k)$,
  - $\ln(v)_{\phi_o} : \Psi(v, \phi) \mapsto e^{i\sqrt{\underline{\Theta}}(\phi - \phi_o)} \ln(v)\Psi(v, \phi_o)$.
- Dynamics:
  - Two classes: ever contracting and ever expanding.
  - The dispersions $\sigma_{\ln(v)_\phi}$ are constant.

WDW dynamics



LQC quantization

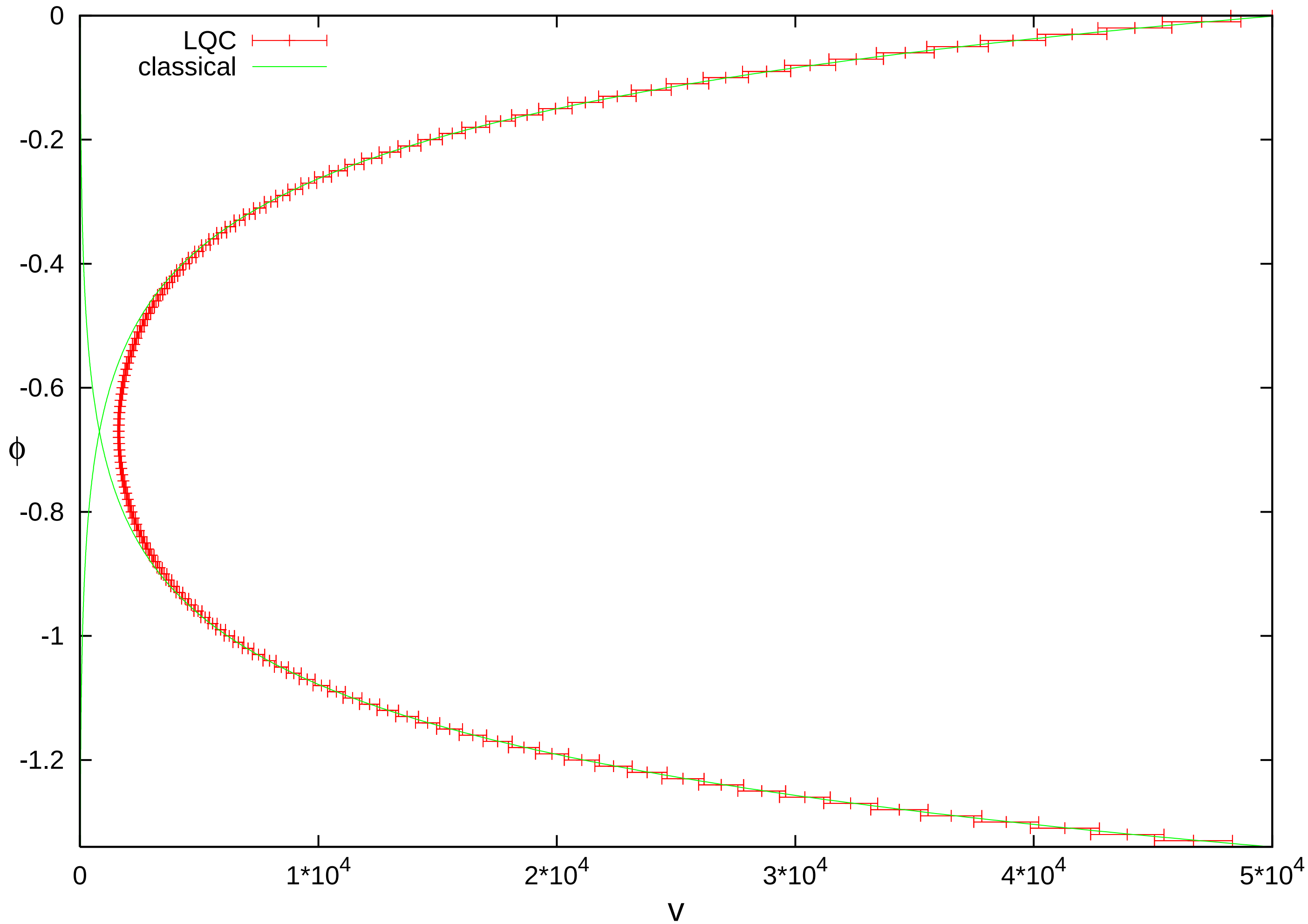
Geometry polymeric, matter standard, structure analogous

- Kinematics:
  - Geometry space: $\mathcal{H}^{\rm grav} = L^2(\mathbb{R}_{\rm Bohr} {\rm d}\mu_{\rm Bohr})$.
  - Geometry basis: $|v\rangle : \langle v|v'\rangle = \delta_{vv'}$.
  - Basic operators: holonomies $h = e^{\int_\gamma A{\rm d}x}$, fluxes $S = \int_S \star E {\rm d}\sigma$.
- The constraint: reexpressed in terms of $\hat{h}$, $\hat{S}$

$$[\Theta\psi](v) = -f^+(v)\psi(v+4) + f^o(v)\psi(v) - f^-(v)\psi(v-4)$$

  for large v: $f^{o,\pm}(v) \propto v^2$.
- The physical states: $\Psi(v,\phi) = \int_{\mathbb{R}+} {\rm d}k \tilde{\Psi}(k) e_k(v) e^{i\omega\phi}$
  $\tilde{\Psi} \in L^2(\mathbb{R}^+, {\rm d}k), \quad \omega = \sqrt{12\pi G}|k|, \quad \Theta e_k(v) = \omega^2(k) e_k(v)$.
- Dirac observables: analogous to WDW.
- The dynamics:
  - in distant future and past agreement with GR,
  - bounce in the Planck regime (energy densities).

LQC dynamics



WDW limit of LQC states

- $f^{o,\pm}$ - real: e_k are standing waves.
- Observation: converge to WDW standing waves.

$$e_k(v) = \underline{\psi}_k(v) + O(|\underline{e}_k(v)|(k/v)^2)$$
$$\underline{\psi}_k(v) := r(k)[e^{i\alpha(k)}\underline{e}_k(v) + e^{-i\alpha(k)}\underline{e}_{-k}(v)]$$

  - Comp. of LQC and WDW norms + self-adjointness of Θ: $r(k) = 2$
- Analytical proof:
  - reformulation of the diff. equation in the 1st order form,
  - (local) decomp. of the LQC eigenf. in terms of WDW ones,
  - asymptotic properties of the resulting transfer matrix.
- Properties of the phase shifts:
  - Analytic results for sLQC: simple analytic form of the eigenfunctions in the momentum of v + stationary phase method
    - $\alpha(k) = -k(\ln|k| - 1) - (3/4)\pi + o(k^0)$
    - $\alpha'(k) = -\ln|k| + O(k^{-1}\ln|k|)$
  - Numerical results for: APS, sLQC, MMO:
    - $\alpha'(k) = -\ln|k| + O(k^{-2})$
    - $|k\alpha''(k)| \le 1$

The phase shifts

Properties analyzed analytically and numerically:

- Analytic results for sLQC: simple analytic form of the eigenfunctions in the momentum of v + stationary phase method
  - $\alpha(k) = -k(\ln|k| - 1) - (3/4)\pi + o(k^0)$
  - $\alpha'(k) = -\ln|k| + O(k^{-1}\ln|k|)$
- Numerical results for: APS, sLQC, MMO:
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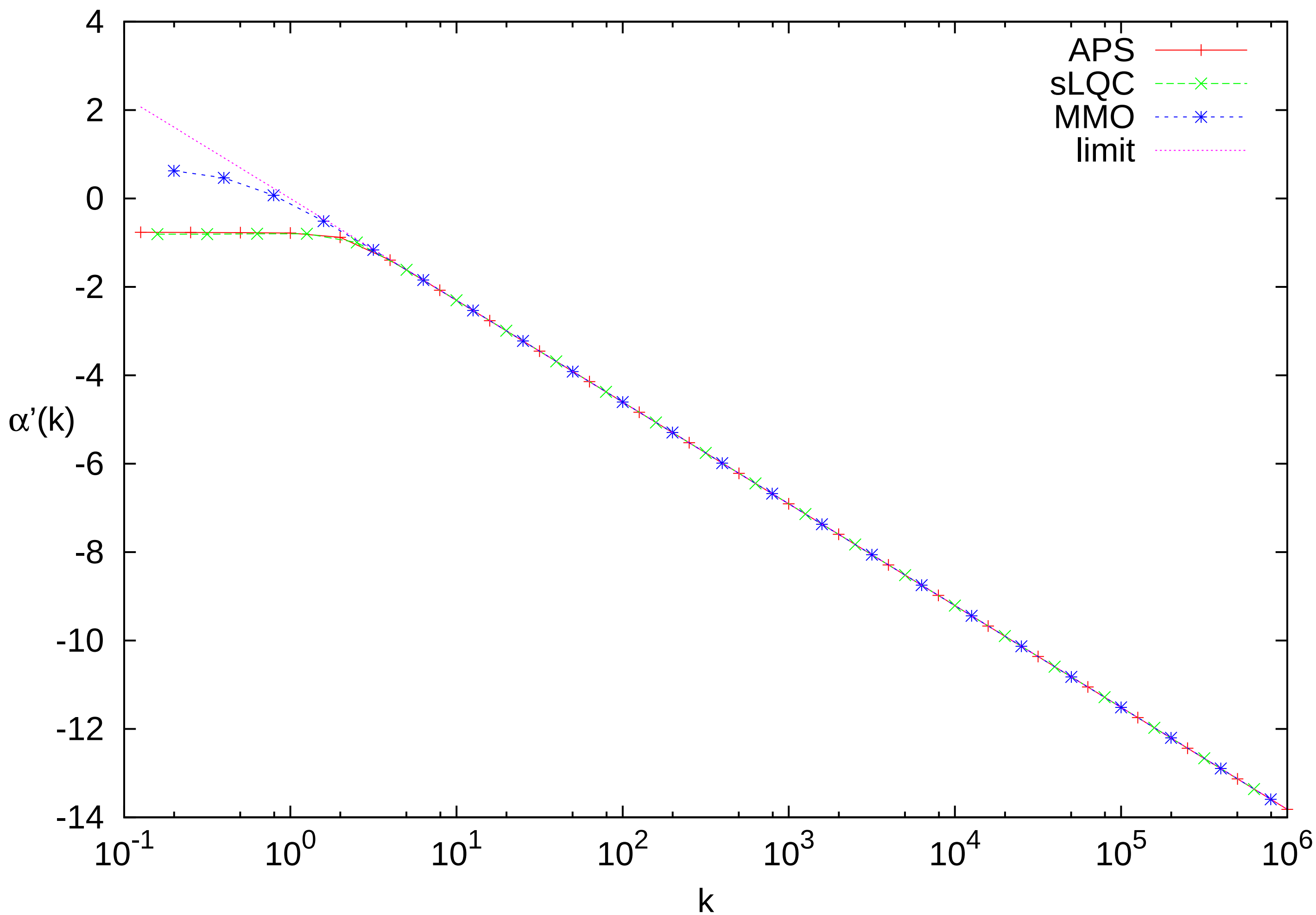
  - $|k\alpha''(k)| \leq 1$

Very regular behavior!

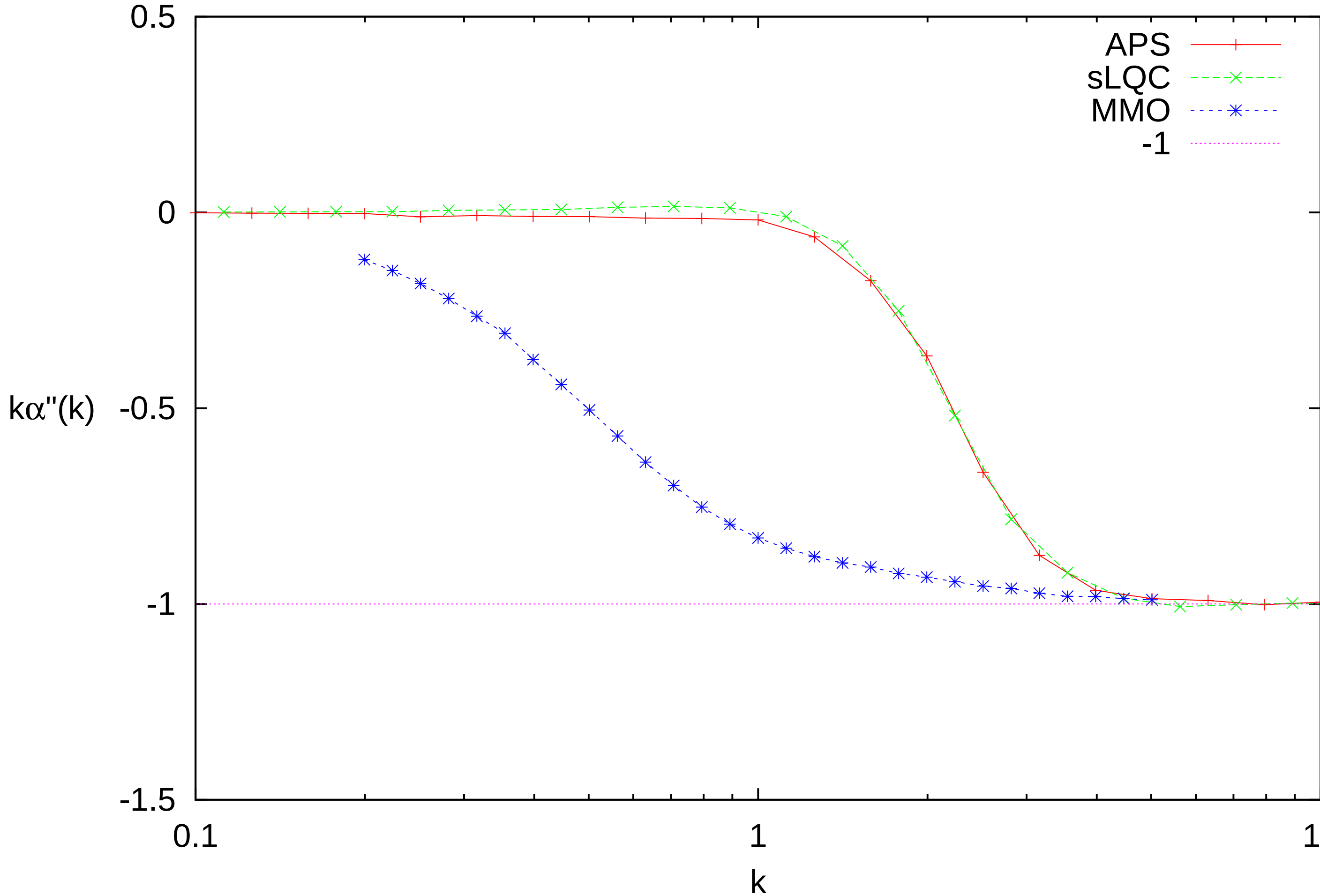
Phase shift properties

$\alpha'(k)$.



Phase shift properties

$k\alpha''(k)$.



The scattering picture

- WDW limit: $e_k(v) \to \underline{\psi}_k(v)$

$$\tilde{\Psi}(|k|) \mapsto \underline{\tilde{\Psi}}(k) = 2e^{i\,\mathrm{sgn}(k)\alpha(|k|)}\,\mathrm{sgn}(k)\tilde{\Psi}(|k|)$$

- Two components: $\underline{\tilde{\Psi}}_{\pm}(k) = \theta(\pm k)\underline{\tilde{\Psi}}(k)$
- The limits of observables:

$$\lim_{\phi\to\pm\infty}\langle\Psi|\ln(v)_\phi|\Psi\rangle = \langle\underline{\Psi}_{\pm}|\ln(v)_\phi|\underline{\Psi}_{\pm}\rangle,$$
$$\lim_{\phi\to\pm\infty}\langle\Psi|\Delta\ln(v)_\phi|\Psi\rangle = \langle\underline{\Psi}_{\pm}|\Delta\ln(v)_\phi|\underline{\Psi}_{\pm}\rangle =: \sigma_{\pm}.$$

- Interpretation as a scattering process:

$$|\underline{\Psi}\rangle_{\mathrm{in}} \mapsto |\underline{\Psi}\rangle_{\mathrm{out}} = \hat{\rho}\,|\underline{\Psi}\rangle_{\mathrm{in}}\,, \qquad \underline{\tilde{\Psi}}_{\mathrm{in}}(k) := \underline{\tilde{\Psi}}_{+}(k),\ \underline{\tilde{\Psi}}_{\mathrm{out}}(k) := \underline{\tilde{\Psi}}_{-}(k)$$

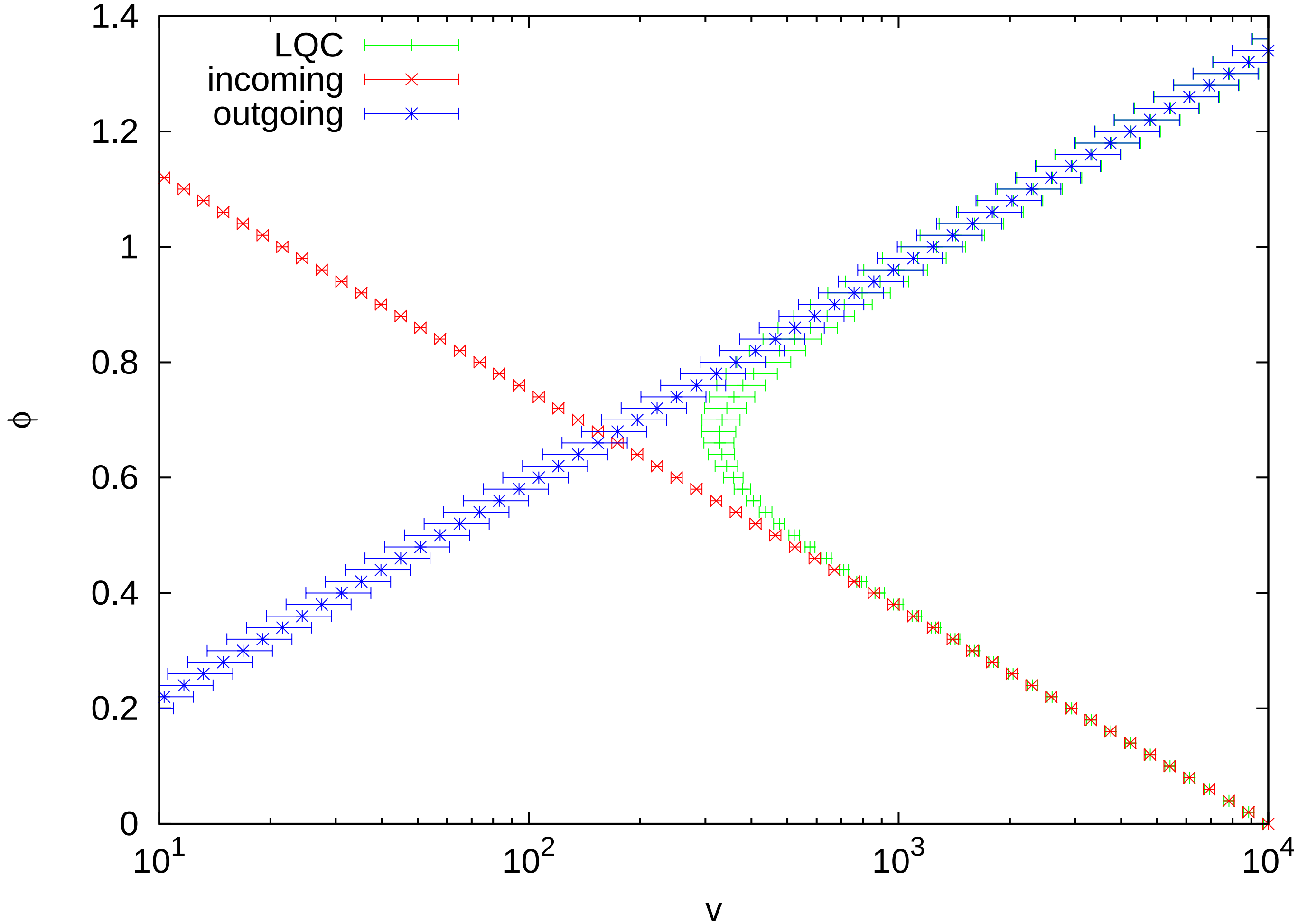
- The scattering matrix:

$$\rho(k,k') = (\underline{e}_k|\hat{\rho}|\underline{e}_{k'}) = e^{-i\,\mathrm{sgn}(k')\alpha(|k'|)}\delta(k+k').$$

- The transformation: total reflection

$$\underline{\tilde{\Psi}}(k) \mapsto U\underline{\tilde{\Psi}}(k) := e^{2i\,\mathrm{sgn}(k)\alpha(|k|)}\underline{\tilde{\Psi}}(-k)$$

The scattering



The dispersion growth

An application: comparizon of the dispersions of $\ln(v)_\phi$.

- Action on $\mathcal{H}^{\rm phy}$: $\ln|\hat{v}|_\phi \underline{\tilde{\Psi}} = [-i\partial_k - (\partial_k\omega(k))\phi\hat{\mathbb{I}}]\underline{\tilde{\Psi}}$
- The relation between action on limits: $\langle\hat{O}\rangle_\pm := \langle\underline{\Psi}_\pm|\hat{O}|\underline{\Psi}_\pm\rangle$

  $$\langle -i\partial_k\rangle_- = \langle U^{-1}[-i\partial_k]U\rangle_+, \qquad \langle\Delta[-i\partial_k]\rangle_- = \langle\Delta U^{-1}[-i\partial_k]U\rangle_+.$$

  where $U^{-1}[-i\partial_k]U = -i\partial_k - 2\alpha'\mathbb{I}$.
- The effects on dispersions: ($\sigma_{A+B} \leq \sigma_A + \sigma_B$ - Schwartz ineq.)

  $$\sigma_- \leq \sigma_+ + 2\langle\Delta\alpha'\mathbb{I}\rangle_+$$
- Estimate via dispersion in ω:
  - General inequality:
    $\langle\Delta\alpha'\mathbb{I}\rangle_+^2 = \langle(\alpha'^2 - \langle\alpha'\rangle_+)^2\,\mathbb{I}\rangle_+ \leq \langle(\alpha'^2 - \alpha'^\star)^2\,\mathbb{I}\rangle_+$.
  - The choice: $\alpha'^\star = \alpha'(\exp(\lambda^\star)), \quad \lambda^\star := \langle\ln(k)\rangle_+$ and props. of α' give: $\langle\Delta\alpha'\mathbb{I}\rangle_+^2 \leq \langle(\ln(\hat{k}) - \lambda^\star\mathbb{I})^2\rangle_+ = \langle\Delta\ln(\hat{k})\rangle_+^2 =: \sigma_\star^2$
- The final inequality:

  $$\sigma_- \leq \sigma_+ + 2\sigma_\star$$

Summary

- The results: Devised method of comparing the properties of distant future and distant past states
  - does not require exact solvability
  - uses only asymptotic properties of phys. Hilbert space basis
  - is general (without restriction to particular types/shapes of states),
  - in genuinely quantum (no semiclassical approximations of any kind),
  - application to FRW with massless scalar:
    general triangle inequalities on dispersions. Strict upper bound on eventual dispersion growth.

    Just the upper bound, the actual dispersion may even shrink.

    Universe's memory has to be indeed very sharp !
  - generalization:
    - isotropic sector of Bianchi I
    - slightly weaker version: vacuum Bianchi I: arXiv:0906.3751
  - further extension: $\Lambda > 0$ (see 2nd part).

$\Lambda > 0$ - *WDW model*

- Hamiltonian constraint:

$$-\partial_\phi^2 = \underline{\Theta}_o - \pi G \gamma^2 \Delta \Lambda \mathbb{I} = \underline{\Theta}_\Lambda$$

  $\underline{\Theta}_\Lambda$ admits 1d family of selfadjoint extensions, labeled by $\beta \in U(1)$

- Each extension $\underline{\Theta}_{\Lambda\beta}$ has continuous spectrum: $\mathrm{Sp}(|\underline{\Theta}_{\Lambda\beta}|) = \mathbb{R}^+$
- Physical states:

$$\underline{\Psi}(v,\phi) = \int_0^\infty \mathrm{d}k \tilde{\Psi}(k) e_k^\beta(v) e^{i\omega\phi}, \quad \omega = \sqrt{12\pi G} k$$

$$e_k^\beta(v) = \frac{1}{\sqrt{|v|}} [c_1(\beta,\Lambda,k) H_{ik}^{(1)}(av) + c_2(\beta,\Lambda,k) H_{ik}^{(2)}(av)],$$

  where $a = \sqrt{\frac{\gamma^2 \Delta \Lambda}{12\pi G}}$ and $H^{(1)}, H^{(2)}$ - Hankel functions.

- For each extension all e_k^β have common asymptotics

$$e_k^\beta = N(\Lambda,\beta,k)|v|^{-1} \cos(\Omega(\Lambda)|v| + \sigma(\Lambda,\beta)) + O(|v|^{-3/2})$$

- Dynamics: follows analytically extended classical trajectory.

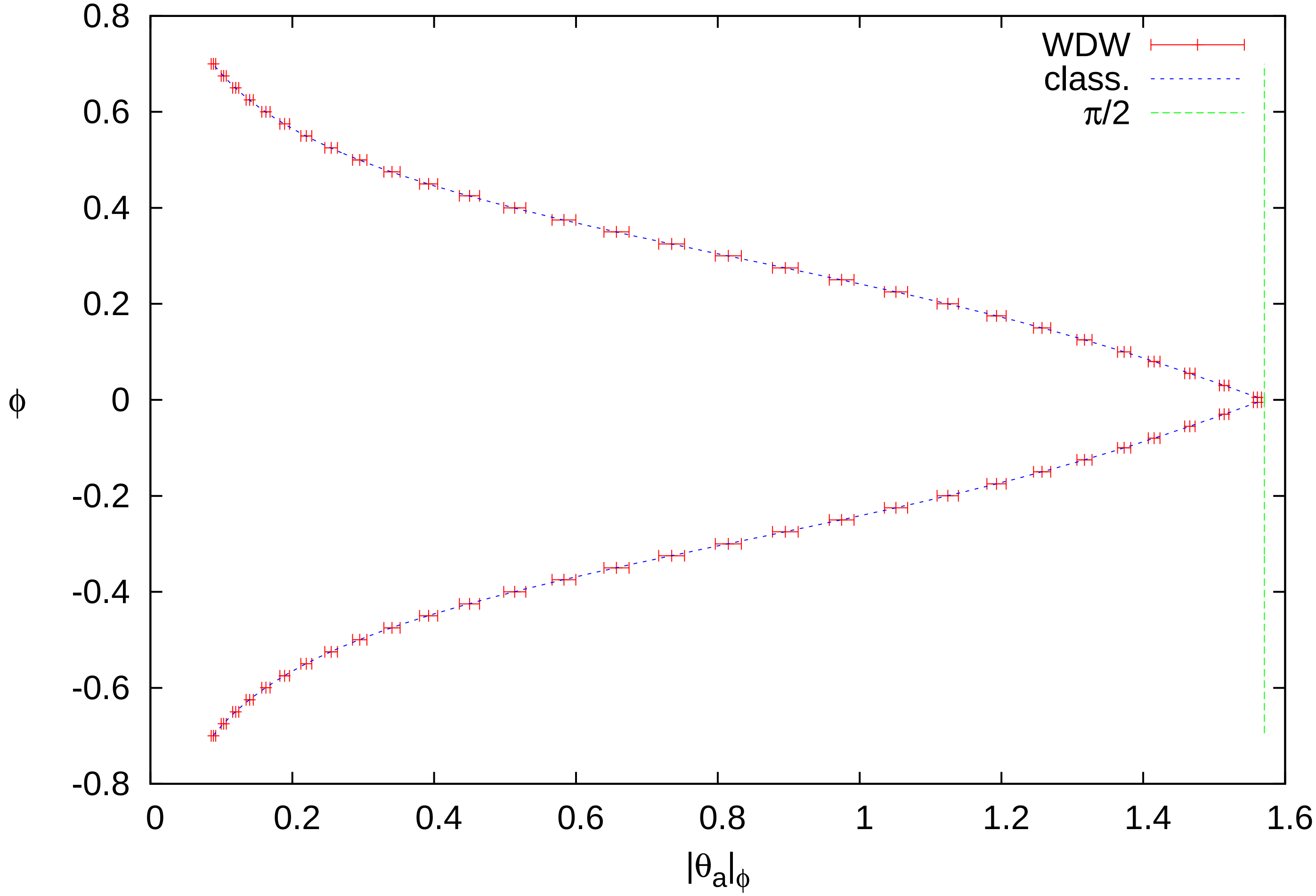
WDW: observables

- Observables: analogous to $\Lambda = 0$ case.
  Problem: the analog of $|v|_\phi$ leads outside of $\mathcal{H}^{\rm phy}$!
- Failure of semiclassical treatment:
  - If one selects canonical pair v, b the requirement $\langle (|\hat{v}|_\phi - \langle |v|_\phi \rangle)^n \rangle < \infty$ on some open $\phi \in \mathcal{O}$ implies $\int \mathrm{d}k \tilde{\Psi}(k) N(\Lambda, \beta, k) e^{i\omega\phi} = 0, \quad \forall \phi \in \mathcal{O}$
  - Since $N(\Lambda, \beta, k) \propto \sqrt{k}$ we have: $\tilde{\Psi}(k) \sim k^{-3/2} \ \Rightarrow \ \langle \Delta p_\phi \rangle = \infty$.
  - For $|v| \ll 1$ $e^\beta_k \approx e^{ik \ln |v|}$, thus at early times

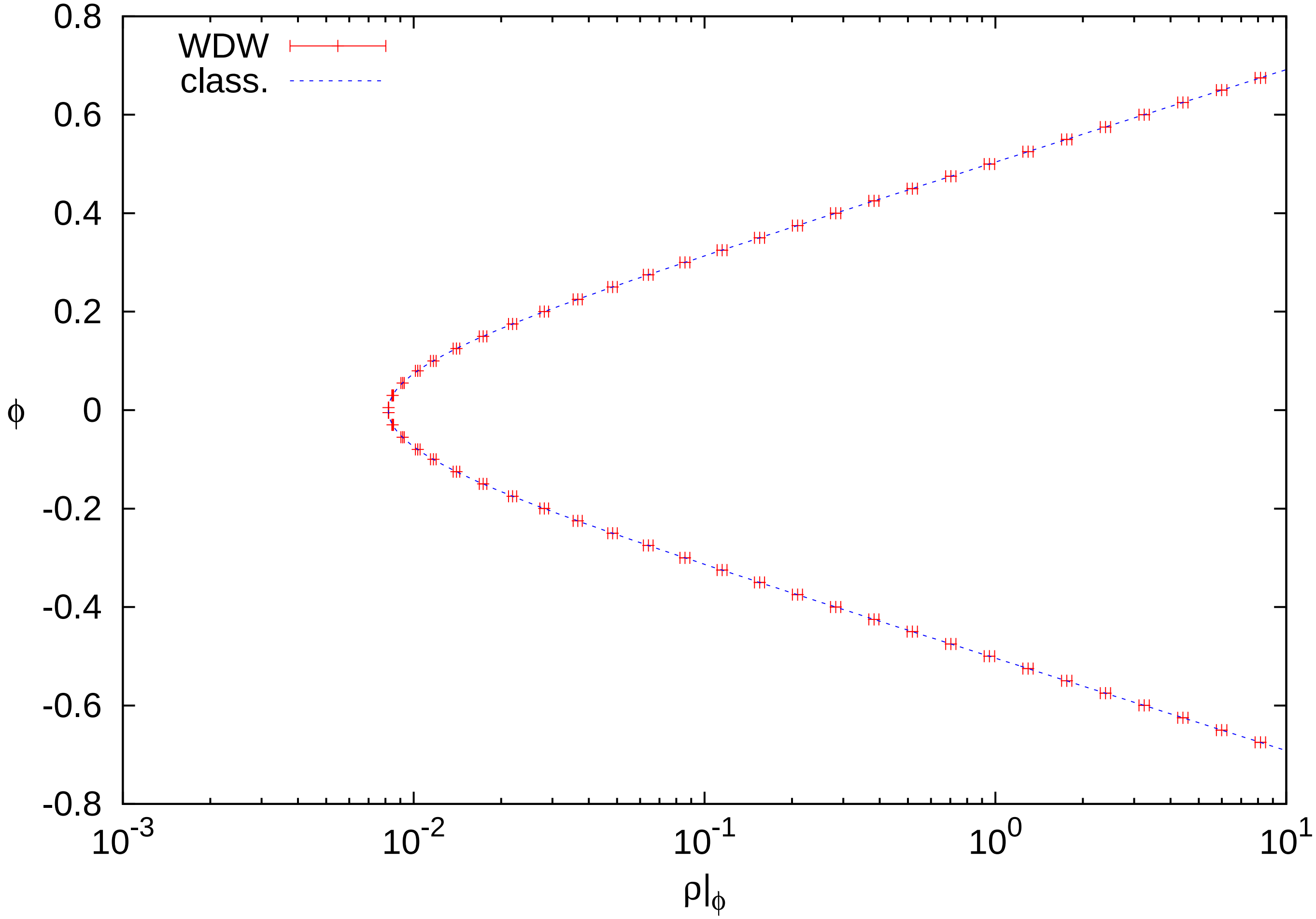
    $$\langle \Delta p_\phi \rangle = \infty \ \Rightarrow \ \langle \Delta b|_\phi \rangle = \infty$$

    Impossible to built states well behaving in both v and b even for a short time!
- Solution:
  - Compactify v, for example use $\theta_a = \arctan(|v|/a)$ or
  - Use truly measurable quantities, like Hubble parameter H or energy density ρ.

WDW dynamics for $\Lambda > 0$



WDW dynamics for $\Lambda > 0$



$\Lambda > 0$: LQC model

- Allowed value $\Lambda \in [0, \Lambda_c]$, where $\Lambda_c = 8\pi G \rho_c$
- The constraint (sLQC prescription):

$$-\partial_\phi^2 = \Theta_o - \pi G \gamma^2 \Delta \Lambda \mathbb{I} = \Theta_\Lambda$$

  Θ_Λ admits 1d family of selfadjoint extensions, labeled by $\beta \in U(1)$

- Each extension $\Theta_{\Lambda\beta}$ has discrete spectrum: $\mathrm{Sp}(\Theta_{\Lambda\beta}) = \{\omega_n^2\}$ where $\omega_n = [\sqrt{12\pi^3 G}/f_1(\Lambda)] \cdot k_n$,

$$\tan(g(\Lambda)k_n) + \tanh[(\pi - g(\Lambda))k_n]\tan(\beta) = 0, \quad g(\Lambda) \in [0, \pi]$$
$$\Rightarrow \quad \omega_n = (n\pi - \beta)/f_2(\Lambda) + O(e^{-2\pi n(\pi - g(\Lambda))/g(\Lambda)})$$

- Physical states:

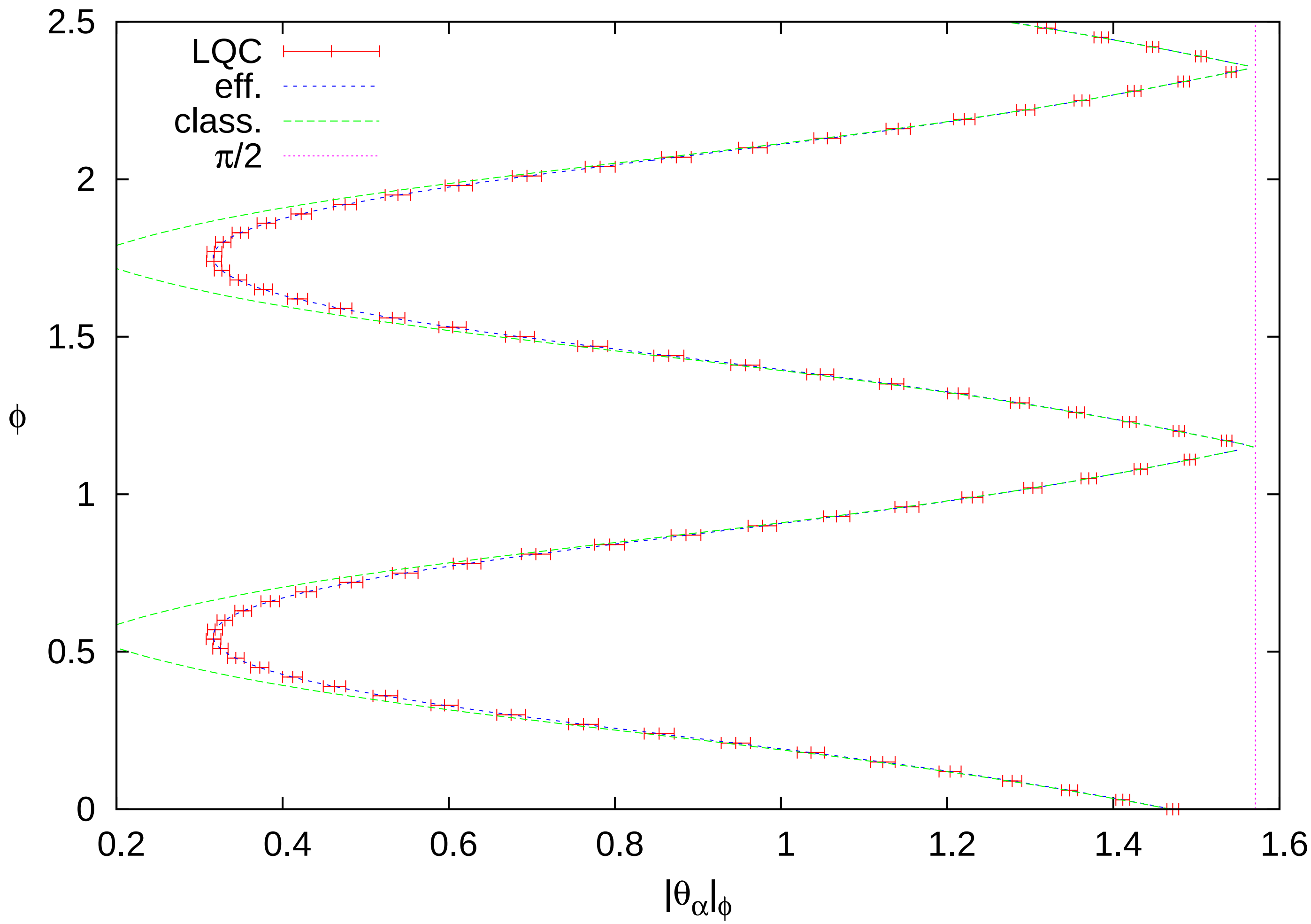
$$\Psi(v, \phi) = \sum_{n=0}^{\infty} \tilde{\Psi}_n e_n^\beta(v) e^{i\omega_n \phi}$$

- For each extensions common leading order asymptotics

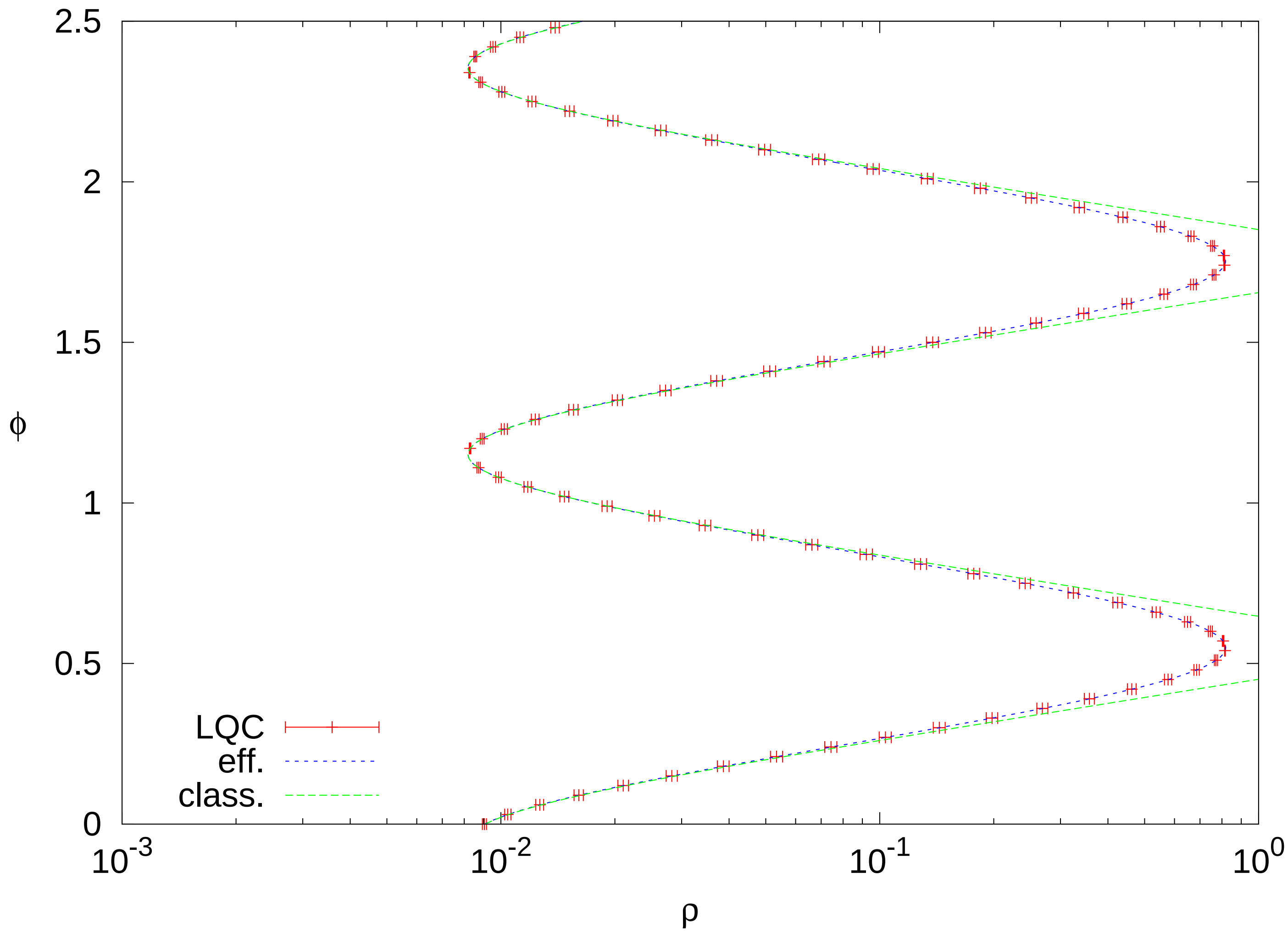
$$e_n^\beta = N_n(\Lambda, \beta)|v|^{-1}\cos(\Omega(\Lambda)|v| + \sigma(\Lambda, \beta)) + O(|v|^{-3/2})$$

- Dynamics:
  - Agreement with GR for low energies.
  - Bounce in Planck regime.

LQC dynamics for $\Lambda > 0$



LQC dynamics for $\Lambda > 0$



2nd order asymptotics

- Analysis of the transfer matrix asymptotics:

$$e_n^\beta(v) = N_n[e^{i\alpha}e_n^+(v) + e^{-i\alpha}e_n^-(v)] + O(v^{-3}) \quad (\star)$$

$$e_n^\pm = |v|^{-1}\, e^{\pm i\Omega|v|} \cdot e^{\pm i\kappa(n,\Lambda,\beta)/|v|}, \quad 1/|v| \approx (1/a)(\theta_a - \pi/2)$$

  where: $\cos(4\Omega(\Lambda)) = 1 - 2\Lambda/\Lambda_c =: 1 - 2\lambda,$

$$\kappa(n,\Lambda,\beta) = \frac{3\pi G(1-2\lambda)+\omega_n^2}{12\pi G\sqrt{\lambda(1-\lambda)}} =: A\omega_n^2 + B$$

- Limit spaces:
  - "Standing wave form" of e_n^β: split onto incoming and outgoing components.
  - For each comp. $e^{\pm i\Omega|v|}$ is a global rotation, only $e_n^\pm$ relevant.
  - $e_n^\pm$ form wave packet regular in θ_a. *Schrödinger type rather that Klein-Gordon.*
- Transformation into limit spaces:
  - Define spaces $\mathcal{H}^\pm$ spanned by $e_n^\pm$ with IP inherited from $\mathcal{H}^{\rm phy}$ through the limit $(\star)$.
  - $\mathcal{H}^{\rm phy} \ni \tilde\Psi_n \mapsto \Phi_n^\pm := \tilde\Psi_n M_n \in \mathcal{H}^\pm$

WDW limit states

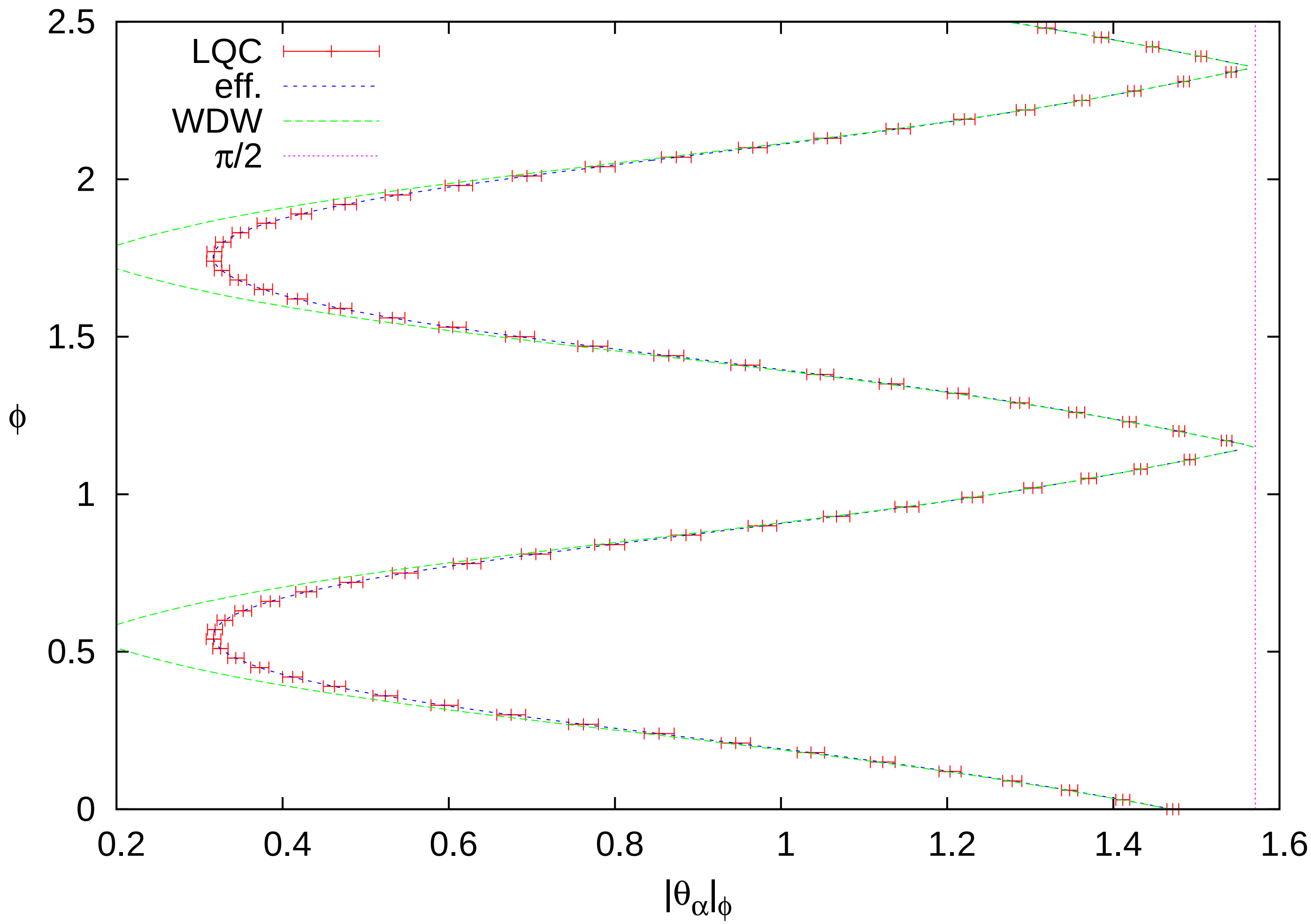
- Taking the limit of the mass gap $\Delta \to 0$ one can derive analogous 2nd order limit in WDW theory.
  - As $\Omega(\Lambda)$ depends implicitly on Δ to ensure uniform convergence one has to allow for flow $\Lambda = \Lambda(\Delta)$.
  - Result: LQC wave packet has the WDW limit spanned by $\underline{e}_k$ such that:
  $$\underline{e}_k^{\beta}(v) = N(k)[e^{i\alpha}\underline{e}_k^{+}(v) + e^{-i\alpha}\underline{e}_k^{-}(v)] + O(v^{-3})$$
  where
  $$\underline{e}_k^{\pm} = |v|^{-1}\, e^{\pm i\Omega|v|} \cdot e^{\pm i\kappa(k,\Lambda,\beta)/|v|}$$
  with the same Ω and $\kappa = \kappa(\omega = \sqrt{12\pi G}k)$
  however the WDW model corresponds to the cosmological constant
  $$\underline{\Lambda}/\Lambda_c = \underline{\lambda} = \arccos(1 - 2\lambda).$$
- Construction of the limit spaces $\underline{\mathcal{H}}^{\pm}$ analogous to LQC.
  Again Schrödinger wave packets near $\theta = \pi/2$.
  - If the wave packet sharply peaked about $\theta = \pi/2$ then (up to higher order corrections)
  $$\langle \Delta | \hat{\theta}_a |_\phi \rangle_{\mathrm{WDW}} = \langle \Delta | \hat{\theta}_a |_\phi \rangle_{\lim}$$

Instantiations

- The continuous limit:
  - We identified the relations between $\mathcal{H}^{\rm phy}$, $\underline{\mathcal{H}}^{\rm phy}$ and the appropriate limit spaces.
  - To build $\mathcal{H}^{\rm phy} \leftrightarrow \underline{\mathcal{H}}^{\rm phy}$ we need $\mathcal{H}^{\pm} \leftrightarrow \underline{\mathcal{H}}^{\pm}$.
  - Problem: Identification of Φ_n and $\Phi(k)$ will produce zero norm WDW states!
- The instantiations:
  - Fix the moment $\phi = \phi_o$. By rotation $\tilde{\Psi}(k) \mapsto \tilde{\Psi}(k)e^{i\omega\phi_o}$ one can bring it to $\phi = 0$.
  - On $\underline{\mathcal{H}}^{\pm}$ the operator $\hat{x} := \hat{\theta}_a - \pi/2$ takes the form $\hat{x} = \frac{ia}{2A\omega}\partial_\omega$
  - On $\mathcal{H}^{\pm}$ one can build "$\sin(cx)/c$" oper. via $h: \; [h\tilde{\Phi}]_n = \tilde{\Phi}_{n-1}$
    - $\sin(cx)/c = \frac{ai}{2A[\Delta\omega](\omega+[\Delta\omega]/2}[h - h^{-1}] + O(e^{-a\omega}),$

    where $\Delta\omega = \lim_{n\to\infty}[\omega_n - \omega_{n-1}]$.
  - If at ϕ_o state sharply peaked about $x = 0$: $\sin(cx)/c$ - good replacement of $\hat{x}$
  - Instantiation: Interpolation of $\tilde{\Phi}_n$ s.th. actions of $\hat{x}^n$ and $[\sin(cx)/c]^n$ agree up to $n = 2$.

Instantiations + scattering



The scattering, decoherence

- The picture:
  - Instantiations provide identification of quasi-periodic state of LQC with non-periodic WDW one at given time ϕ_o.
  - Once can build a sequence of instantiations at ϕ_n where $\langle\theta\rangle \approx \pi/2$.
  - Transfer $\phi_n \to \phi_{n+1}$: scattering of WDW state into another.
  - Evolution: sequence of scatterings (determined by instantiations).
    - Since $\omega_n = (n\pi - \beta)/f_2(\Lambda) + \delta\omega_n$ one cycle corresponds to

$$\tilde{\Psi}_n \mapsto e^{2\pi i \delta\omega_n} \tilde{\Psi}_n =: U \tilde{\Psi}_n$$

- Application:
  - $\langle \Delta \sin(cx)/c \rangle_{\mathcal{H}^{\rm phy}} = \langle \Delta \sin(cx)/c \rangle_{\mathcal{H}^{\pm}} + O(v^{-3})$.
  - After $N \gg 1$ cycles

$$\langle \Delta \sin(cx)/c \rangle_{\phi_o + N\Delta\phi} \leq \langle \Delta \sin(cx)/c \rangle_{\phi_o} + N \langle \Delta \frac{2\pi \partial_\omega \delta\omega}{\omega} \rangle$$

  where the last term $\langle \Delta \frac{2\pi \partial_\omega \delta\omega}{\omega} \rangle \leq C \langle \Delta \frac{2\pi e^{-a\omega}}{\omega} \rangle$.

Summary

- Scattering picture successfully extended to the model with $\Lambda > 0$.
- Evolution between pure deSitter epochs - chain of Wheeler-DeWitt universes consecutively scattered one into another.
- The instantiation procedure allowed to relate the dispersion in compactified volume θ_a of the LQC state and its WDW limit at given moment ϕ_o (in pure deSitter epoch).
- The scattering corresponds to unitary rotation by $e^{2\pi i \delta\omega_n}$, where the deviation $\delta\omega_n$ decays exponentially.
- Consequence: The decoherence of the state between large number N of pure deSitter epochs is bounded from above by (up to a known constant) the dispersion of the operator $Ne^{-a\omega}/\omega$ infinitesimal for the states peaked about large $p_\phi^\star$.

Appendix: The transfer matrix method

- $f^{o,\pm}$ - real: e_k are standing waves.
- Observation: converge to WDW standing waves.
- Verification: transfer matrices
  - 1st order form of the difference equation:

$$\vec{e}_k(v) = \begin{bmatrix} e_k(v) \\ e_k(v-4) \end{bmatrix}, \qquad A(v) = \begin{bmatrix} \frac{f_o(v)-\omega^2(k)}{f_+(v)} & -\frac{f_-(v)}{f_+(v)} \\ 1 & 0 \end{bmatrix}$$

$$\vec{e}_k(v+4) = A(v)\vec{e}_k(v)$$

  - Expressing in WDW basis:

$$\vec{e}_k(v) = B(v)\vec{\chi}_k(v), \qquad B(v) := \begin{bmatrix} \underline{e}_k(v+4) & \underline{e}_{-k}(v+4) \\ \underline{e}_k(v) & \underline{e}_{-k}(v) \end{bmatrix}.$$

  - Final form: $\vec{\chi}_k(v+4) = B^{-1}(v)A(v)B(v-4)\vec{\chi}_k(v) =: M(v)\vec{\chi}_k(v)$
- Limit of the transfer matrix: $M(v) = \mathbb{I} + O(v^{-3}) \quad \Rightarrow$

$$e_k(v) = \underline{\psi}_k(v) + O(|\underline{e}_k(v)|(k/v)^2)$$

$$\underline{\psi}_k(v) := r(k)[e^{i\alpha(k)}\underline{e}_k(v) + e^{-i\alpha(k)}\underline{e}_{-k}(v)]$$

- Comparizon of norms: $r(k) = 2$

Appendix: Comparizon between the norms

- Evolution: mapping $\mathbb{R} \ni \phi \mapsto \psi(\cdot) := \Psi(\cdot, \phi) \in \mathcal{H}^{\mathrm{grav}}$
- Inner products:
  $\langle\psi|\chi\rangle = \sum_{\mathcal{L}_0^+} \overline{\psi(v)}\chi(v)$, $\langle\underline{\psi}|\underline{\chi}\rangle = \int_{\mathbb{R}+} \mathrm{d}v\, \overline{\underline{\psi}(v)}\underline{\chi}(v)$
- Distributional estimates:
  - splitting the domain:
    $X(k,k') := \langle e_{k'}|e_k\rangle = \sum_{\mathcal{L}_0^+\cap[1,\infty]} \overline{e_{k'}(v)} e_k(v)$
  - extracting WDW limits:
    $X(k,k') = +\sum_{\mathcal{L}_0^+\cap[1,\infty]} [\overline{\underline{\psi}_{k'}(v)}\underline{\psi}_k(v) + O(v^{-5/2})]$
  - estimating the sum by the integral:
    $\sum \overline{\underline{\psi}_{k'}(v)}\underline{\psi}_k(v) = (1/4)\int_{[1,\infty[} \mathrm{d}v\, \left[\overline{\underline{\psi}_{k'}(v)}\underline{\psi}_k(v) + O(v^{-3/2})\right]$
  - relation: $\int_{\mathbb{R}+} \mathrm{d}x\ e^{ikx} = \frac{1}{2}\left(\int_{\mathbb{R}} \mathrm{d}x\ e^{ikx} - \frac{i}{\pi k}\right)$
- Final relation: $X(k,k') = (r^2(k)/8)\langle\underline{\psi}_k|\underline{\psi}_{k'}\rangle + F(k,k')$
- Orthonormality: $F(k,k') = 0$, $r(k) = 2$.

Appendix: The limits of observables

Argumentation for past limit, problem symmetric in time.

- Assumed localization in $\ln|v|_\phi$ of WDW limit: $\langle \ln|v|_\phi\rangle < \infty$, $\langle \Delta \ln|v|_\phi\rangle < \infty$.
- Past wave packet follows the trajectory $\bar{x}(\phi) = x_o - \beta(\phi - \phi_o)$, where $\beta = \sqrt{12\pi G}$.
- $f(v)_\phi$ - multiplication operator on ID's $\Psi(\cdot, \phi) \in \mathcal{H}^{\rm grav}$, expectations - local sums: $\langle f(v)_\phi\rangle = \sum_{\mathcal{L}_0^+} f(v)|\Psi(v,\phi)|^2$, analogous situation in WDW.
- introduce $\tilde{x}(\phi) = x_o - (\beta/2)(\phi - \phi_o)$ and split the local sums along it.
- the following properties:
  - falloff conditions due to localization: parts for $x > \tilde{x}$ converge (sufficiently fast) to complete sums.
  - relation between LQC and WDW norms: convergence of WDW and LQC partial sums for states localized in k.

  imply that:

$$\lim_{\phi\to-\infty}\langle\Psi|\ln(v)_\phi|\Psi\rangle = \langle\underline{\Psi}_-|\ln(v)_\phi|\underline{\Psi}_-\rangle,$$
$$\lim_{\phi\to-\infty}\langle\Psi|\Delta\ln(v)_\phi|\Psi\rangle = \langle\underline{\Psi}_-|\Delta\ln(v)_\phi|\underline{\Psi}_-\rangle.$$