Big bounce and inhomogeneities: a hybrid approach.

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Motivation

- Loop quantization of the isotropic/homogeneous cosmological models ⇒ changes of the dynamics at near-Planck densities causing the Big Bounce.

- Robustness of the picture:
  - Does it survive in presence of the inhomogeneities?
  - If yes: how the bounce affects the structure of the inhomogeneities?

- Very preliminary answer in context of the Gowdy $T^3$ model with linearly polarized gravitational waves.
Outline

The model specification:
- Definition, partial gauge fixing
- Degrees of freedom

The hybrid quantization:
- Loop quantization of the homogeneous “background”
- Fock quantization of the “gravitational waves”

The effective dynamics:
- Heuristic construction
- Equations of motion

Effects of inhomogeneities on the bounce point
- Effect through the “energy”: finite number of the degrees of freedom

Asymptotic structure of inhomogeneities
- Statistical behavior of the energy of the modes.
The Gowdy $T^3$ model

- The model specification:
  - Manifold structure: $T^3 \times \mathbb{R}$
  - Two spatial Killing fields: hypersurface orthogonal

- Elements of the metric:
  - norm of one of the Killing fields
  - area of the Killing orbits
  - scale factor of 1-metric ind. on the orbit manifold

- Coordinates: $(t, \sigma, \rho, \theta)$ where $\partial_\sigma, \partial_\rho$ – KVF’s.

- Gauge fixing: System can be deparametrized completely, here just partial gauge fixing.
  - orbit area and the scale factor are homogeneous.

- Canonical formalism:
  - Most of constraints automatically satisfied
  - The only remaining: spatial averages of
    - the diffeomorphism (in \(\theta\)) and
    - the Hamiltonian constraint.
Degrees of freedom

Qualitatively structure of the perturbation theory:

Homogeneous:
- Phase space of Bianchi I model
- Ashtekar variables: $A^i_a = \frac{c^i}{2\pi} \delta^i_a$, $E^a_i = \frac{p^a}{4\pi^2} \delta^a_i$

Inhomogeneous:
- The metric field $\xi(\theta)$ + canonical momentum $P_\xi$,
- Fixing the conformal rescaling freedom of $\xi$:
  - Upon complete deparametrization $\xi$ can be interpreted as the local scalar field evolving wrt time $p_\theta$.
  - then the standard Fock quantization can be implemented
  - There is unique conformal rescaling ensuring the unitary evolution of that quantum field.
- Fourier modes $\xi_m, P^m_\xi \rightarrow$
  
  $$a_m = \sqrt{\frac{\pi}{8G|m|}} \left( |m| \xi_m + \frac{4iG}{\pi} P^m_\xi \right)$$

where: $\{a_m, a^*_m\} = -i\delta_{m,m'}$.

Despite the choice of DOF’s the complete nonlinear structure of the system implemented.
Metric and the constraints

**Metric:**
\[ ds^2 = \frac{|p_\theta p_\sigma p_\delta|}{4\pi^2} \left[ e^{\tilde{\gamma}} \left( -\frac{\tilde{N}^2}{(2\pi)^4} dt + \frac{d\theta^2}{p_\theta^2} \right) + e^{-\frac{2\pi \xi}{\sqrt{p_\theta}}} \frac{d\sigma^2}{p_\sigma^2} + e^{\frac{2\pi \xi}{\sqrt{p_\theta}}} \frac{d\delta^2}{p_\delta^2} \right] , \]
\[ \tilde{\xi}(\theta) = \sum_{m \neq 0} \frac{\sqrt{G}}{\pi \sqrt{|m|}} (a_m + a^*_m)e^{im\theta} , \]
\[ \tilde{\gamma}(\theta) = \left( \frac{2c_\delta p_\delta}{c_\sigma p_\sigma + c_\delta p_\delta} - 1 \right) \frac{2\pi}{\sqrt{|p_\theta|}} \tilde{\xi}(\theta) - \frac{\pi^2}{|p_\theta|} \tilde{\xi}^2(\theta) - \frac{8\pi G \gamma}{c_\sigma p_\sigma + c_\delta p_\delta} \zeta(\theta) , \]
\[ \zeta(\theta) = i \sum_{m, \tilde{m} \neq 0} \text{sgn}(m + \tilde{m}) \frac{\sqrt{|m + \tilde{m}| |\tilde{m}|}}{m} \]
\[ \times (a_{-\tilde{m}} - a^*_{\tilde{m}})(a_{m + \tilde{m}} + a^*_m)(m + \tilde{m})e^{im\theta} . \]

**Diffeomorphism:**
\[ C_\theta = \sum_{m=1}^{\infty} m(a^*_m a_m - a^*_{-m} a_{-m}) = 0 , \]

**Hamiltonian:**
\[ C = C_H + C_I , \]
\[ C_H = -\frac{2}{\gamma V} (\Theta_\theta \Theta_\sigma + \Theta_\theta \Theta_\rho + \Theta_\rho \Theta_\sigma) , \quad \Theta_i = c_i p_i , \quad V = \sqrt{|p_\theta p_\rho p_\sigma|} , \]
\[ C_I = \frac{G}{V} \left[ \frac{(\Theta_\rho + \Theta_\sigma)^2}{\gamma^2 |p_\theta|} H^\xi_{\text{int}} + 32\pi^2 |p_\theta| H^\xi_o \right] , \]
\[ H^\xi_o = \sum_{m \neq 0} |m| a^*_m a_m , \]
\[ H^\xi_{\text{int}} \sum_{m \neq 0} \frac{1}{2|m|} \left[ 2a_m a_m + a_m a_{-m} + a^*_m a^*_{-m} \right] . \]
The Hybrid quantization

Main idea: M. Martin-Benito, L. Garay, G. Mena Marugan

- Perturbation-like treatment: inhomogeneities treated as “fields” living on the homogeneous background.
- Different quantization of the homogeneous and inhomogeneous DOFs.

Loop quantization of the homogeneous DOFs (Bianchi I):

- Selected original LQC lapse $VN$
- $C_H$ expressed as $\int_{\Sigma} d^3x e^{-1} \epsilon^{ij} E_i^a E_j^b F_{ab}^k$
- (where $e = \sqrt{|\det E|}$ and $F_{ab}^k := 2\partial_a A_b^k + \epsilon^{kij} A^i_a A^j_b$).
- Components reexpressed in terms of holonomies and $\hat{p}_i$.
- Lengths $\lambda_i$ of holonomies fixed by setting the area of the rectangular loop used for rewriting $F$ to equal area gap $\Delta$.
- Used old prescription in which $\lambda_i = \lambda_i(p_i)$. Not correct for noncompact models.
- Applied factor ordering in which the triad orientations always separate.
- $C_H$ densitized: lapse $N$ restored on the quantum level

$$C_H \rightarrow C_H = (1/V)^{-1/2} C_H (1/V)^{-1/2}.$$
The Hybrid quantization (2)

- Fock quantization of the inhomogeneous DOFs:
  - Selected complex structure naturally associated with the identification of $a_m, a_m^*$ as creation and annihilation operators.
  - Constructed Fock space $a_m, a_m^*$ in $C_\theta$ and $C_I$ promoted to operators
  - The quantization of the homogeneous elements of $C_\theta$ and $C_I$ transferred from the previous point
  - densitization of $C_I$ analogous to $C_H$

- Final form:
  $$\hat{C} = \hat{C}_H + \hat{C}_I$$
  $$\hat{C}_H = -\frac{2}{\gamma^2} [\hat{\Theta}_\theta \hat{\Theta}_\sigma + \hat{\Theta}_\theta \hat{\Theta}_\rho + \hat{\Theta}_\rho \hat{\Theta}_\sigma]$$
  $$\hat{\Theta}_i = (3/2)\hat{v}_i \sin(\hat{b}_i), \quad \hat{v}_i = 2/(3\sqrt{\Delta})\hat{p}_i \sqrt{|\hat{p}_i|}$$
  where $b_i := \sqrt{\Delta}c_i/\sqrt{p_i}$ is a canonical momentum of $v_i$.
  $$\hat{C}_I = \ell_{P1}^2 \left[ \frac{(\hat{\Theta}_\rho + \hat{\Theta}_\sigma)^2}{\gamma^2} \left( \frac{1}{\sqrt{|p_\theta|}} \right)^2 \hat{H}_\int + 32\pi^2|\hat{p}_\theta|\hat{H}_\xi \right]$$
The effective dynamics

Heuristic method:

- the operators $\hat{v}_i$, $\sin(\hat{b}_i)$, $\hat{\Theta}$, $\hat{a}_m$, $\hat{a}_m^\dagger$ replaced by their expectation values,

- neglected regularization of $1/\sqrt{|p_\theta|}$

Why we hope it can work:

- successfully tested with homogeneous models (one field) – expected to work with finite number,

- countable number of “matter” DOFs,

- specific structure – coupled only in pairs and through the total energy.

The effective constraint: $C = C_H + C_I$

$C_H = -\frac{2}{\gamma^2} [\Theta_\theta \Theta_\sigma + \Theta_\theta \Theta_\rho + \Theta_\rho \Theta_\sigma]$, \quad $\Theta_i = (3/2)v_i \sin(b_i)$,

$C_I = \ell_P^2 \left[ \frac{(\Theta_\rho + \Theta_\sigma)}{\gamma^2 |p_\theta|} H_\int^\xi + 32\pi^2 |p_\theta| H_\sigma^\xi \right]$. 
Equations of motion

Hamilton’s equations:

\[
\dot{v}_\theta = \frac{24\pi G}{\gamma} (\Theta_\rho + \Theta_\sigma) v_\theta \cos b_\theta,
\]

\[
\dot{b}_\theta = -\frac{24\pi G}{\gamma} (\Theta_\rho + \Theta_\sigma) \sin b_\theta - \frac{16\pi G^2}{3\gamma} \left( \frac{2}{3\sqrt{\Delta}} \right)^{2/3} (\Theta_\rho + \Theta_\sigma)^2 \frac{\text{sgn}(v_\theta)}{|v_\theta|^{5/3}} H_\text{int}^\xi
\]

\[
+ \frac{512}{3} \pi^3 G^2 \gamma \left( \frac{3M}{2} \right)^{2/3} \frac{\text{sgn}(v_\theta)}{|v_\theta|^{1/3}} H_0^\xi,
\]

\[
\dot{a}_m = -32\pi^2 iG|m| \left( \frac{3\sqrt{\Delta}}{2} \right)^{2/3} |v_\theta|^{2/3} a_m
\]

\[
- \frac{iG}{\gamma^2|m||v_\theta|^{2/3}} \left( \frac{2}{3\sqrt{\Delta}} \right)^{2/3} (\Theta_\rho + \Theta_\sigma)^2 (a_m + a_{-m}^*),
\]

\[
\dot{v}_\rho = \frac{24\pi G}{\gamma} \left( \Theta_\theta + \Theta_\sigma - \left( \frac{2}{3\sqrt{\Delta}} \right)^{2/3} G (\Theta_\rho + \Theta_\sigma) \frac{H_\text{int}^\xi}{|v_\theta|^{2/3}} \right) v_\rho \cos b_\rho,
\]

\[
\dot{b}_\rho = -\frac{24\pi G}{\gamma} \left( \Theta_\theta + \Theta_\sigma - \left( \frac{2}{3\sqrt{\Delta}} \right)^{2/3} G (\Theta_\rho + \Theta_\sigma) \frac{H_\text{int}^\xi}{|v_\theta|^{2/3}} \right) \sin b_\rho.
\]

Reduce to Bianchi I for \( a_m = 0 \).

\( \Theta_\rho, \Theta_\sigma \) – constants of motion.

Effective diffeomorphism constraint: \( C_\theta = \sum_{m=1}^\infty mK_m \)

where \( K_m \equiv a_m a_{m}^* - a_{-m} a_{-m}^* \) – const. of motion.
The dynamical trajectories

An example of a dynamical trajectory – the solution to EOMs

\[ \log_{10}(p_\rho/l_{Pl}^2) \]

\[ \log_{10}(p_\theta/l_{Pl}^2) \]
The bounce

For $a_m = 0$: position of the bounce: $v_i^b = (2/3)\Theta_i$.

The directions $\sigma, \rho$: from eq. for $\dot{v}_\rho$, at the bounce $\cos(b_{\rho}) = 0$ thus (see def. $\Theta_\rho$) value of $v_{\rho}^b$ unchanged.

The direction $\theta$: more complicated ...

Maximal solution to:

$$\frac{9}{4}(\Theta_\rho + \Theta_\sigma)^2 v_{\theta}^b v_{\theta}^b - \left[F(v_{\theta}^b) - \Theta_\rho \Theta_\sigma\right]^2 = 0$$

$$F(v_{\theta}) = 16\pi^2 \gamma^2 G \left(\frac{3\sqrt{\Delta}}{2}\right)^{2/3} v_{\theta}^{2/3} H_{0}^\xi$$

$$+ \frac{G}{2} (\Theta_\delta + \Theta_\sigma)^2 \left(\frac{2}{3\sqrt{\Delta}}\right)^{2/3} \frac{H_{\text{int}}^\xi}{v_{\theta}^{2/3}}$$

(root of the polynomial of the 5th order).

Two cases:

For $\Theta_\rho \Theta_\sigma < 0$: inhomogeneities increase [...] – $v_{\theta}^b > (2/3)\Theta_\theta$.

For $\Theta_\rho \Theta_\sigma > 0$: the terms inside [...] compete – numerical studies necessary.
Numerical studies of the bounce

- Inhomogeneities enter only through $H^\xi_0$, $H^\xi_{\text{int}}$ – finite number of DOFs!

- Partial compactification: reparametrization
  
  $H^\xi_{\text{int}} =: \alpha H^\xi_0$, $\alpha \in [0, 2]$,  
  
  $\Theta_\delta =: \frac{3}{2} \frac{v_H}{\beta}$, $\Theta_\sigma =: \frac{3}{2} \frac{v_H}{1-\beta}$, $\beta \in [0, 1]$.  

- Only two dimensions noncompact: $H^\xi_0$, $v_H$.

- Numerical methodology:

  - For each point $(v_H, H^\xi_0, \alpha, \beta)$ numerically finding $v^b_\theta$ as the maximal root of the 5th order polynomial.

  
  $v^b_\theta(v_H, H^\xi_0, \alpha, \beta) \rightarrow v^b_\theta(v_H, \alpha, \beta) = \min_{H^\xi_0} v^b_\theta(v_H, H^\xi_0, \alpha, \beta)$. Domain necessary for numerical scanning determinable analytically.

  - Result scanned over the compact $\alpha, \beta$.

  - Also verified the limit $v_H \rightarrow \infty$ for fixed $\alpha, \beta$.

- THE RESULT: $v^b_\theta \geq 0.05v_H$ !!!
Numerical scan of the bounce (1)

Roots for fixed $v_H, \alpha, \beta$. 

![Graph showing roots for fixed $v_H, \alpha, \beta$.]
Numerical scan of the bounce (2)

Minimum $v^b_{\theta}(v_H, \alpha, \beta)$ for fixed $\alpha$. 
Numerical scan of the bounce (3)

Minimum $v^b_\theta(v_H, \alpha, \beta)$ for fixed $\beta$. 

![Graph showing a 3D visual of the bounce scan](image-url)
Numerical scan of the bounce (4)

The slice of $v_H, \alpha, \beta$ for fixed $\alpha, \beta$. 
Caveats

On dynamical trajectories $H^\xi_o \neq \text{const} – \text{oscillations!}$

However...

- oscillations are small – small change of $v^b_\theta$ in the range of achieved values of $H^\xi_o$ for given trajectory.

All the roots are reflective boundaries. Thus in principle possible for the trajectory to go “down the throat” to the singularity!

However...

- The process requires an infinite number of reflections, thus fine tuning (critical phenomena).
- Only very small region of the parameter space can violate the predicted bound on $v^b_\theta$.

Conclusions:

- The bounce is generic.
- There is a upper 5% bound on the position of the bounce, which is violated only in a very small region of the parameter space.
Dynamical trajectories (1)

An illustration of the oscillations of $a_m$ in the time evolution.

![Graph showing oscillations of $a_m$ over time](image-url)
Dynamical trajectories (2)

Near-critical dynamical trajectory (multiple bounces and recollapses).
The energy of inhomogeneities

- $H_0^ξ$ – convenient measure of the energy of the inhomogeneities,
- For large $v_θ$ approaches constants different for the future and the past.
- Contribution of a single mode encoded in $|a_m|^2$.
- Change between asymptotic values of $|a_m|^2$ depends on the phases between $a_m$ and $a_{−m}$.

Idea: neglect the information about the phases: The deterministic system becomes statistical.

Question: How energy distribution changes through the bounce?

Answer: Through the numerical Monte-Carlo methods.

- Distribution of the energies: values of $|a_m|$ with Gaussian probability.
- Distribution of the phases: uniform distribution.
**The evolution of the modes**

**Observation:** Fix all other parameters and probe the amplitude change $\Delta|a_m|$ between asymptotic past and future as a function of the phase $\phi$.

![Graphs showing $\Delta|a_m|$ vs $\phi$.]

**Two domains:**
- **Inhomogeneity dominated:** The average of $\Delta|a_m|$ over $\phi$ vanishes. **Statistical preservation.**
- **Near vacuum:** The average is positive: **amplification of the inhomogeneities.**
The energy change distribution

The relative change in the energy of a single mode as a function of $v^b_\theta$

$$\log_{10}(\Delta H^\xi_o/H^\xi_o)$$

$$\log_{10}(v^b_\theta/v_H)$$
Conclusions

The fate of the bounce:

In considered model the bounce is generic.

The position $v$ of the bounce in the Killing directions is not affected by the inhomogeneities.

In the “inhomogeneous” direction the bounce position $v^b_\theta$ is bounded by an analogous value of the homogeneous universe $v^b_\theta \geq 0.05v_H$. The bound can be violated only in some very small region of the parameter space near the critical solutions.

The evolution of the inhomogeneities:

Once we neglect the phases between $a_m$ and $a_{-m}$ the change of the energies of the inhomogeneities between $t \to -\infty$ and $t \to +\infty$ attains the probabilistic character.

Two regions of a distinct qualitative behavior:

- inhomogeneity dominated: where the amplitudes of the gravitational wave modes are statistically preserved, and
- near vacuum, where the amplitudes are statistically amplified.