# Black Hole Entropy and SU(2) Chern Simons Theory

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Based on joint work with Jonathan Engle and Karim Noui

# The plan

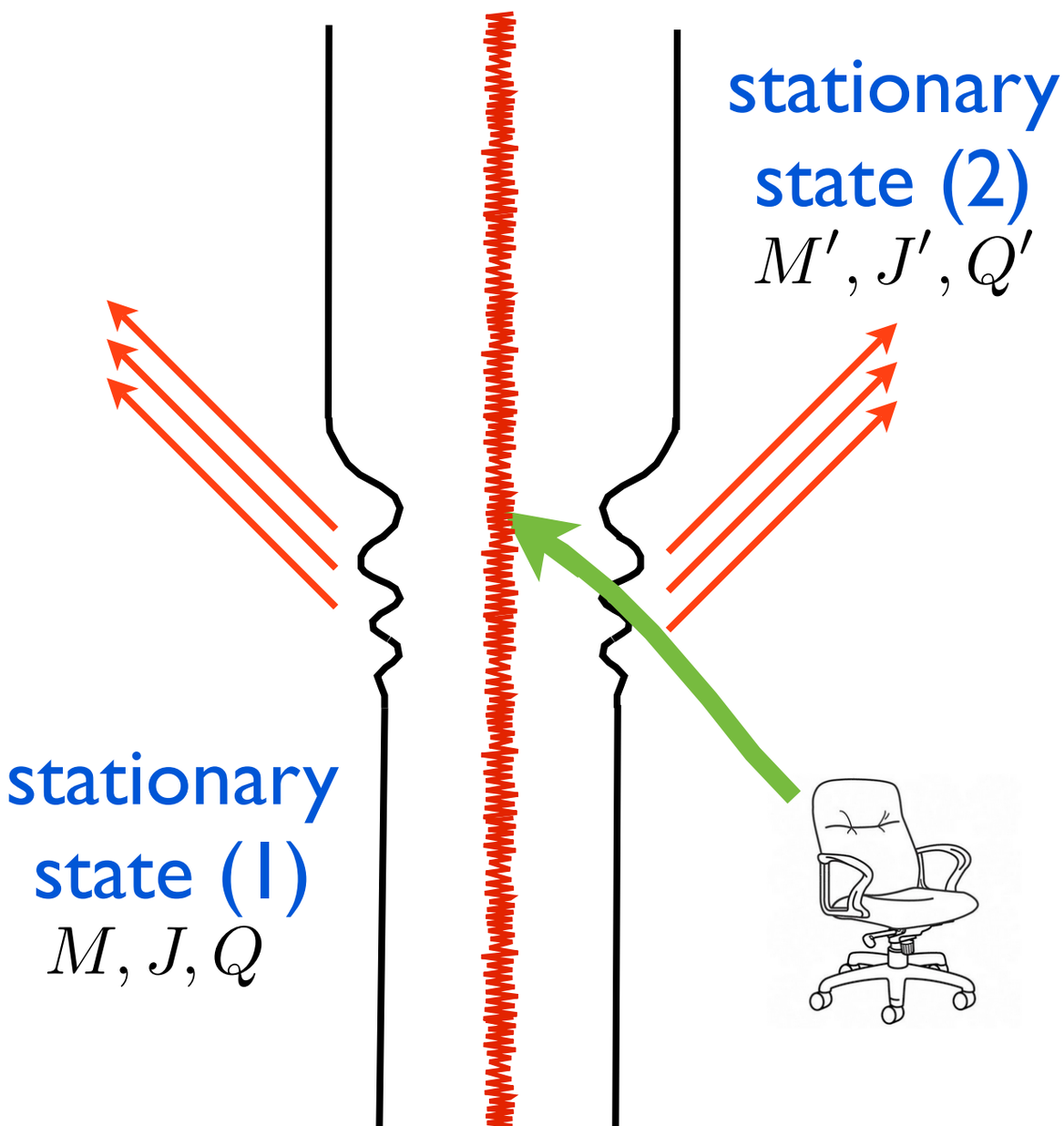
- Brief motivation and introduction
- The main idea in Ashtekar-Krasnov-Corichi treatment of the problem
- Isolated horizons (in the ACK formulation)
- Connection variables (the SU(2) invariant canonical formulation)
- Quantization
- Relation between the U(1) and SU(2)
- Counting

# Introduction and motivation

## Black Hole Thermodynamics

**Some definitions**

$$\begin{cases} \Omega \equiv \text{horizon angular velocity} \\ \kappa \equiv \text{surface gravity ('grav. force' at horizon)} \\ \text{If } \ell^a = \text{killing generator, then } \ell^a \nabla_a \ell^b = \kappa \ell^b. \\ \Phi \equiv \text{electromagnetic potential.} \end{cases}$$




**0th law:** the surface gravity $\kappa$ is constant on the horizon.

**1st law:**

$$\delta M = \frac{\kappa}{8\pi}\delta A + \underbrace{\Omega \delta J + \Phi \delta Q}_{\text{work terms}}$$

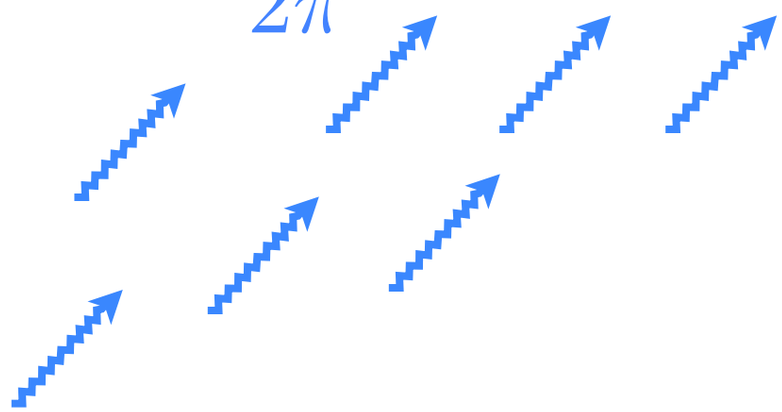
**2nd law:**

$$\delta A \geq 0$$

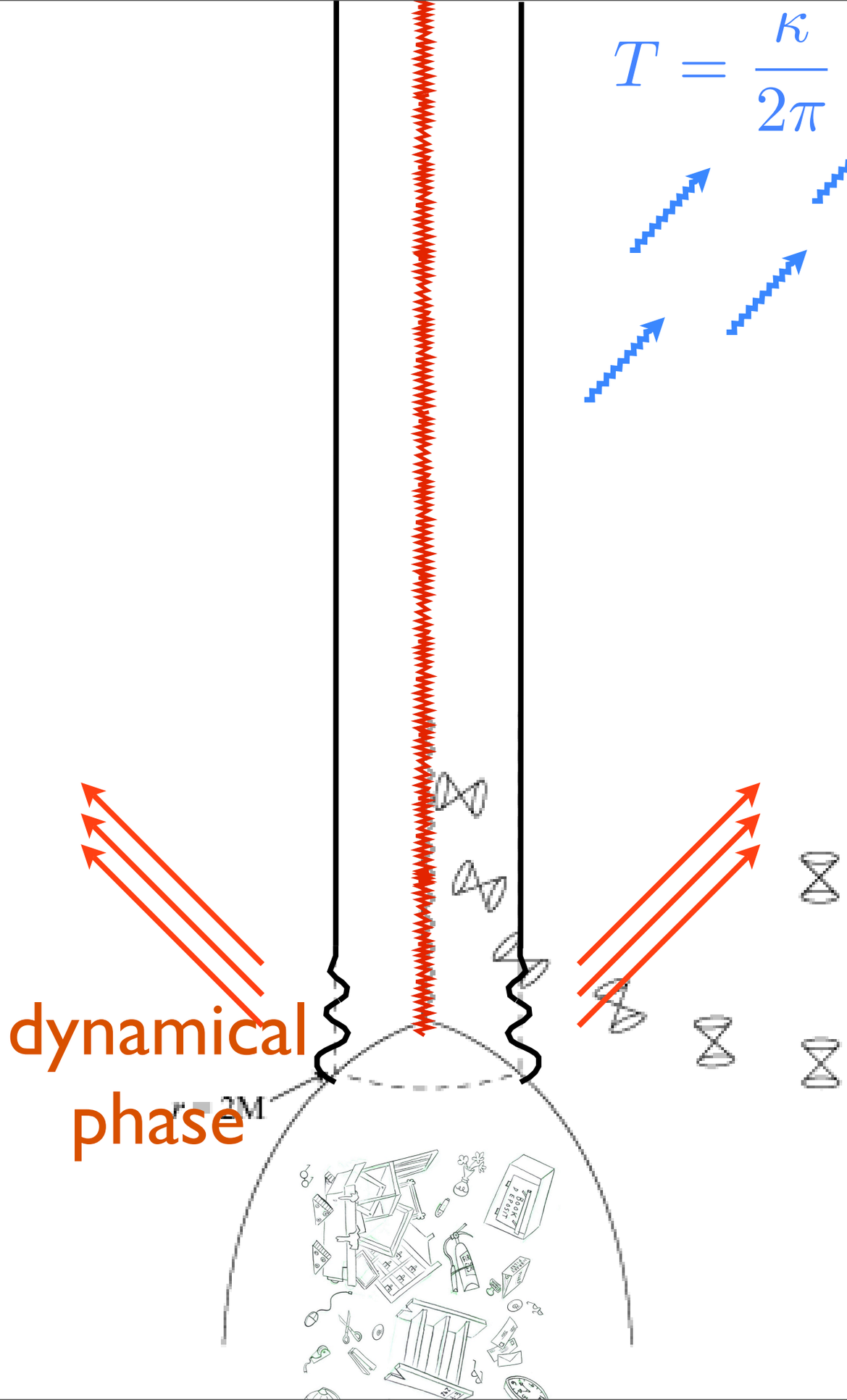
**3rd law:** the surface gravity value $\kappa = 0$ (extremal BH) cannot be reached by any physical process.

# Hawking Radiation

$$T = \frac{\kappa}{2\pi}$$



$$\boxed{S = \frac{1}{4}A}$$

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# Introduction and motivation

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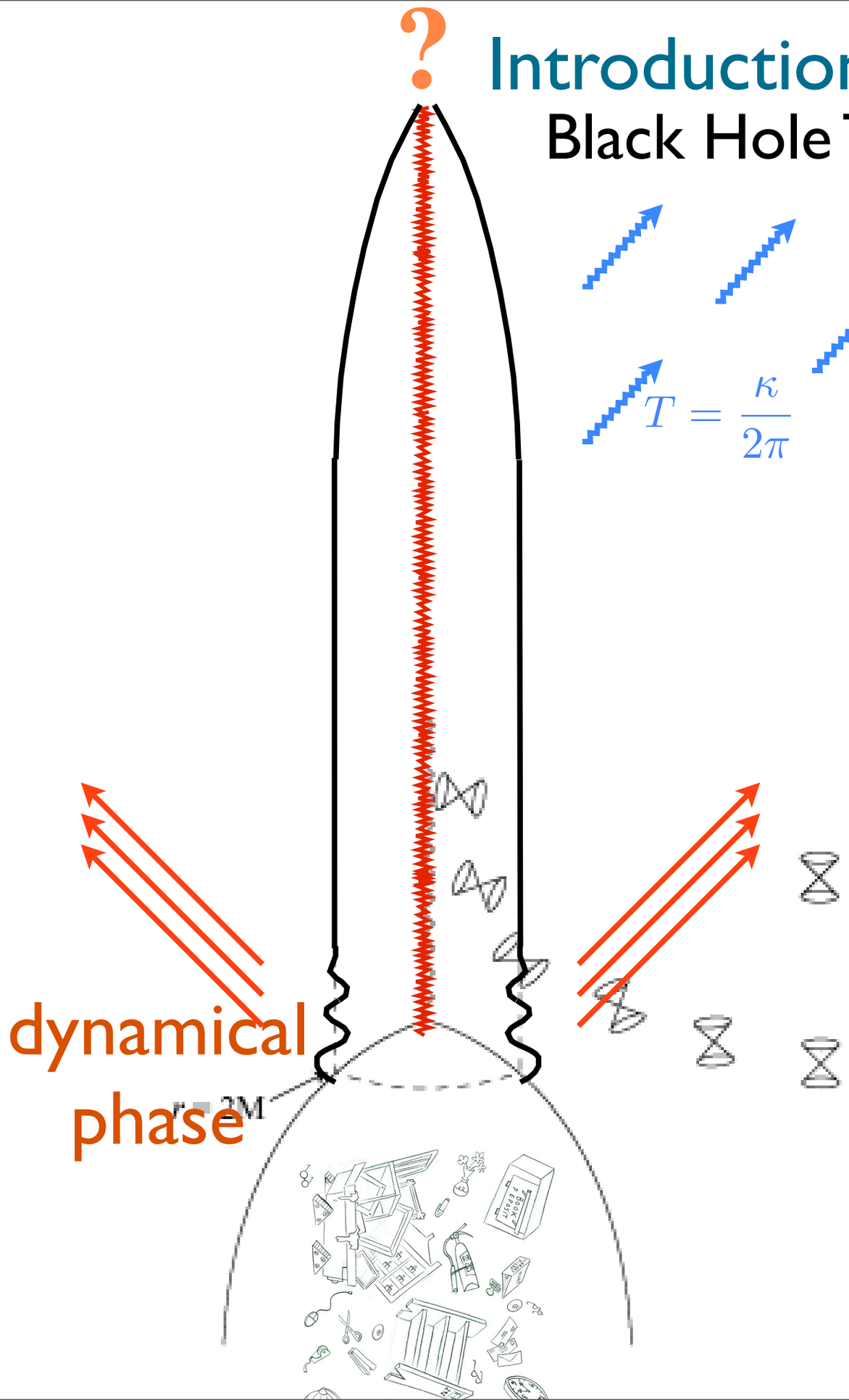



## Black Holes Evaporate

### Questions for Quantum Gravity

$$S = \frac{1}{4}A$$

1) Microscopic origing of the entropy.

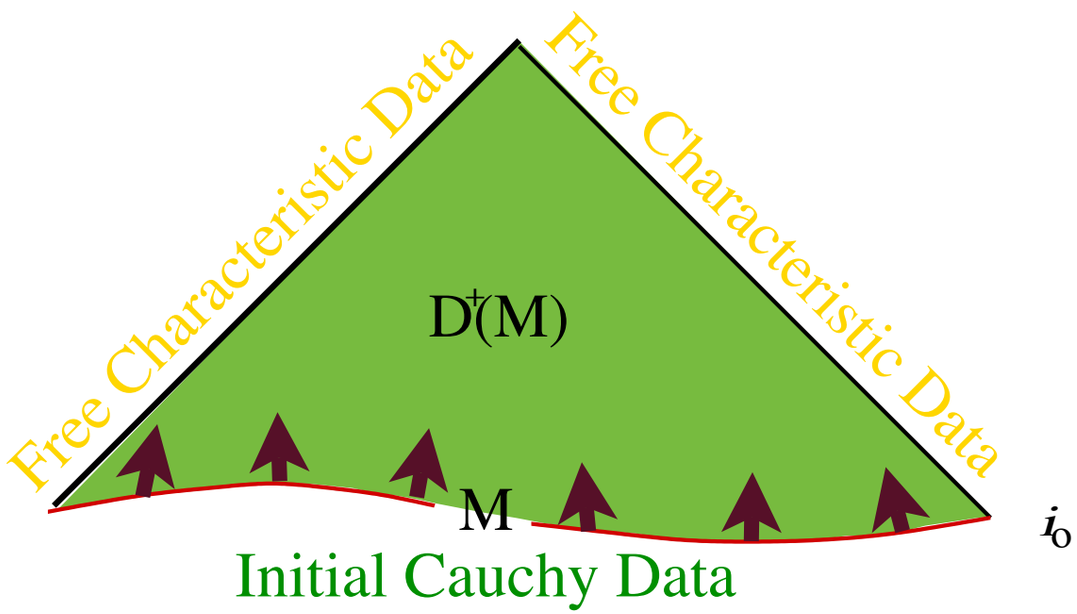
2) Dynamics of black hole evaporation.

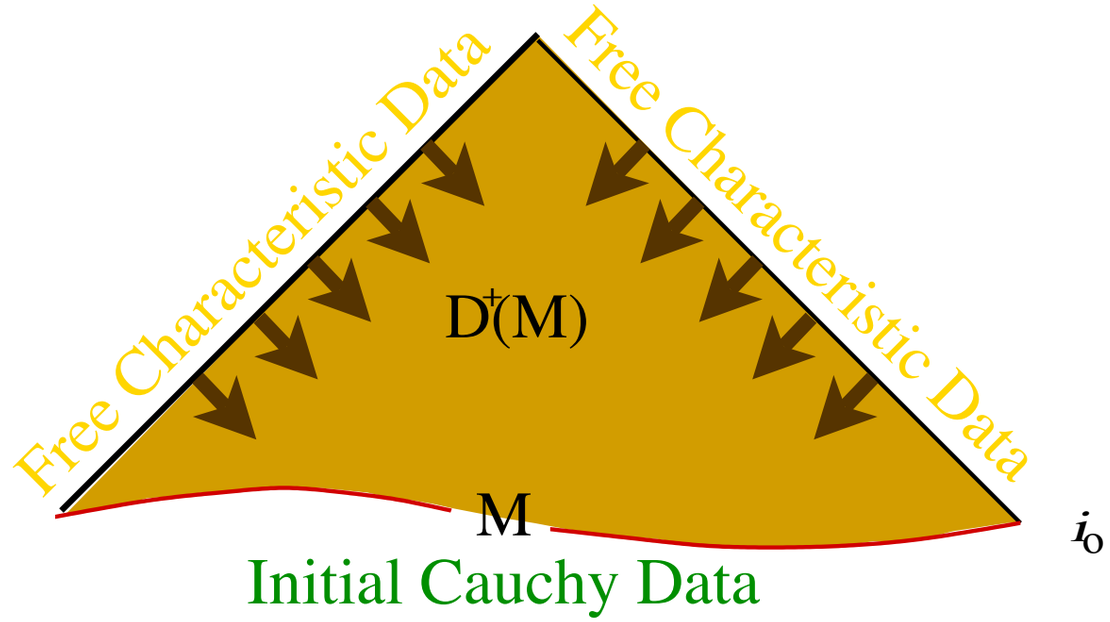
3) What replaces the classical singularity.

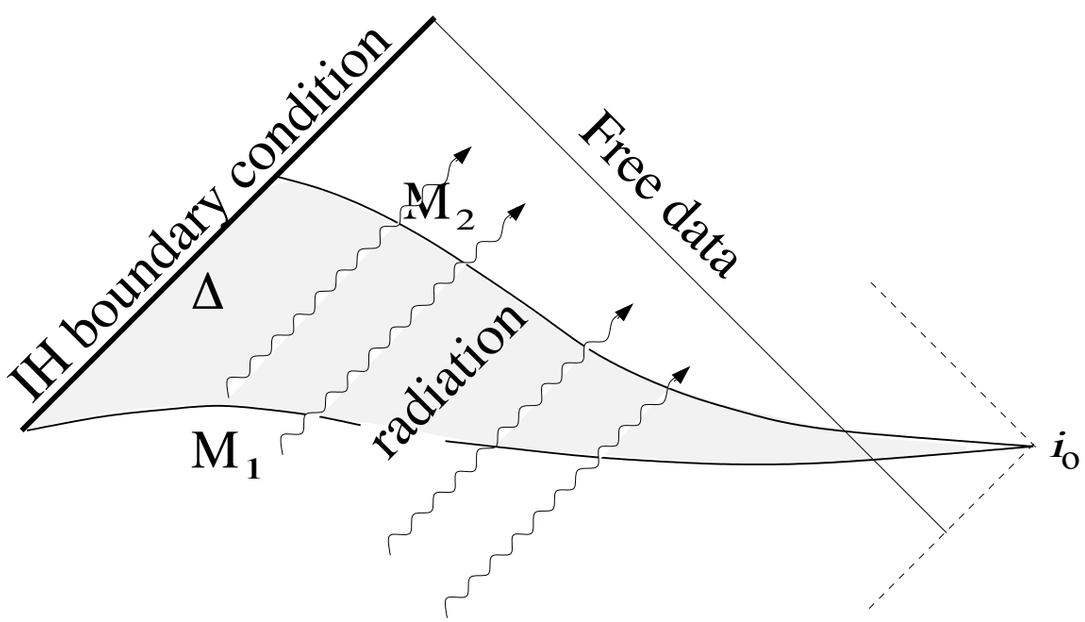
4) What happens when a BH completely evaporates.

# Quasilocal definition of BH

## From the Characteristic formulation of GR to isolated horizons




**ACK IH boundary condition**

- $\Delta = S^2 \times R$
- Einstein's equations hold on $\Delta$
- For $\ell^a$ the null normal $\rho = 0$ and $\sigma = 0$.
- For the null vector field $n^a$, such that $n \cdot \ell = 1$, $\sigma' = 0$ and $\rho' = -f(v)$

# ACK isolated horizons

## Implications in terms of connection variables

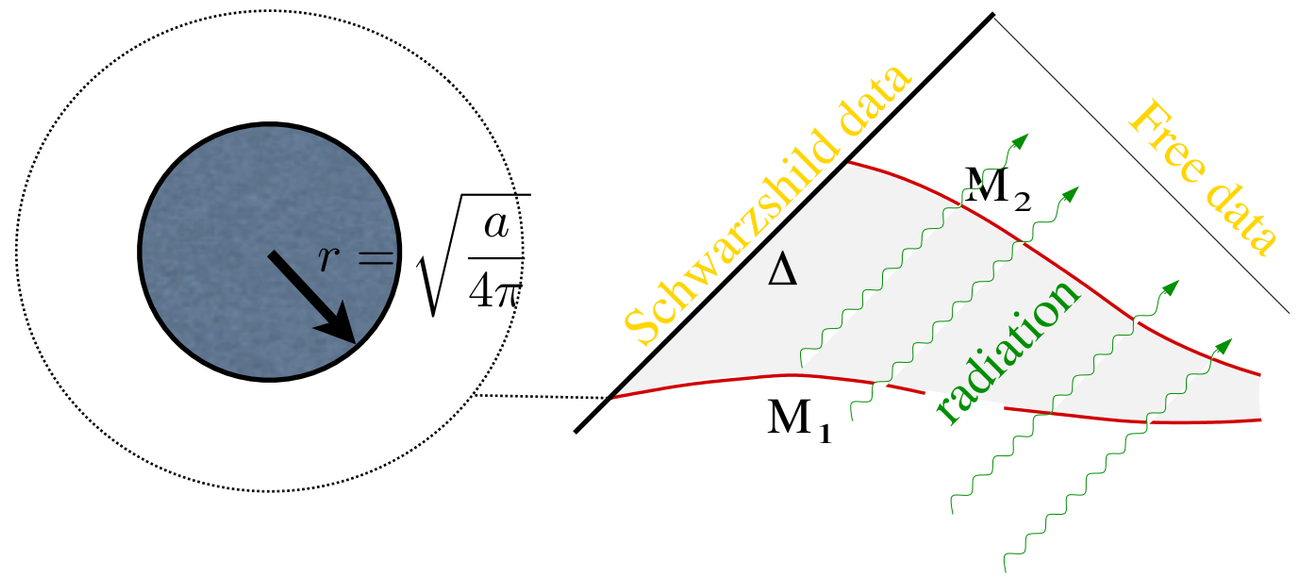
## The main equations:

equations below are valid on the appropriate thermal slicing where 0th and 1st laws hold.

$$\Sigma^i = -\frac{a}{\pi} F^i(\Gamma) \qquad K^i = -\sqrt{\frac{2\pi}{a}} e^i$$

$$A^i = \Gamma^i + \gamma K^i$$

$$F_{ab}{}^i(A) = -\frac{\pi(1-\gamma^2)}{a} \Sigma_{ab}{}^i$$




In the gauge normal gauge (where $e^1$ is normal to H) the main equation can be written in AKC form, plus two additional conditions on $\Sigma^i$

$$\Longrightarrow \quad V_a = 2\Gamma^1, \quad dV = -\frac{2\pi}{a}\Sigma^1 = -\frac{2\pi}{a}\epsilon^2$$

$$\Sigma^2 = \Sigma^3 = 0$$

# Fixing of the null generator

**In IH there is no unique notion of $\kappa$**

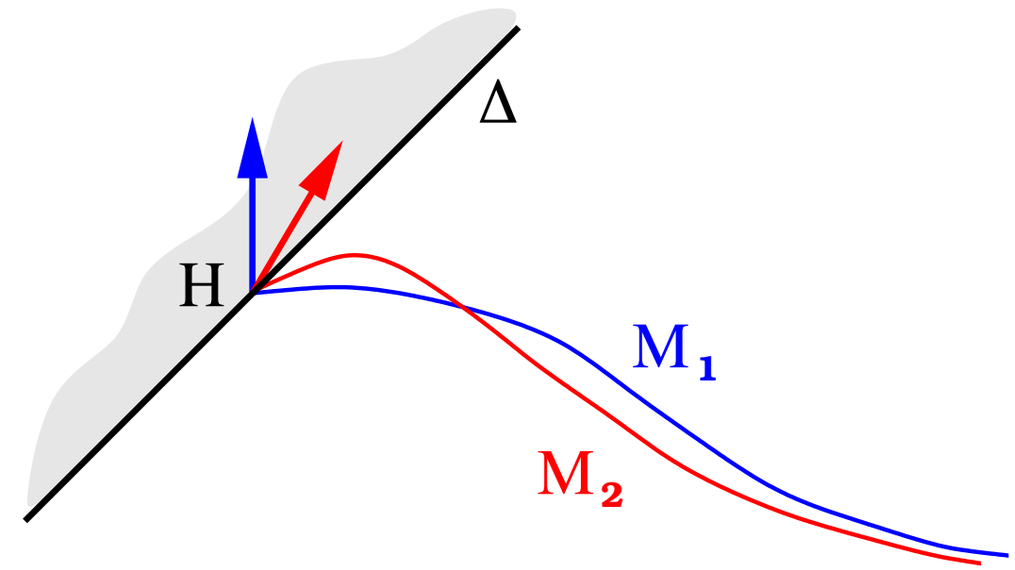
$$\ell^a \nabla_a \ell^b = \kappa\ \ell^b$$

**Lorentz Transf. above**

$$\ell^a \to \tilde{\ell}^a = \exp(-(\alpha(x)))\ell^a$$

$$n^a \to \tilde{n}^a = \exp((\alpha(x)))n^a$$

$$normal \to normal$$




A preferred normal direction to H: the allowed histories must satisfy (Ashtekar)

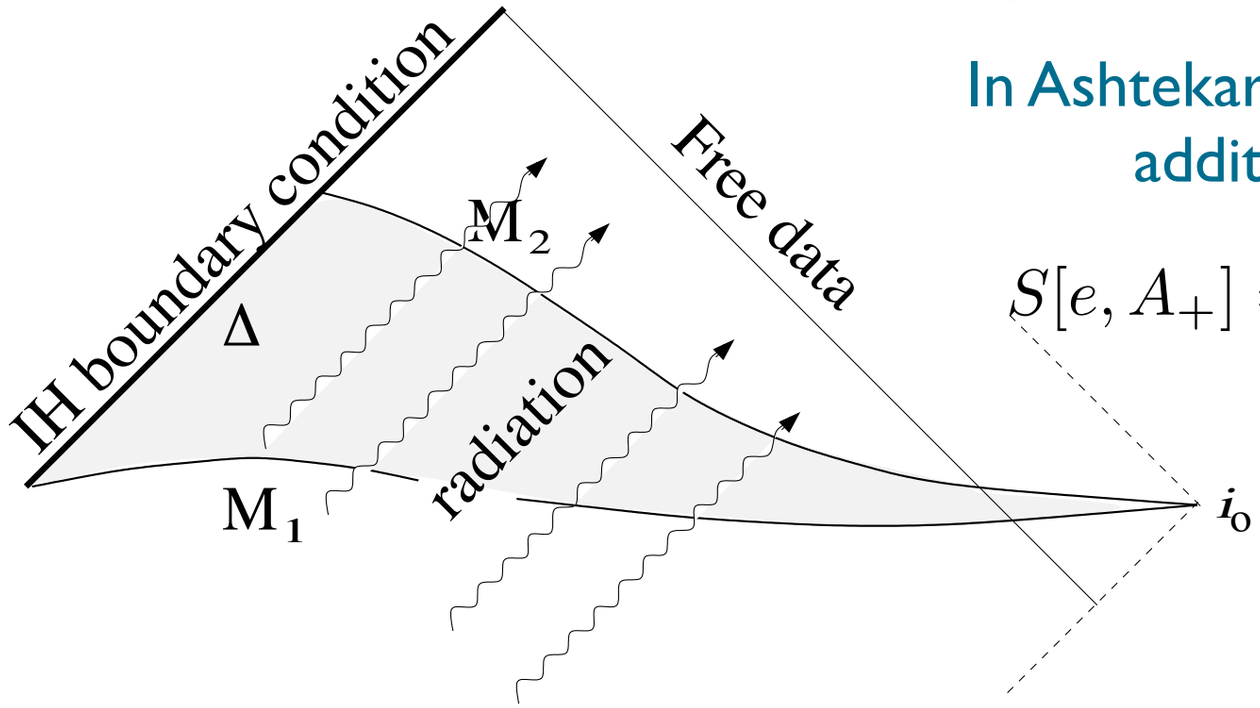
$$q^{ab} r_a r_b = 1 \qquad \Longleftrightarrow \qquad \sqrt{\delta^{ij} E^a_i E^b_j r_a r_b} = \sqrt{q} \tag{a}$$

**Indeed this choice is the one that makes the zero, and first law of IH look just as the corresponding laws of stationary black hole mechanics [ACK, Ashtekar-Beetle-Fairhurst-Krishnan].**

$$K^i_a = -\frac{f(v)}{\sqrt{2}} e^i_a \to \boxed{K^i_a = -\sqrt{\frac{2\pi}{a}}\ e^i_a} \tag{b}$$

# Isolated horizon dynamical system

## Covariant phase space formulation




In Ashtekar variables, due to the IH boundary condition, no additional boundary term need to be included.

$$S[e, A_+] = -\frac{i}{\kappa}\int_M \Sigma_i(e)\wedge F^i(A_+) + \frac{i}{\kappa}\int_{\tau_\infty} \Sigma_i(e)\wedge A^i_+$$

**The presymplectic current:** $$J(\delta_1,\delta_2) = -\frac{2i}{\kappa}\delta_{[1}\Sigma_i\wedge\delta_{2]}A^i_+ \quad \forall \;\; \delta_1,\delta_2\in T_p\Gamma \qquad (a)$$

**Einsteins Equations imply:** $$dJ(\delta_1,\delta_2) = 0 \qquad (b)$$

$$0 = \frac{i\kappa}{2}\int_{\partial B} J(\delta_1,\delta_2) = \underbrace{\int_\Delta \delta_{[1}\Sigma_i\wedge\delta_{2]}A^i_+} + \int_{M_1}\delta_{[1}\Sigma_i\wedge\delta_{2]}A^i_+ - \int_{M_2}\delta_{[1}\Sigma_i\wedge\delta_{2]}A^i_+ \qquad (c)$$

$$\overbrace{\frac{a_H}{4\pi}\int_{H_2}\delta_{[1}A_{+i}\wedge\delta_{2]}A^i_+ - \frac{a_H}{4\pi}\int_{H_1}\delta_{[1}A_{+i}\wedge\delta_{2]}A^i_+}$$

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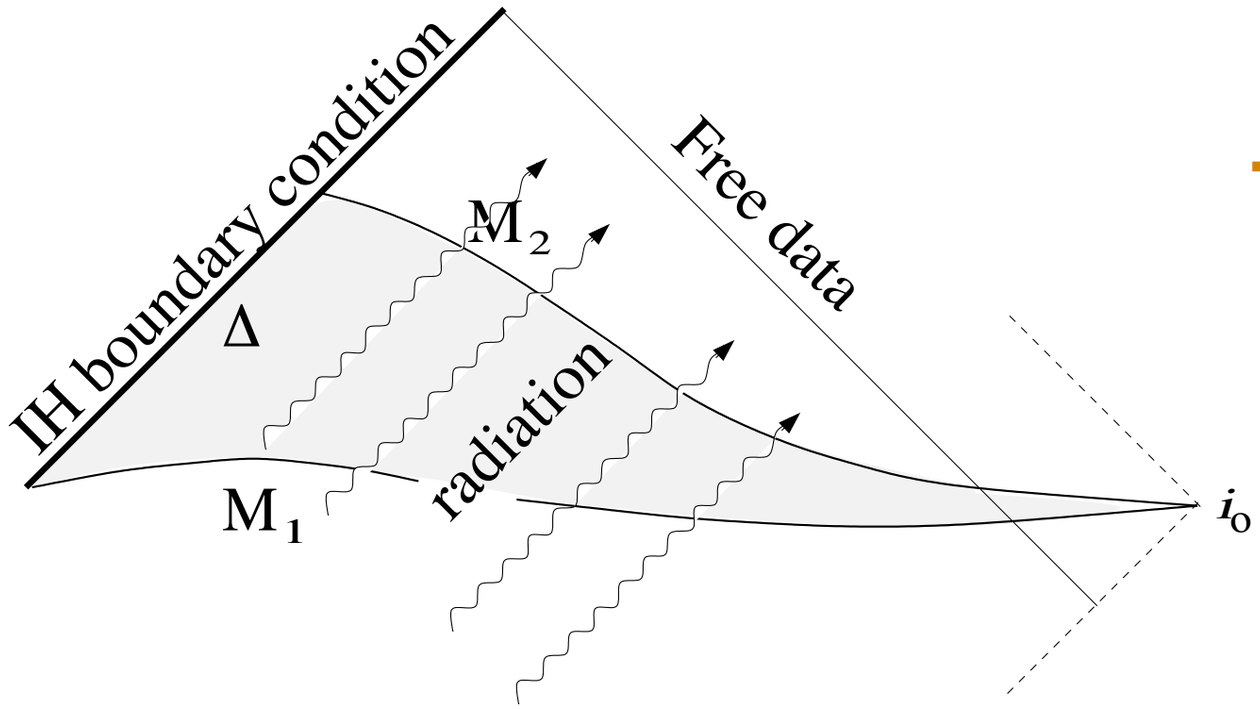



**The conserved presymplectic form:**

$$\Omega_{M_1}(\delta_1, \delta_2) = \Omega_{M_2}(\delta_1, \delta_2) \qquad (a)$$

**The conserved presymplectic form:**

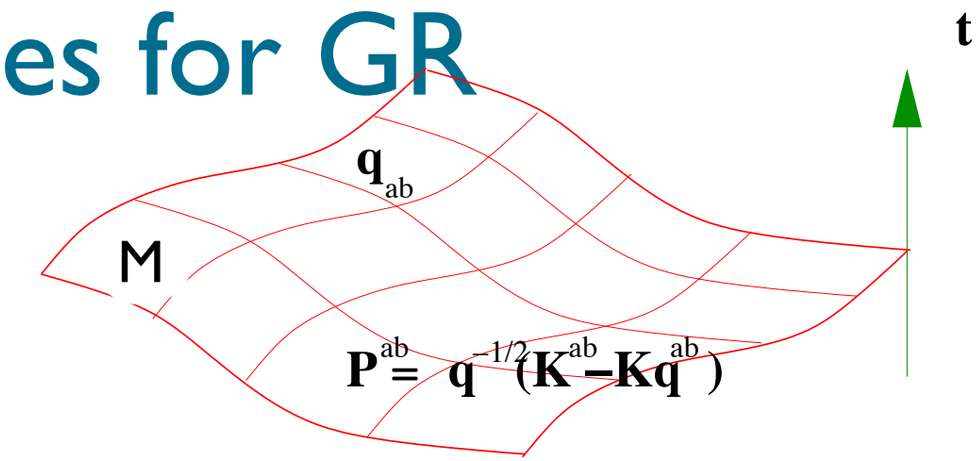
$$i\kappa\,\Omega_M(\delta_1, \delta_2) = \int_M 2\delta_{[1}\Sigma_i \wedge \delta_{2]}A^i_+ - \frac{a_H}{2\pi}\int_H \delta_{[1}A_{+i} \wedge \delta_{2]}A^i_+ \qquad (b)$$

**Reality conditions:**

$$A^i_+ = \Gamma^i + iK^i \qquad \Longrightarrow \qquad \mathrm{Im}[\Omega_M(\delta_1, \delta_2)] = 0 \qquad (c)$$

$$\kappa\Omega_M(\delta_1, \delta_2) = \kappa\mathrm{Re}[\Omega_M(\delta_1, \delta_2)] = \int_M 2\delta_{[1}\Sigma_i \wedge \delta_{2]}K^i \qquad (d)$$

# Connection variables for GR




The Einstein Hilbert Action

$$S[e,A] = \frac{1}{\kappa}\int_{\mathcal{M}} \sqrt{-g}\; R[g]$$

The (pre) Symplectic Form

$$\Omega(\delta_1,\delta_2) = \frac{1}{\kappa}\int_M (\delta_1 q_{ab}\;\delta_2\Pi^{ab} - \delta_2 q_{ab}\;\delta_1\Pi^{ab}) \quad \forall \;\; \delta_1,\delta_2 \in T(\Gamma^{IH})$$

Introduce a triad (SU(2) gauge symm.)

$$q_{ab} = e^i_a e^j_b \delta_{ij}$$

New Canonical Variables

$$\Sigma^i = \epsilon^{ijk}(e_j \wedge e_k)$$

$$K^i_a = e^b_j K_{ab}\delta^{ij}$$

Second fundamental form

New Connection Variables

$$\Sigma^i = \epsilon^{ijk}(e_j \wedge e_k)$$

$$A^i = \gamma K^i + \Gamma(e)^i$$

Torsion free connection

$$\Omega(\delta_1,\delta_2) = \frac{1}{8\pi G}\int_M \delta_{[1}\Sigma^i \wedge \delta_{2]}K^i = \frac{1}{8\pi G\gamma}\int_M \delta_{[1}\Sigma^i \wedge \delta_{2]}A^i, \qquad (a)$$

The Symplectic Potential

$$8\pi G\gamma\Theta(\delta) = \int_M \Sigma_i \wedge (\gamma\delta K^i) = \int_M \Sigma_i \wedge \delta(\Gamma^i + \gamma K^i) - \int_M \Sigma_i \wedge \delta\Gamma^i$$

$$\Sigma_i \wedge \delta\Gamma^i = -d(e_i \wedge \delta e^i)$$

# Isolated horizon dynamical system

## Covariant phase space formulation

The conserved presymplectic form:

$$\Omega_M(\delta_1,\delta_2) = \frac{1}{\kappa}\int_M [\delta_1\Sigma^i \wedge \delta_2 K_i - \delta_2\Sigma^i \wedge \delta_1 K_i] \quad \forall \quad \delta_1,\delta_2 \in T(\Gamma_{cov})$$

The Symplectic Potential:

$$8\pi\gamma\Phi(\delta) = \int_M \Sigma_i \wedge \delta(\gamma K^i) + \int_M \Sigma_i \wedge \delta\Gamma^i - \int_M \Sigma_i \wedge \delta\Gamma^i \qquad (a)$$

Rewriting the Symplectic Potential:

$$\begin{aligned} 8\pi\gamma\Phi(\delta) &= \int_M \Sigma_i \wedge \delta(\gamma K^i + \Gamma^i) - \int_M \Sigma_i \wedge \delta\Gamma^i \\ &= \int_M \Sigma_i \wedge \delta A + \int_H e_i \wedge \delta e^i \qquad (b) \end{aligned}$$

The presymplectic form in connection variables:

$$\Omega_M(\delta_1,\delta_2) = \frac{1}{\kappa\gamma}\int_M [\delta_1\Sigma^i \wedge \delta_2 A_i - \delta_2\Sigma^i \wedge \delta_1 A_i] + \frac{1}{\kappa\gamma}\int_H \delta_1 e_i \wedge \delta_2 e^i \qquad (c)$$

$$\Sigma(\alpha,S) \equiv \int_{S\subset H} \mathrm{Tr}[\alpha\Sigma] \qquad \{\Sigma(\alpha,S),\Sigma(\beta,S)\} = \Sigma([\alpha,\beta],S)$$

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No boundary term here either

$$\Omega_M(\delta_1,\delta_2) = \frac{1}{\kappa}\int_M [\delta_1\Sigma^i \wedge \delta_2 K_i - \delta_2\Sigma^i \wedge \delta_1 K_i] \quad \forall \quad \delta_1,\delta_2 \in T(\Gamma_{cov})$$

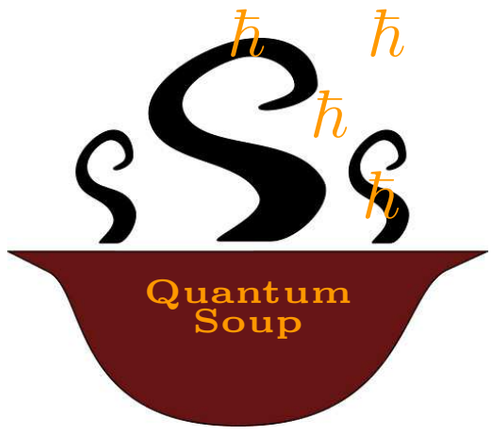
The Symplectic Potential

$$\begin{aligned} 8\pi\gamma\Phi(\delta) &= \int_M \Sigma_i \wedge \delta(\gamma K^i + \Gamma^i) - \frac{a}{2\pi(1-\gamma^2)}\int_H A_i \wedge \delta A^i \\ &= \int_M \Sigma_i \wedge \delta(\gamma K^i) - \underbrace{\int_H e_i \wedge \delta e^i + \frac{a}{2\pi(1-\gamma^2)} A_i \wedge \delta A^i}_{\text{closed one form in } \Gamma_{cov}} \end{aligned}$$

$$F_{ab}{}^i(A) = -\frac{\pi(1-\gamma^2)}{a}\,\Sigma_{ab}{}^i \longrightarrow$$

$$\Omega_M(\delta_1,\delta_2) = \frac{1}{\kappa\gamma}\int_M [\delta_1\Sigma^i \wedge \delta_2 A_i - \delta_2\Sigma^i \wedge \delta_1 A_i] + \frac{a}{\pi(1-\gamma^2)\kappa\gamma}\int_H \delta_1 A_i \wedge \delta_2 A^i$$

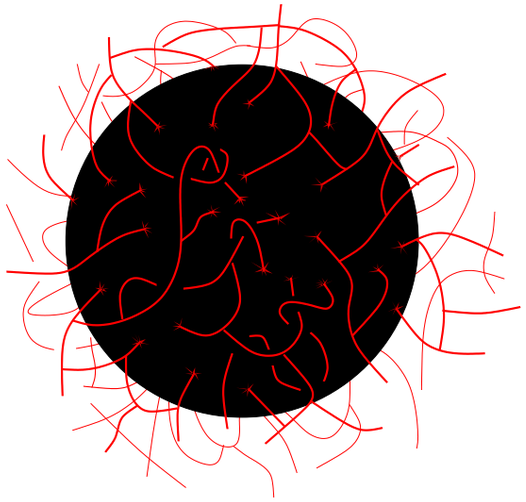
# The quantum Soup

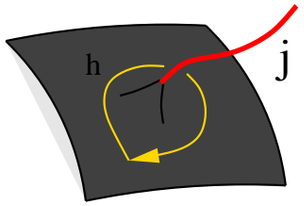



**(1) Quantize and solve the IH boundary condition**

**Quantum $SU(2)$ Chern-Simons theory with defects**[**Witten, Smolin**, Krasnov]

$$\left[\frac{a_H}{\pi(1-\gamma^2)}\hat{F}+\hat{\Sigma}\right]|\Psi\rangle = 0$$






**(2) Count states using Witten's results:** $S = \log(\mathscr{N})$ **with** $\mathscr{N} = \sum_{n;(j)_1^n} \dim[\mathscr{H}^{CS}(j_1 \cdots j_n)]$ [**Kaul-Majumdar (first), (then)Carlip**] **with labels** $j_1 \cdots j_p$ **of the punctures constrained by the condition** $\sum_{p=1}^{n} \sqrt{j_p(j_p+1)} \leq \frac{a_H}{8\pi\beta\ell_p^2} \equiv k_0$ **(precise counting done resently)** [**Agullo-Barbero-Borja-Diaz-Polo-Villasenor**][Noui et al. (new ideas)]

# The single intertwiner BH model

$$\dim[\mathscr{H}^{CS}(j_1 \cdots j_n)] \leq \dim[\mathrm{Inv}(j_1 \otimes \cdots \otimes j_n)]$$

**The area constraint** $\sum_{p=1}^{n} \sqrt{j_p(j_p+1)} \leq \frac{a_H}{8\pi\beta\ell_p^2} \Rightarrow$

$$\boxed{\dim[\mathscr{H}^{CS}(j_1 \cdots j_n)] = \dim[\mathrm{Inv}(j_1 \otimes \cdots \otimes j_n)]}$$

**We can model the IH system by a single $SU(2)$ intertwiner [Livine-Terno, Rovelli-Krasnov]**

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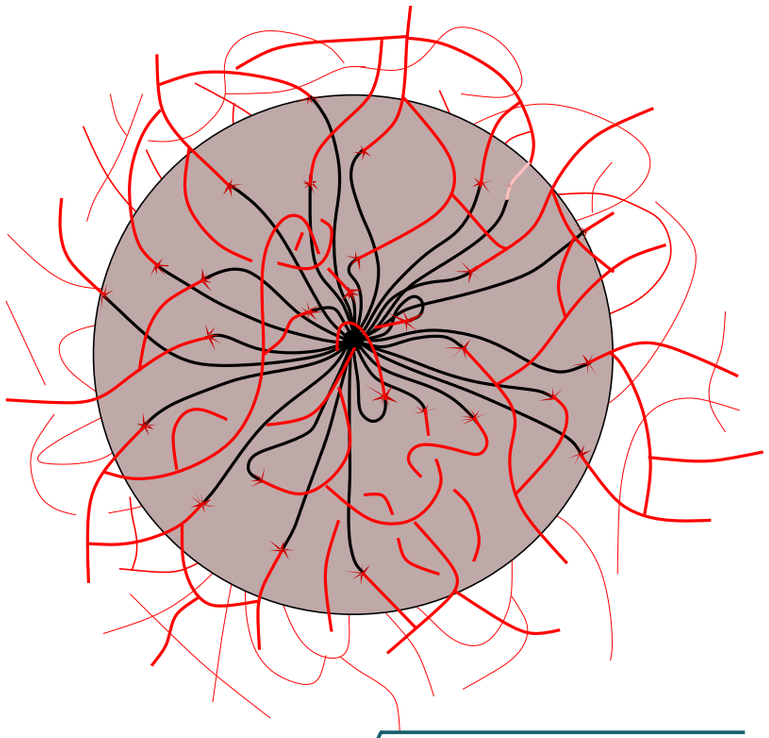

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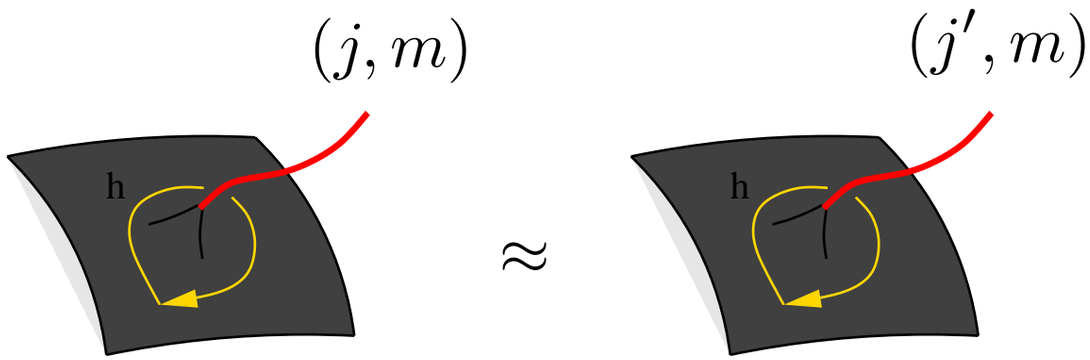
# Relation to U(1) treatment

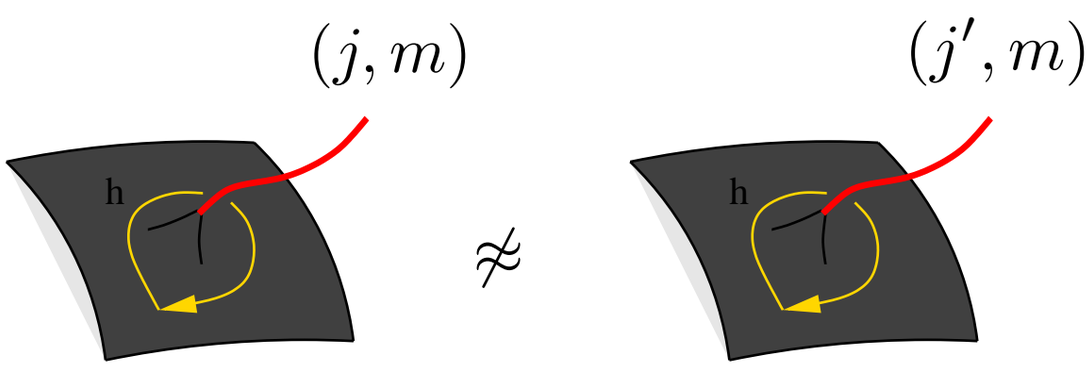
## Domagala-Lewandowski counting




Density matrix constructed by tracing out "bulk" degrees of freedom where only the connection is a surface observable

$$\left(\frac{k}{4\pi}\hat{F} - \hat{\Sigma}^i r_i\right)|\Psi\rangle = 0 \qquad \langle\Psi'|\hat{\Sigma}^i(\delta_{ij} - r_i r_j)|\Psi\rangle = 0$$

## Ghosh-Mitra counting




Density matrix constructed by tracing out "bulk" degrees of freedom where both connection and the area density are a surface observables

(see Ashtekar-Engle-Van den Broeck)

$$\left(\frac{k}{4\pi}\hat{F} - \hat{\Sigma}^i r_i\right)|\Psi\rangle = 0 \qquad \langle\Psi'|\hat{\Sigma}^i(\delta_{ij} - r_i r_j)|\Psi\rangle = 0$$

# The issue of fixing the null generator in the SU(2) treatment

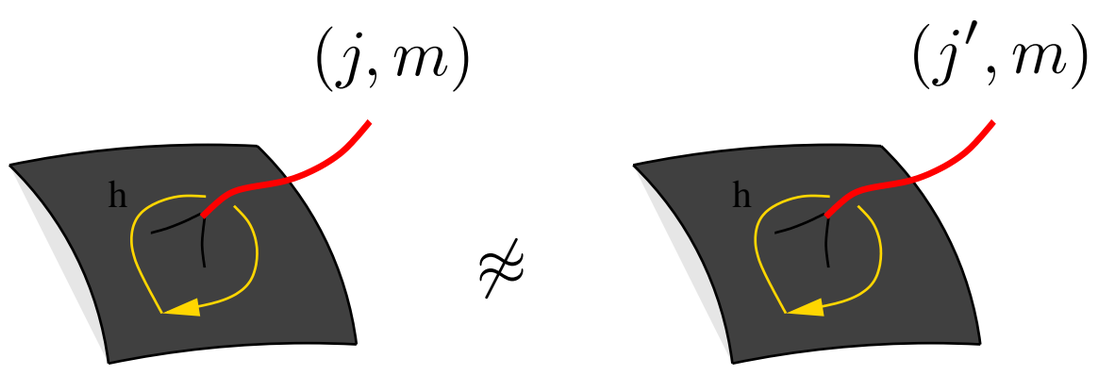
counting remains unchanged after imposing the condition in the quantum theory

**Counting:** Density matrix constructed by tracing out "bulk" degrees of freedom where both connection and the area density are a surface observables

$$\boxed{(\frac{k}{4\pi}\widehat{F}^i + \widehat{\Sigma}^i)|\Psi >= 0}$$

$$\sqrt{\delta^{ij} E^a_i E^b_j r_a r_b} = \sqrt{q}$$

$$(\widehat{\sqrt{h} - \sqrt{q}})|\Psi\rangle = 0 \qquad (a)$$




Assuming there is at least one solution to equation (a) per surface state the counting is unaffected as (a) only restricts bulk d.o.f.

# Conclusions

- **IH's [Ashtekar et al.] admits an $SU(2)$ invariant treatment. The d.o.f. of the horizon are described in terms of an $SU(2)$ Chern-Simons theory with level $k = \frac{a_H}{2\pi\ell_p^2\gamma(1-\gamma^2)}$ [meets old ideas by Smolin].** (done in the ACK definition, more work needed for general IH)

- **For $|\gamma| \leq \sqrt{3}$, $k_0 \leq k$: $S = \frac{\gamma_0 a_H}{4\ell_p^2\gamma} - \frac{3}{2}\log(a_H)$ with $\gamma_0 = 0.274067...$ [Agullo-Barbero-Borja-Diaz-Polo-Villasenor, Kaul-Majumdar (for $-3/2\log$)].**

- **Leading term in entropy coinsides with the $U(1)$ treatment with Ghosh-Mitra counting.** (number of physical states smaller than U(1) case)

- **Can model black hole as a single intertwiner! Vindicates Rovelli-Krasnov's arguments for computting BH entropy in full theory.**

- **BH entropy from topological limit of GR [Liu-Montesinos-Perez]**

# Future

- Put our long paper on the arxiv. (ENP).
- Extend to general IHs (avoid the fixing of the null generators).
- Treat distorted and rotating case.
- Dynamics, dynamics.