Black Hole
Entropy and SU(2) Chern Simons Theory

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Based on joint work with Jonathan Engle and Karim Noui
The plan

- Brief motivation and introduction
- The main idea in Ashtekar-Krasnov-Corichi treatment of the problem
- Isolated horizons (in the ACK formulation)
- Connection variables (the SU(2) invariant canonical formulation)
- Quantization
- Relation between the U(1) and SU(2)
- Counting
Introduction and motivation
Black Hole Thermodynamics

Some definitions

\[ \Omega \equiv \text{horizon angular velocity} \]
\[ \kappa \equiv \text{surface gravity ('grav. force' at horizon)} \]
If \( \ell^a \) = killing generator, then \( \ell^a \nabla_a \ell^b = \kappa \ell^b \).
\[ \Phi \equiv \text{electromagnetic potential}. \]

0th law: the surface gravity \( \kappa \) is constant on the horizon.

1st law:
\[ \delta M = \frac{\kappa}{8\pi} \delta A + \Omega \delta J + \Phi \delta Q \]
work terms

2nd law:
\[ \delta A \geq 0 \]

3rd law: the surface gravity value \( \kappa = 0 \) (extremal BH) cannot be reached by any physical process.
Hawking Radiation

\[ T = \frac{\kappa}{2\pi} \]

\[ S = \frac{1}{4} A \]

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Black Holes Evaporate

Questions for Quantum Gravity

\[ T = \frac{\kappa}{2\pi} \]

\[ S = \frac{1}{4} A \]

1) Microscopic origing of the entropy.
2) Dynamics of black hole evaporation.
3) What replaces the classical singularity.
4) What happens when a BH completely evaporates.
Quasilocal definition of BH
From the Characteristic formulation of GR to isolated horizons

ACK IH boundary condition

- $\Delta = S^2 \times R$
- Einstein’s equations hold on $\Delta$
- For $\ell^a$ the null normal $\rho = 0$ and $\sigma = 0$.
- For the null vector field $n^a$, such that $n \cdot \ell = 1$, $\sigma' = 0$ and $\rho' = -f(v)$
The main equations:

The equations below are valid on the appropriate thermal slicing where 0th and 1st laws hold.

\[ \Sigma^i = -\frac{a}{\pi} F^i (\Gamma) \]
\[ K^i = -\sqrt{\frac{2\pi}{a}} e^i \]
\[ A^i = \Gamma^i + \gamma K^i \]
\[ F_{ab}^i (A) = -\frac{\pi (1 - \gamma^2)}{a} \Sigma_{ab}^i \]

In the gauge normal gauge (where \( e^1 \) is normal to \( H \)) the main equation can be written in AKC form, plus two additional conditions on \( \Sigma^i \)

\[ V_\alpha = 2\Gamma^1, \quad dV = -\frac{2\pi}{a} \Sigma^1 = -\frac{2\pi}{a} \epsilon^2 \]
\[ \Sigma^2 = \Sigma^3 = 0 \]
Fixing of the null generator

In IH there is no unique notion of $\kappa$
\[ \ell^a \nabla_a \ell^b = \kappa \ell^b \]
Lorentz Transf. above
\[ \ell^a \rightarrow \tilde{\ell}^a = \exp(-(\alpha(x)))\ell^a \]
\[ n^a \rightarrow \tilde{n}^a = \exp((\alpha(x)))n^a \]

A preferred normal direction to $H$: the allowed histories must satisfy
(Ashtekar)
\[ q^{ab} r_a r_b = 1 \iff \sqrt{\delta^{ij} E^a_i E^b_j} r_a r_b = \sqrt{q} \]

Indeed this choice is the one that makes the zero, and first law of IH look just as the corresponding laws of stationary black hole mechanics [ACK, Ashtekar-Beetle-Fairhurst-Krishnan].

\[ K^i_a = -\frac{f(v)}{\sqrt{2}} e^i_a \rightarrow K^i_a = -\sqrt{\frac{2\pi}{a}} e^i_a \]
Isolated horizon dynamical system

Covariant phase space formulation

In Ashtekar variables, due to the IH boundary condition, no additional boundary term need to be included.

\[ S[e, A_+] = -\frac{i}{\kappa} \int_M \Sigma_i(e) \wedge F^i(A_+) + \frac{i}{\kappa} \int_{\tau_\infty} \Sigma_i(e) \wedge A^i_+ \]

The presymplectic current:

\[ J(\delta_1, \delta_2) = -\frac{2i}{\kappa} \delta_{[1} \Sigma_i \wedge \delta_{2]} A^i_+ \quad \forall \ \delta_1, \delta_2 \in T_p \Gamma \]

Einstein's Equations imply:

\[ dJ(\delta_1, \delta_2) = 0 \]

\[ 0 = \frac{i\kappa}{2} \int_{\partial B} J(\delta_1, \delta_2) = \int_{\Delta} \delta_{[1} \Sigma_i \wedge \delta_{2]} A^i_+ + \int_{M_1} \delta_{[1} \Sigma_i \wedge \delta_{2]} A^i_+ - \int_{M_2} \delta_{[1} \Sigma_i \wedge \delta_{2]} A^i_+ \]

\[ \frac{a_H}{4\pi} \int_{H_2} \delta_{[1} A_{+i} \wedge \delta_{2]} A^i_+ - \frac{a_H}{4\pi} \int_{H_1} \delta_{[1} A_{+i} \wedge \delta_{2]} A^i_+ \]
Isolated horizon dynamical system

Covariant phase space formulation

The conserved presymplectic form:

\[ \Omega_{M_1}(\delta_1, \delta_2) = \Omega_{M_2}(\delta_1, \delta_2) \]  \hspace{1cm}(a)

\[ i \kappa \Omega_M(\delta_1, \delta_2) = \int_M 2 \delta[1 \Sigma_i \wedge \delta_2] A^i_+ - \frac{a_H}{2\pi} \int_H \delta[1 A_{+i} \wedge \delta_2] A^i_+ \]  \hspace{1cm}(b)

Reality conditions:

\[ A^i_+ = \Gamma^i + iK^i \hspace{1cm} \implies \hspace{1cm} \text{Im}[\Omega_M(\delta_1, \delta_2)] = 0 \]  \hspace{1cm}(c)

\[ \kappa \Omega_M(\delta_1, \delta_2) = \kappa \text{Re}[\Omega_M(\delta_1, \delta_2)] = \int_M 2 \delta[1 \Sigma_i \wedge \delta_2] K^i \]  \hspace{1cm}(d)
Connection variables for GR

The Einstein Hilbert Action

\[ S[e, A] = \frac{1}{\kappa} \int_{\mathcal{M}} \sqrt{-g} \, R[g] \]

The (pre) Symplectic Form

\[ \Omega(\delta_1, \delta_2) = \frac{1}{\kappa} \int_{\mathcal{M}} (\delta_1 q_{ab} \delta_2 \Pi^{ab} - \delta_2 q_{ab} \delta_1 \Pi^{ab}) \quad \forall \, \delta_1, \delta_2 \in T(\Gamma^{IH}) \]

Introduce a triad (SU(2) gauge symm.)

\[ q_{ab} = e^a_i e^j_b \delta_{ij} \]

New Canonical Variables

\[ \Sigma^i = e^{ijk} (e_j \wedge e_k) \]
\[ K^i_a = e^b_j K_{ab} \delta^{ij} \]

New Connection Variables

\[ \Sigma^i = e^{ijk} (e_j \wedge e_k) \]
\[ A^i = \gamma K^i + \Gamma(e)^i \]

Second fundamental form

\[ \Omega(\delta_1, \delta_2) = \frac{1}{8\pi G} \int_{\mathcal{M}} \delta [\Sigma^i \wedge \delta_2] K^i = \frac{1}{8\pi G \gamma} \int_{\mathcal{M}} \delta [\Sigma^i \wedge \delta_2] A^i, \quad (a) \]

Torsion free connection

The Symplectic Potential

\[ 8\pi G \gamma \Theta(\delta) = \int_{\mathcal{M}} \Sigma_i \wedge (\gamma \delta K^i) = \int_{\mathcal{M}} \Sigma_i \wedge (\gamma \delta K^i - \delta \Gamma^i) - \int_{\mathcal{M}} \Sigma_i \wedge \delta \Gamma^i \]

\[ \Sigma_i \wedge \delta \Gamma^i = -d(e_i \wedge \delta e^i) \]
Isolated horizon dynamical system

Covariant phase space formulation

The conserved presymplectic form:

\[ \Omega_M(\delta_1, \delta_2) = \frac{1}{\kappa} \int_M [\delta_1 \Sigma^i \wedge \delta_2 K_i - \delta_2 \Sigma^i \wedge \delta_1 K_i] \quad \forall \quad \delta_1, \delta_2 \in T(\Gamma_{\text{cov}}) \]

The Symplectic Potential:

\[ 8\pi \gamma \Phi(\delta) = \int_M \Sigma_i \wedge \delta(\gamma K^i) + \int_M \Sigma_i \wedge \delta \Gamma^i - \int_M \Sigma_i \wedge \delta \Gamma^i \]

Rewriting the Symplectic Potential:

\[ 8\pi \gamma \Phi(\delta) = \int_M \Sigma_i \wedge \delta(\gamma K^i + \Gamma^i) - \int_M \Sigma_i \wedge \delta \Gamma^i \]
\[ = \int_M \Sigma_i \wedge \delta A + \int_H e_i \wedge \delta e^i \]

The presymplectic form in connection variables:

\[ \Omega_M(\delta_1, \delta_2) = \frac{1}{\kappa \gamma} \int_M [\delta_1 \Sigma^i \wedge \delta_2 A_i - \delta_2 \Sigma^i \wedge \delta_1 A_i] + \frac{1}{\kappa \gamma} \int_H \delta_1 e_i \wedge \delta_2 e^i \]

\[ \Sigma(\alpha, S) \equiv \int_{S \subset H} \text{Tr}[\alpha \Sigma] \]
\[ \{\Sigma(\alpha, S), \Sigma(\beta, S)\} = \Sigma([\alpha, \beta], S) \]
Isolated horizon dynamical system

Covariant phase space formulation

\[ S[e, A] = \frac{1}{\kappa} \int_M \sqrt{-g} \, R[g] \]

Due to the IH boundary condition no additional boundary term need to be included in the action

No boundary term here either

\[ \Omega_M(\delta_1, \delta_2) = \frac{1}{\kappa} \int_M \left[ \delta_1 \Sigma^i \wedge \delta_2 K_i - \delta_2 \Sigma^i \wedge \delta_1 K_i \right] \quad \forall \quad \delta_1, \delta_2 \in T(\Gamma_{cov}) \]

The Symplectic Potential

\[ 8\pi \gamma \Phi(\delta) = \int_M \Sigma_i \wedge \delta (\gamma K^i + \Gamma^i) - \frac{a}{2\pi(1 - \gamma^2)} \int_H A_i \wedge \delta A^i \]

\[ = \int_M \Sigma_i \wedge \delta (\gamma K^i) - \int_H e_i \wedge \delta e^i + \frac{a}{2\pi(1 - \gamma^2)} A_i \wedge \delta A^i \]

Closed one form in \( \Gamma_{cov} \)

\[ F_{ab}^i (A) = -\frac{\pi(1 - \gamma^2)}{a} \Sigma_{ab}^i \]

\[ \Omega_M(\delta_1, \delta_2) = \frac{1}{\kappa \gamma} \int_M \left[ \delta_1 \Sigma^i \wedge \delta_2 A_i - \delta_2 \Sigma^i \wedge \delta_1 A_i \right] + \frac{a}{\pi(1 - \gamma^2) \kappa \gamma} \int_H \delta_1 A_i \wedge \delta_2 A^i \]
The quantum Soup

(1) Quantize and solve the IH boundary condition

Quantum SU(2) Chern-Simons theory with defects [Witten, Smolin, Krasnov]

\[ \left[ \frac{a_H}{\pi(1-\gamma^2)} \hat{F} + \hat{\Sigma} \right] |\Psi\rangle = 0 \]

(2) Count states using Witten’s results: \( S = \log(\mathcal{N}) \) with

\( \mathcal{N} = \sum_{n; (j_1 \cdots j_n)} \dim[\mathcal{H}^{CS}(j_1 \cdots j_n)] \) [Kaul-Majumdar (first), (then)Carlip]

with labels \( j_1 \cdots j_p \) of the punctures constrained by the condition

\[ \sum_{p=1}^{n} \sqrt{j_p(j_p+1)} \leq \frac{a_H}{8\pi \beta \ell_p^2} \equiv k_0 \] (precise counting done resently)

[Agullo-Barbero-Borja-Diaz-Polo-Villasenor][Noui et al. (new ideas)]
The single intertwiner BH model

$$\text{dim}[\mathcal{H}^{CS}(j_1 \cdots j_n)] \leq \text{dim}[\text{Inv}(j_1 \otimes \cdots \otimes j_n)]$$

The area constraint

$$\sum_{p=1}^{n} \sqrt{j_p(j_p+1)} \leq \frac{a_H}{8\pi\beta\ell_p^2} \Rightarrow$$

$$\dim[\mathcal{H}^{CS}(j_1 \cdots j_n)] = \dim[\text{Inv}(j_1 \otimes \cdots \otimes j_n)]$$

We can model the IH system by a single SU(2) intertwiner [Livine-Terno, Rovelli-Krasnov]
The single intertwiner BH model

$$\dim[\mathcal{H}^{CS}(j_1 \cdots j_n)] \leq \dim[\text{Inv}(j_1 \otimes \cdots \otimes j_n)]$$

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$$\sum_{p=1}^{n} \sqrt{j_p(j_p + 1)} \leq \frac{a_H}{8\pi\beta\ell^2_p} \Rightarrow$$

$$\dim[\mathcal{H}^{CS}(j_1 \cdots j_n)] = \dim[\text{Inv}(j_1 \otimes \cdots \otimes j_n)]$$

We can model the IH system by a single $SU(2)$ intertwiner [Livine-Terno, Rovelli-Krasnov]
Relation to $U(1)$ treatment

**Domagala-Lewandowski counting**

\[
\left( j, m \right) \approx \left( j', m \right)
\]

\[
\left( \frac{k}{4\pi} \hat{F} - \hat{\Sigma}^i r_i \right) |\Psi\rangle = 0
\]

\[
\langle \Psi'|\hat{\Sigma}^i (\delta_{ij} - r_i r_j)|\Psi\rangle = 0
\]

**Ghosh-Mitra counting**

\[
\left( j, m \right) \not\approx \left( j', m \right)
\]

\[
\left( \frac{k}{4\pi} \hat{F} - \hat{\Sigma}^i r_i \right) |\Psi\rangle = 0
\]

\[
\langle \Psi'|\hat{\Sigma}^i (\delta_{ij} - r_i r_j)|\Psi\rangle = 0
\]

Density matrix constructed by tracing out "bulk" degrees of freedom where only the connection is a surface observable.

Density matrix constructed by tracing out "bulk" degrees of freedom where both connection and the area density are a surface observables.

(see Ashtekar-Engle-Van den Broeck)
The issue of fixing the null generator in the SU(2) treatment counting remains unchanged after imposing the condition in the quantum theory

**Counting:** Density matrix constructed by tracing out “bulk” degrees of freedom where both connection and the area density are a surface observables

\[
\left( \frac{k}{4\pi} \hat{F}^i + \hat{\Sigma}^i \right) \Psi \geq 0
\]

\[
\sqrt{\delta^{ij} E^a_i E^b_j r^a r^b} = \sqrt{q}
\]

\[
(\sqrt{h} - \sqrt{q})|\Psi\rangle = 0
\] (a)

Assuming there is at least one solution to equation (a) per surface state the counting is unaffected as (a) only restricts bulk d.o.f.
Conclusions

- IH’s [Ashtekar et al.] admits an $SU(2)$ invariant treatment. The d.o.f. of the horizon are described in terms of an $SU(2)$ Chern-Simons theory with level $k = \frac{a_H}{2\pi \ell_p^2 \gamma (1-\gamma^2)}$ [meets old ideas by Smolin]. (done in the ACK definition, more work needed for general IH)

- For $|\gamma| \leq \sqrt{3}$, $k_0 \leq k$: $S = \frac{\gamma_0 a_H}{4\ell_p^2 \gamma} - \frac{3}{2} \log(a_H)$ with $\gamma_0 = 0.274067...$ [Agullo-Barbero-Borja-Diaz-Polo-Villasenor, Kaul-Majumdar (for $-3/2 \log$)].

- Leading term in entropy coinsides with the $U(1)$ treatment with Ghosh-Mitra counting. (number of physical states smaller than $U(1)$ case)

- Can model black hole as a single intertwiner! Vindicates Rovelli-Krasnov’s arguments for computing BH entropy in full theory.

- BH entropy from topological limit of GR [Liu-Montesinos-Perez]
Future

• Put our long paper on the arxiv. (ENP).
• Extend to general IHs (avoid the fixing of the null generators).
• Treat distorted and rotating case.
• Dynamics, dynamics.