Coherent spin-networks

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Collaboration with E. Bianchi, E. Magliaro

- They are introduced via a heat kernel technique
- The labels are the ones used in Spin Foam semiclassical calculations
- The set of labels can be viewed as an $SL(2,\mathbb{C})$ element $\rightarrow$ coincide with Thiemann's complexifier coherent states, with a suitable choice of complexifier and of the heat-kernel time
- We study the properties of semiclassicality an find, surprisingly, that they reproduce a superposition over spins with nodes labeled by Livine-Speziale coherent intertwiners. The weight associated to spins is a Gaussian with phase, as originally proposed by Rovelli
- Good semiclassical properties: e.g. volume operator
- Coherence properties
- Holomorphic representation of Loop Quantum Gravity
- Discussion and concusions

A key ingredient in the semiclassical calculations are the semiclassical states. They are peaked on a prescribed intrinsic and extrinsic geometry of space. The original idea of Rovelli was to take a superposition over spins of spin-network states, with a simple ansatz for the weight associated to each link:

$$c_j(j_0, \xi) = \exp\Big(-\frac{(j-j_0)^2}{2\sigma_0}\Big)\exp(-i\xi j) \tag{1}$$

The spin j_0 is the classical value of the area of the surface cut by the link. The angle ξ is the variable conjugate to the spin, the 4-dimensional dihedral angle coding the extrinsic curvature.

The dispersion is chosen to be given by $\sigma_0 \approx (j_0)^k$ (with $0 < k < 2$) so that, in the large j_0 limit, both variables have vanishing relative dispersions. Those kind of states where used for the calculations with the Barrett-Crane SFM.

On the other hand, Rovelli and Speziale introduced an ansatz for the semiclassical tetrahedron (superposition over virtual spins at each node):

$$c_k(k_0, \phi) = \exp\Big(-\frac{(k-k_0)^2}{2\tau_0}\Big)\exp(-i\phi k) \tag{2}$$

The virtual spin k_0 is the classical value of the 3-dimensional dihedral angle between two faces of the tetrahedron. The phase ϕ is needed to peak on the correct value all the dihedral angles.

Livine-Speziale coherent intertwiner

The Rovelli-Speziale ansatz can be introduced through the mathematical theory of coherent states for $SU(2)$. A coherent state is defined by:

$$\vec{J} \cdot \vec{n} \, |j, \vec{n}\rangle = j \, |j, \vec{n}\rangle \tag{3}$$

There is a phase ambiguity $|j, \vec{n}\rangle \to e^{i\alpha}|j, \vec{n}\rangle$. CS (3) minimize the uncertainty

$$\langle J^2 \rangle - \langle J \rangle^2 = j \tag{4}$$

Livine-Speziale (N-valent) coherent intertwiner is

$$|j_a, \vec{n}_a\rangle = \int g \triangleright \otimes_{a=1}^{N} |j_a, \vec{n}_a\rangle \mathrm{d}g \tag{5}$$

Their components on the usual *virtual spin* basis are:

$$\Phi_i(\vec{n}_a) = v_i \cdot \Big(\otimes_{a=1}^{N} |j_a, \vec{n}_a\rangle \Big) \tag{6}$$

When $N = 4$, the coefficients (6) reproduce the Speziale-Rovelli ansatz in the large spin limit. The states (5) have good semiclassical properties, e.g.

$$\langle j_a, \vec{n}_a | V(\vec{J}) | j_a, \vec{n}_a \rangle = V(j\vec{n}) + corr. \tag{7}$$

The volume operator gives the classical volume of a tetrahedron, for $j \gg 1$.

Graviton propagator from the new Spin Foams

The new Spin Foam models give non trivial dynamics to the intertwiner d.o.f. To test their semiclassical limit, we can use a boundary state with a Gaussian weight associated to spins, and nodes labeled by Livine-Speziale intertwiners. Our candidate semiclassical state (BMP) was:

$$|\Psi\rangle = \sum_{j_{ab}, i_a} \exp\Big(- \frac{(j_{ab} - j_0)^2}{2\sigma_0} \Big) \exp(-i\xi j_{ab}) \Phi_{i_a}(\vec{n}_{ab}) |j_{ab}, i_a\rangle \tag{8}$$

Recall our definition of LQG graviton propagator. It is the connected 2-point function of electric-flux operators (indices omitted) acting at 2 different nodes a, b

$$G(a,b) = \langle E_a \cdot E_a E_b \cdot E_b \rangle - \langle E_a \cdot E_a \rangle \langle E_b \cdot E_b \rangle \tag{9}$$

With the state (8), and in the case of a single 4-simplex, we found, in the large j_0 limit with γj_0 fixed:

$$\tilde{G}(a,b) = \frac{M}{l^2} + corr. \tag{10}$$

with M the tensorial structure of the standard propagator of perturbative quantum gravity. In other words, we showed at least in this simplified context that the new SFM, together with our ansatz for the boundary semiclassical state, overcome the problem of BC SFM.

1st question: can we find a top-down derivation of our states (8) ?

On the other hand, within the canonical framework, Thiemann and collaborators have strongly advocated the use of complexifier coherent states.
When restricted to a single graph, they are labeled by an $SL(2,\mathbb{C})$ element per each link. Their peakedness properties have been studied in detail. However the geometric interpretation of the $SL(2,\mathbb{C})$ labels and the relation with the states used in Spin Foams remained unexplored.

Moreover, Thiemann and Flori concluded that:

- "coherent states whose complexifiers are squares of area operators are not an appropriate tool with which to analyze the semiclassical properties of the volume operator"
- "the expectation value of the volume operator with respect to coherent states depending on a graph with only N-valent vertices reproduces its classical value at the phase space point at which the coherent state is peaked only if N=6"

These statements contributed to increase the tension between the canonical and the covariant frameworks.

2nd question: can we understand the origin of this tension ?

1st answer = 2nd answer

Consider the heat-kernel of $L^2(\mathbb{R}^n, \mathrm{d}x)$ defined by:

$$K_t(x, x') = e^{-\frac{t}{2}\Delta_x}\delta(x, x') \tag{11}$$

The phase-space of a particle in $\mathbb{R}^n$ is $\mathbb{R}^{2n} = \mathbb{C}^n$, the complexification of the Abelian group $\mathbb{R}^n$. Consider now the unique analytical continuation of the heat-kernel w.r.t. the variable x'. We have thus defined the family of wave functions

$$\psi_z^t(x) = K_t(x, z) \qquad z \in \mathbb{C}^n \tag{12}$$

The states (12) are *coherent* in the following mathematical sense:

1. They are annihilated by the annihilation operator $\hat{z} = \hat{x} + it\hat{p}$
2. they saturate the Heisenberg uncertainty relation $\Delta x \Delta p = \frac{1}{2}$
3. they form an overcomplete basis of $L^2(\mathbb{R}^n, \mathrm{d}x)$

$$\int \overline{\psi_z^t(x)}\psi_z^t(x')\mathrm{d}^n z = \delta(x, x') \tag{13}$$

Hall's coherent states for $SU(2)$

Hall generalized the previous coherent states for $SU(2)$. These are the coherent states associated with the generalization of the Segal-Bargmann transform to Lie groups (talk about it later).

Consider the Laplace-Beltrami operator on $SU(2)$ with Riemannian structure given by the unique bi-invariant metric. Apply the heat-kernel evolution to the Dirac delta distribution over the group:

$$K_t(h, h') = e^{-t\Delta}\delta(h, h') \tag{14}$$

Take the unique analytic continuation w.r.t. the variable h'. Now we have wave-functions labeled by an element H in the complexification of $SU(2)$ which is $SL(2, \mathbb{C})$

$$\psi_H^t(h) = K_t(h, H) \qquad H \in SL(2, \mathbb{C}) \tag{15}$$

Being $SU(2)$ simply connected, $SU(2)^{\mathbb{C}}$ is defined via exponentiation of the complexification of the Lie algebra.

The wave-function (15) can be thought as an LQG loop state associated to a loop γ, the group element g being the holonomy along γ. It and has the following Peter-Weyl expansion in $SU(2)$ irreducible representations

$$K_t(h, H) = \sum_j (2j+1)e^{-j(j+1)t}\chi^{(j)}(h^{-1}H) \tag{16}$$

Consider now a general graph Γ. Coherent spin-networks are defined as follows: we consider the gauge-invariant projection of a product over links of heat kernels,

$$\Psi_{\Gamma,H_{ab}}(h_{ab}) = \int \Big(\prod_a dg_a\Big) \prod_{ab} K_{t_{ab}}(h_{ab},\, g_a\, H_{ab}\, g_b^{-1}) \tag{17}$$

The notation refers to a complete graph, just for convenience. These are the coherent states associated to the Segal-Bargmann transform for theories of connections introduced by Ashtekar, Lewandowski, Marolf, Mourao, Thiemann in 1994. We rediscovered them following a completely different path.

Our proposal consists in:

- give the $SL(2,\mathbb{C})$ labels a geometrical interpretation
- the heat-kernel time is fixed in terms of the other labels, in order to recover the semiclassical limit

The main observation is the following: every $SL(2,\mathbb{C})$ element can written in terms of

$$(\eta, \vec{n}, \vec{n}', \xi) \tag{18}$$

A positive real number η, two unit vectors $\vec{n}$, $\vec{n}'$, an angle ξ. $\vec{n}$ is the (unit-)flux of the electric field $\vec{E}$ through a surface intersected by the link, as viewed from the first node. $\vec{n}'$ the flux viewed from the second node. Finally, η is related to modulus of the electric field, namely to the area of the surface. Exactly the labels used in Spin foams!

An element $H \in SL(2,\mathbb{C})$ can be written in terms of a positive real number η and two $SU(2)$ group elements g and $\tilde{g}$:

$$H = g\, e^{\eta\, \tau_3}\, \tilde{g}^{-1} \tag{19}$$

In turn, a $SU(2)$ group element can be uniquely written in terms of an angle $\tilde{\phi}$ and a unit-vector $\vec{n}$, once we choose a phase phase convention (a section of the Hopf fiber bundle). Let us define $\vec{n}$ via its inclination and azimuth

$$\vec{n} = \big(\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta\big) \tag{20}$$

and introduce the associated group element $n \in SU(2)$ defined as

$$n = e^{-i\phi\tau_3}\, e^{-i\theta\tau_2} \tag{21}$$

Then a general $SU(2)$ group element g is given by $g = n\, e^{+i\beta\tau_3}$. Using such parametrization we finally find

$$H = n\, e^{-iz\tau_3}\, \tilde{n}^{-1} \tag{22}$$

with $z = \xi + i\eta$ and $\xi = \beta - \tilde{\beta}$. This was for a single link. When we consider the full graph, we have as a set of labels:

$$(\eta_{ab}, n_{ab}, n_{ba}, \xi_{ab}) \tag{23}$$

So we have an area and an angle labeling each link, and a set of V unit vectors labeling each node (V is the valence).

Thus what we have just found is the remarkable correspondence between the (covering of) Lorentz group and the classical geometric labels

$$H \leftrightarrow (\eta, \vec{n}, \vec{n}', \xi) \tag{24}$$

These are the labels of the states used in our ansatz for the definition of the semiclassical 2-point function. Is it only a coincidence?
On the other hand we have the recent work by Speziale and Freidel. They show that there is a map

$$H \in \mathcal{T}^* SU(2) \leftrightarrow (\eta, \vec{n}, \vec{n}', \xi) \in \mathcal{T}^* \mathbb{R} \times S^2 \times S^2 \tag{25}$$

and they prove this map is a symplectomorphism, with the natural symplectic structure on $\mathcal{T}^* SU(2)$ and a natural, in fact unique, symplectic structure on the label space. In particular, the area η and the angle ξ (which codes part of the extrinsic curvature) are canonically conjugate.

But notice that

$$\mathcal{T}^* SU(2) \simeq SL(2, \mathbb{C}) \,! \tag{26}$$

It follows that our map, and FS's one are the same. The big surprise comes in the next slide, where we compute the asymptotics of coherent spin-networks, in a particular limit.

Coherent spin-networks (4)

The coherent spin-network can be expanded on the spin-network basis

$$\Psi_{\Gamma,H_{ab}}(h_{ab}) = \sum_{j_{ab}} \sum_{i_a} f_{j_{ab},i_a} \, \Psi_{\Gamma,j_{ab},i_a}(h_{ab}) \tag{27}$$

with components

$$f_{j_{ab},i_a} = \Big(\prod_{ab} (2j_{ab}+1) e^{-j_{ab}(j_{ab}+1)t_{ab}} \Pi^{(j_{ab})}(H_{ab}) \Big) \cdot \Big(\prod_{a} v_{i_a} \Big) \tag{28}$$

We are interested in its asymptotics for $\eta_{ab} \gg 1$. The crucial observation is that in this limit, we have the following asymptotic behavior

$$\overset{j_{ab}}{\Pi}(e^{-iz_{ab}\tau_3})_{mm'} = \delta_{mm'} e^{-imz_{ab}} \sim \delta_{mm'} \, e^{+\eta_{ab}j_{ab}} \, \delta_{m,j_{ab}} e^{-i\xi_{ab}j_{ab}} \tag{29}$$

Therefore, introducing the projector $P_+ = |j_{ab}, +j_{ab}\rangle\langle j_{ab}, +j_{ab}|$ on the highest magnetic number, we can write (29) as

$$\overset{j_{ab}}{\Pi}(e^{-iz_{ab}\tau_3}) \sim e^{-i\xi_{ab}j_{ab}} e^{+\eta_{ab}j_{ab}} P_+ \tag{30}$$

The projection on the highest magnetic number is the key for the link with coherent states of $SU(2)$, hence with the Livine-Speziale intertwiners: $SU(2)$ coherent states are defined as the (rotations of the) highest magnetic number states.

Next step: notice that

$$-j(j+1)t + j\,\eta = -\big(j - \frac{\eta - t}{2t}\big)^2\,t + \frac{(\eta - t)^2}{4t} \tag{31}$$

so defining

$$(2j^0_{ab} + 1) \equiv \frac{\eta_{ab}}{t_{ab}} \quad \text{and} \qquad \sigma^0_{ab} \equiv \frac{1}{2t_{ab}} \tag{32}$$

we find the following asymptotics for coherent spin-networks:

$$f_{j_{ab},i_a} \sim \Big(\prod_{ab} \exp\big(-\frac{(j_{ab} - j^0_{ab})^2}{2\sigma^0_{ab}}\big)\, e^{-i\xi_{ab}j_{ab}}\Big)\Big(\prod_a \Phi_{i_a}(n_{ab})\Big) \tag{33}$$

These are exactly that Elena, Eugenio and I considered as boundary semiclassical states in the analysis of the graviton propagator. In particular, their intertwiners are the Livine-Speziale ones. The parameters ξ were chosen so to reproduce the extrinsic curvature, so also in this more general context ξ must be interpreted an an extrinsic angle, as originally proposed by Rovelli.

This result confirms the geometric interpretation of our variables and extends the validity of the semiclassical states used in Spin Foams well beyond the simplicial setting: coherent spin-networks are defined in full LQG.

In the large η limit, the expectation value of the area operator is easily computed

$$\langle A \rangle = \frac{(\Psi_{\gamma,\xi+i\eta},\, \hat{A}\,\Psi_{\gamma,\xi+i\eta})}{(\Psi_{\gamma,\xi+i\eta},\, \Psi_{\gamma,\xi+i\eta})} = \gamma L_P^2 \,\sqrt{j_0(j_0+1)} \tag{34}$$

and confirms the interpretation of η as the quantity that prescribes the expectation value of the area. The Wilson loop operator acts on basis vectors as

$$\hat{W}_\gamma\, \chi^{(j)}(h) = \chi^{(\frac{1}{2})}(h)\, \chi^{(j)}(h) = \chi^{(j+\frac{1}{2})}(h) + \chi^{(j-\frac{1}{2})}(h) \tag{35}$$

As a result, we find

$$\langle W_\gamma \rangle = 2\, \cos(\xi/2)\; e^{-\frac{t}{8}} \tag{36}$$

Therefore, in the limit $t \to 0$ compatible with η large, the parameter ξ can be interpreted as the conjugacy class of the group element h_0 where the Ashtekar-Barbero connection is peaked on. Similarly

$$\Delta A \equiv \sqrt{\langle A^2 \rangle - \langle A \rangle^2} = \frac{1}{2}\gamma L_P^2\, \sqrt{2\sigma_0} \tag{37}$$

$$\Delta W_\gamma \equiv \sqrt{\langle W_\gamma^2 \rangle - \langle W_\gamma \rangle^2} = \sin(\xi/2)\, \frac{1}{\sqrt{2\sigma_0}} \tag{38}$$

If we require that the relative dispersions vanish in the large j_0 limit, this fixes the scaling

$$t \sim (j_0)^k \qquad 0 < k < 1 \tag{39}$$

Coherence properties

Coherent spin-networks satisfy the following coherence properties

1. Are eigenstates of the annihilation operator associated to a link e

$$\hat{H}_e = e^{-t\hat{A}_e^2}\,\hat{h}_e\,e^{t\hat{A}_e^2} \tag{40}$$

2. Saturate the associated Heisenberg relations
3. Form an overcomplete basis

Point 3 is very important, because it means that every LQG state can be expressed as a superposition of states with semiclassical labels. Of course, coherent spin-networks do not have in general a semiclassical interpretation unless some constraints on their labels are satisfied (i.e. closure condition at each node, and gluing constraints for a "Regge" interpretation).

The resolution of identity (we give it for a loop state but is general) is

$$\int \overline{\Psi_t^H(h)}\Psi_t^H(h')\mathrm{d}\nu_t(H) = \delta(h,h') \tag{41}$$

The measure $\mathrm{d}\nu_t$ is related to the Haar measure $\mathrm{d}H$ on $SL(2,\mathbb{C})$

$$d\nu_t = \Omega_{2t}(H)\mathrm{d}H \tag{42}$$

and Ω_t is the $SU(2)$-averaged heat kernel of $SL(2,\mathbb{C})$ (not the an. cont. of the $SU(2)$ one).

Holomorphic representation for LQG

Coherent s.n.'s lead naturally to an holomorphic representation for LQG. Consider a single copy of $SU(2)$, to simplify the notation. The scalar product in $\mathcal{H} = L^2(SU(2), \mathrm{d}g)$ defines a correspondence between a state $\Psi \in \mathcal{H}$ and a holomorphic function Ξ, defined

$$\Xi : H \longmapsto \langle \Psi_t^H, \Psi \rangle \tag{43}$$

There is more. The correspondence is a unitary map (isometric, onto)

$$L^2(SU(2), \mathrm{d}g) \longmapsto \mathscr{H} L^2(SL(2, \mathbb{C}), \Omega_t \mathrm{d}H) \tag{44}$$

To be more explicit, the $SU(2)$-averaged kernel is

$$\Omega_t(H) = \int_{SU(2)} F_t(Hg)\, dg \tag{45}$$

where F_t is the heat kernel over $SL(2, \mathbb{C})$. Ω_t can be viewed as the heat kernel on $SL(2, \mathbb{C})/SU(2)$.

What is now available is a representation (Ashtekar et al. 1994) for Loop Quantum Gravity where states are holomorphic functions of classical variables H_{ab} that admit a clear geometric interpretation in terms of areas, extrinsic angles and normals,

$$(\eta_{ab}, n_{ab}, n_{ba}, \xi_{ab}) \tag{46}$$

the variables generally used in the Spin Foam setting.

Discussion

- We have discussed a proposal of coherent states for Loop Quantum Gravity and shown that, in a specific limit, they reproduce the states used in the Spin Foam framework.
- These states coincide with Thiemann's complexifier coherent states with the natural choice of complexifier operator, a rather specific choice of heat-kernel time and a clear geometrical interpretation for their $SL(2,\mathbb{C})$ labels.
- The negative conclusions of Thiemann and Flori can be probably traced back to the fact they did not take the same semiclassical limit we considered here.
- It is possible that coherent spin-networks can be obtained via geometric quantization (Freidel-Speziale, to appear), and that the latter coincide with a subset of the states discussed here via heat kernel methods. This would be an instance of Guillemin-Sternberg's 'quantization commutes with reduction'.

Conclusion

- Coherent spin-networks are candidate semiclassical states for full Loop Quantum Gravity. Given a space-time metric (e.g. Minkowski or Schwarzschild), we can smear the Ashtekar-Barbero connection on links of the graph and the electric field on surfaces dual to links. This finite amount of data that can be used as labels for the coherent state.
- The fact that in the large spin limit they are 'effectively' labeled by Livine-Speziale coherent intertwiners guaranties that they are actually peaked on a classical expectation value of non-commuting geometric operators, e.g. the volume operator.
- A surprising property of the states we have discussed is that they bring together so many (apparently conflicting) ideas that have been proposed in the search for semiclassical states in Loop Quantum Gravity. We consider this convergence to be a measure of the robustness of the theory.

Thanks !