The Feynman diagramatics for the spin foam models

Based on *arXiv:1107.5185*

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Motivation

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Our goal was to

1. Introduce a framework in which one can consider spin foams without need to use $4d$ imagination.
2. Find a class of 2-complexes that are best for covariant Loop Quantum Gravity

While doing this we have also

- Introduced a useful representation of the vertex amplitudes as *contractors*
- Found all spin foams with one vertex and with boundary given by Rovelli's Dipole Cosmology graph.

Preliminaries: the Hilbert spaces

In order to define the contractor we need to fix some notions:

In case of graphs

- Link Hilbert space: $\mathcal{H}_\ell := \mathcal{H}_{\rho_\ell}$
- Node Hilbert space:

$$\mathcal{H}_n := \mathrm{Inv}\left(\bigotimes_{\ell_i\,:\,n=s(\ell_i)} \mathcal{H}_{\ell_i} \otimes \bigotimes_{\ell_j\,:\,n=t(\ell_j)} \mathcal{H}^*_{\ell_j}\right) \subset \bigotimes_{\ell_i\,:\,n=s(\ell_i)} \mathcal{H}_{\ell_i} \otimes \bigotimes_{\ell_j\,:\,n=t(\ell_j)} \mathcal{H}^*_{\ell_j}$$

- Graph Hilbert space $\mathcal{H}_\Gamma := \bigotimes_{n\in\Gamma} \mathcal{H}_n$

In case of 2-complexes

- Face Hilbert space $\mathcal{H}_f := \mathcal{H}_{\rho_f}$
- Edge Hilbert space $\mathcal{H}_e := \mathcal{H}_{e,v_1} \otimes \mathcal{H}_{e,v_2}$, where

$$\mathcal{H}_{e,v} := \mathrm{Inv}\left(\bigotimes_{f_i \text{ in}} \mathcal{H}_{f_i} \otimes \bigotimes_{f_j \text{ out}} \mathcal{H}^*_{f_j}\right) \subset \bigotimes_{f_i \text{ in}} \mathcal{H}_{f_i} \otimes \bigotimes_{f_j \text{ out}} \mathcal{H}^*_{f_j}$$

- Vertex Hilbert space $\mathcal{H}_v := \bigotimes_{e \text{ at } v} \mathcal{H}_{e,v}$

The definition of a contractor

Definition

A contractor A_v of a vertex v is a vector in $\mathcal{H}_v^*$.

Since structure of vertex v may be encoded in a graph Γ_v (see [Kamiński,Kisielowski,Lewandowski, arxiv:0909.0939]), we will equivalently say about contractors of a graph.

Examples of contractors

1. *BF*-contractor: *Evaluate the spin network at identity* - i.e. for each link ℓ (face f) put ${\delta^{(\rho_\ell)}}^A{}_B$ and contract with appropriate indexes of intertwiners.
2. Euclidean EPRL: *inject $SU(2)$ spin network into space of gauge invariant $SO(4)$ spin networks and then use BF contractor* - i.e. for each link ℓ instead of ${\delta^{(\rho_\ell)}}^A{}_B = {\delta^{(k_\ell)}}^A{}_B$ use

$$\left(Y_{(k_\ell;\, j^+_\ell, j^-_\ell)}\right)^A{}_{C^+C^-} \; {\delta^{(j^+_\ell, j^-_\ell)}}^{C^+C^-}{}_{D^+D^-} \left(Y^\dagger_{(k_\ell;\, j^+_\ell, j^-_\ell)}\right)^{D^+D^-}{}_B$$

3. Lorentzian EPRL: *inject $SU(2)$ spin network into space of $SL(2,\mathbb{C})$ spin networks and then use BF contractor* - , i.e. instead of $Y_{(k_\ell;\, j^+_\ell, j^-_\ell)}$ use $Y_{(k_\ell \to (p(k_\ell), k_\ell))}$

Operator Spin Foams and Spin Foam Operators

Using the contractors we define an Operator Spin Foam as a colored 2-complex: $\mathcal{F} = (\kappa;\ \rho, P, A)$, where

- ρ colors faces with representations: $f \to \rho_f$
- P colors edges with operators: $e \to P_e \in \mathcal{H}_e$
- A colors vertices with contractors: $v \to A_v \in \mathcal{H}_v^*$

Having an Operator Spin Foam we define its Spin Foam Operator as

$$\mathbb{P}_{\mathcal{F}} := (\bigotimes_v A_v) \lrcorner (\bigotimes_e P_e) \cdot \prod_{f\in\kappa} A_f \cdot \prod_{e\in\partial\kappa} A_e$$

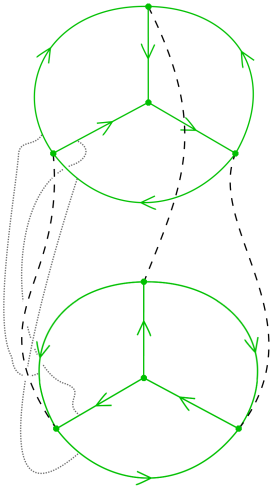
where A_f and A_e are respectively the face amplitude and the boundary edge amplitude.

Let $\Gamma_{\partial\kappa}$ be the boundary graph of the spin foam, together with the induced coloring. The Spin Foam Operator $\mathbb{P}_{\mathcal{F}}$ is an element of dual to the boundary graph Hilbert space:

$$\mathbb{P}_{\mathcal{F}} \in \mathcal{H}^*_{\Gamma_{\partial\kappa}}$$

Graph diagram - an example

Before the strict definition lets see an example



Graph diagram

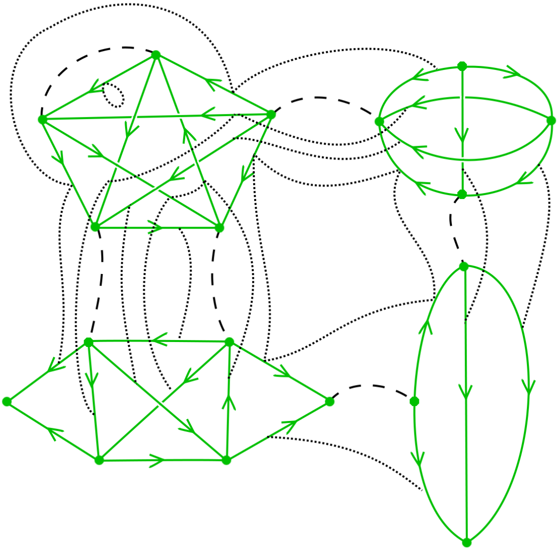
Definition

A graph diagram is given by:

- A set of graphs $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_N\}$ (we assume they are closed and connected)
- A family of relations $\mathcal{R}$ defined as follows:
  - $\mathcal{R}_{\mathrm{node}}$ - a symmetric relation in the set of nodes of the graphs which we call the node relation, such that each node n either is in relation with precisely one $n' \neq n$ or it is unrelated (and then it is called boundary node).
    Two related nodes must have the same valency.
  - $\mathcal{R}_{\mathrm{link}}$ - a family of symmetric relations in the set of links of the graphs which we call collectively the link relation. If a node n of a graph Γ_I is in relation with a node n' of a graph $\Gamma_{I'}$, then one defines a bijective map between incoming / outgoing links of Γ_I at n, with outgoing / incoming links of $\Gamma_{I'}$ at n'; two links identified with each other by the bijection are called to be in the relation $\mathcal{R}_{\mathrm{link}}^{(n,n')}$ at the pair of nodes (n, n')
    A link of Γ_I / $\Gamma_{I'}$ which intersects n / n' twice, emerges in the relation twice: once as incoming and once as outgoing link.

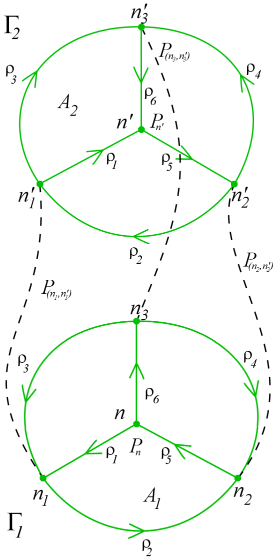
Graph diagram - another example

Another example of graph diagram:



Operator Spin Network Diagram - an example

Again before the strict definition it is good to see an example. In fact OSN-diagram is a colored graph diagram:



Operator Spin Network Diagram

Definition

An Operator Spin Network Diagram is

- A graph diagram $\mathcal{D}$
- A coloring (ρ, P, A) of elements of $\mathcal{D}$
  - ρ labels *links* ℓ of the graphs with representations ρ_ℓ
  - P labels *nodes* n of the graphs with operators $P_n \in \mathcal{H}_n \otimes \mathcal{H}_n^*$
  - A labels *graphs* Γ with contractors $A_\Gamma \in \mathcal{H}_\Gamma^* = \left(\bigotimes_{n\in\gamma} \mathcal{H}_n\right)^*$

The coloring is consistent with the relations $\mathcal{R}$, i.e.

- l and l' are in relation $\mathcal{R}_{\mathrm{link}}^{n,n'}$ at some pair of nodes $\Rightarrow \rho_l = \rho_{\ell'}$
- n and n' are in relation $\mathcal{R}_{\mathrm{node}} \Rightarrow P_n = P_{n'}$

The Edge-Relation

Despite all the information needed is encoded in $\mathcal{R}_{\mathrm{node}}$ and $\mathcal{R}_{\mathrm{link}}^{(n,n')}$s, it is convenient to introduce so called *edge-* and *face-* relations. These new relations are called so, because they will be used to reconstruct edges and faces of the 2-complex corresponding to the graph diagram.

The edge relation $\mathcal{R}_{\mathrm{edge}}$ is defined as the extension of $\mathcal{R}_{node}$ to the equivalence relation, i.e.
(n and n' are in $\mathcal{R}_{\mathrm{edge}}$) *if and only if* ($n = n'$) *or* ($n$ and n' are in $\mathcal{R}_{\mathrm{node}}$).
As a result there are two types of equivalence classes of $\mathcal{R}_{\mathrm{edge}}$:

- one element equivalence classes $\{n\}$ for n being unrelated with anything by $\mathcal{R}_{\mathrm{node}}$
- two element equivalence classes $\{n, n'\}$ for n being related with n' by $\mathcal{R}_{\mathrm{node}}$

The Face-Relation

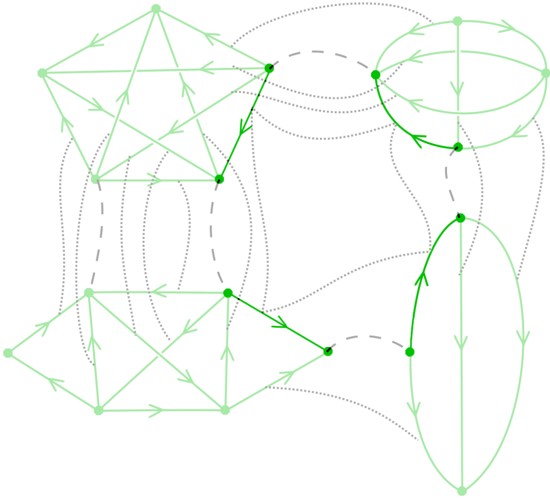
The face relation $\mathcal{R}_{\mathrm{face}}$ is fusion of all $\mathcal{R}_{\mathrm{link}}^{(n,n')}$ relation extended to equivalence relation.
To be specific it is defined as the minimal equivalence relation with the property that whenever ℓ and ℓ' are in $\mathcal{R}_{\mathrm{link}}^{(n,n')}$ relation at some pair of nodes, they are also in $\mathcal{R}_{\mathrm{face}}$.

There are two main types of equivalence classes of $\mathcal{R}_{\mathrm{face}}$

- An equivalence class is called *closed* iff one may form a series of its elements, such that: ℓ_1 and ℓ_2 are in $\mathcal{R}_{\mathrm{link}}^{(n_1,n_1')}$, ℓ_2 and ℓ_3 are in $\mathcal{R}_{\mathrm{link}}^{(n_2,n_2')}$, etc, and ℓ_k and ℓ_1 are in $\mathcal{R}_{\mathrm{link}}^{(n_k,n_k')}$.
- An equivalence class is called *open* iff one may form a series of its elements, such that: ℓ_1 and ℓ_2 are in $\mathcal{R}_{\mathrm{link}}^{(n_1,n_1')}$, ℓ_2 and ℓ_3 are in $\mathcal{R}_{\mathrm{link}}^{(n_2,n_2')}$, etc, *but there is no such* (n_k, n_k') *that* ℓ_k and ℓ_1 are in $\mathcal{R}_{\mathrm{link}}^{(n_k,n_k')}$.

Example of the face relation

An example of equivalence class of the face relation



Motivation

What is an OSN-diagram?

Contractors

OSN-diagram - the definition

OSN-diagram's elements

Properties: reading an OSN-diagram

The Spin Foam encoded in OSN-diagram

The boundary spin network

Examples: writing OSN-diagrams

A trivial example

One-vertex-interaction

OSN-diagrams in use: Finding all spin foam applicable to Dipole Cosmology

Summary & conclusions

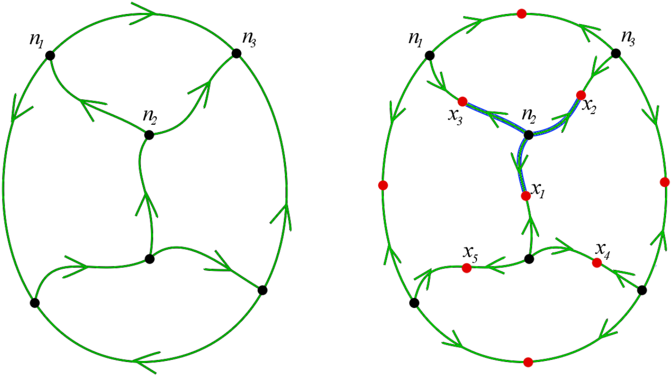
From a graph diagram to 2-complex I: Turn your graphs into squid-graphs

Now we shell see how to construct a 2-complex from an arbitrary graph diagram.
We do it in four steps

From a graph diagram to 2-complex I: Turn your graphs into squid-graphs

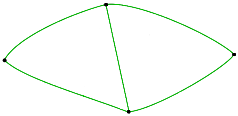
Now we shell see how to construct a 2-complex from an arbitrary graph diagram.
We do it in four steps

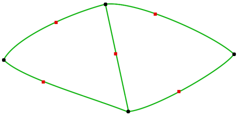
1. Firstly introduce some auxiliary nodes at each graph $\Gamma \in \mathcal{G}$ turning it into so called *squid-graphs*.

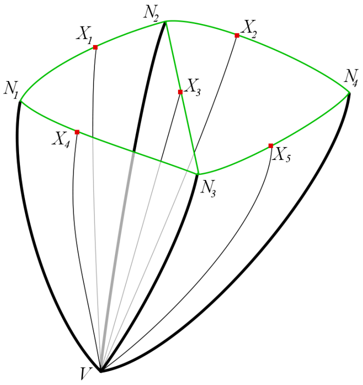


From a graph diagram to 2-complex II: Little balls around each vertex

2. Then shrink each suid graph to a point to obtain a piece of 2-complex corresponding to the little ball around a vertex. (compare [Kamiński,Kisielowski,Lewandowski, arxiv:0909.0939])

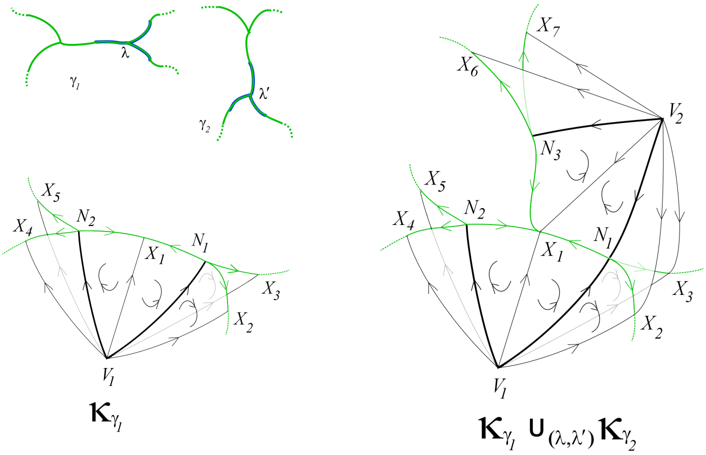






From a graph diagram to 2-complex III: Gluing

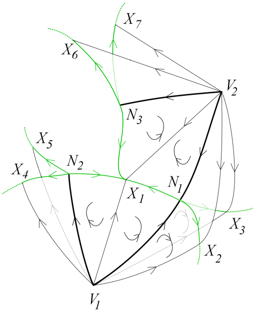
3. For each pair of nodes n, n' being in relation $\mathcal{R}_{\text{node}}$ we glue the squids correspinding to the nodes.

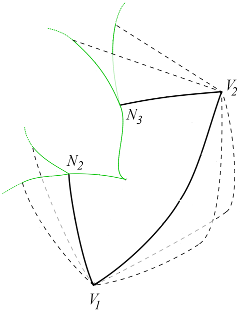


From a graph diagram to 2-complex IV: Erase what's auxiliary and spell out the coloring

4. And finally we erase the auxiliary nodes and all the traces they have left, i.e.:

- The VX edges
- The internal NX edges
- The X vertices
- The internal N vertices





From a graph diagram to 2-complex: summary

We have obtained the 2-complex such that

- For each equivalence class of the face relation $\mathcal{R}_{\text{face}}$ there is a face. If the equivalence class is *open*, the face has a boundary edge.
- For each equivalence class of the edge relation $\mathcal{R}_{\text{edge}}$ there is an internal edge. If the equivalence class is one-element, one of the edge's ends is a boundary node.
- For each graph of the graph diagram there is an internal vertex.

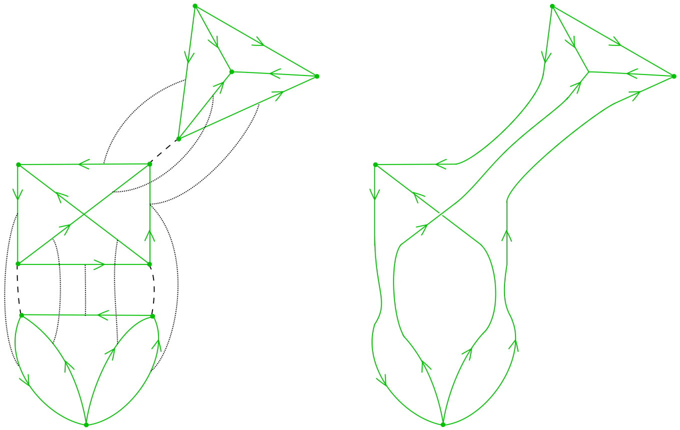
The coloring is straightforward.

Claim

We claim that the class of all 2-complexes that may be constructed in that way is the right class to define the spin foam models.

The boundary graph of a graph diagram

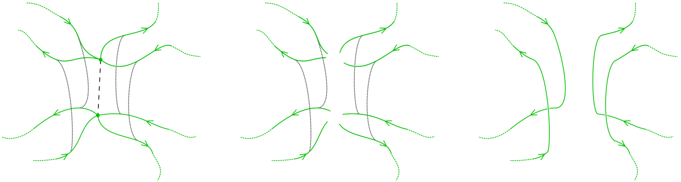
One of features of graph-diagram framework is an easy algorithm to find the boundary graph of the spinfoam corresponding to the diagram:



The boundary graph of a graph diagram: the algorithm

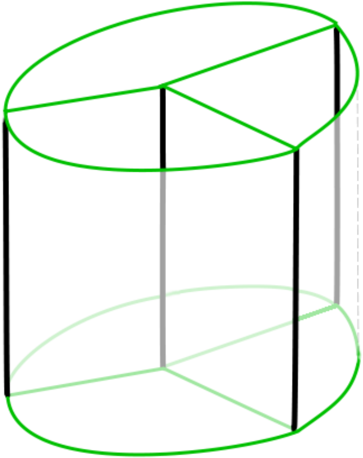
The algorithm is a simple reflection of the gluing procedure:

1. Turn all the graphs into the squid-graphs
2. Remove all the nodes related with another nodes via $\mathcal{R}_{\mathrm{node}}$ relation, together with the squids they belong to
3. Glue the remaining halfs of the links according to the $\mathcal{R}_{\mathrm{link}}$ relation
4. Remove the extra nodes i.e. come back to unsquided graph.



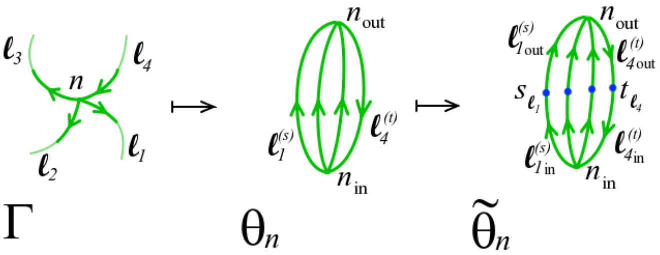
A trivial example

As a very first example we are constructing a trivial diagram i.e. the diagram representing a spin foam that does not change the spin network:



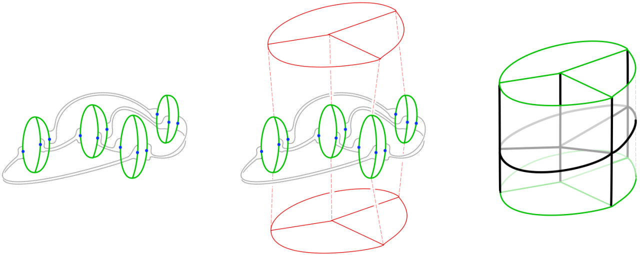
Construction

To construct such a diagram we firstly introduce a θ-like graph for each node of the underlying spin network:



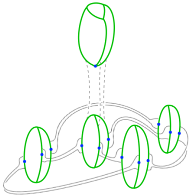
Construction

Then we define the link relation in a way compatible with the underlying graph. The result can be seen below:



A simple extension

Consider an OSN-diagram from previous example extended by one additional graph:



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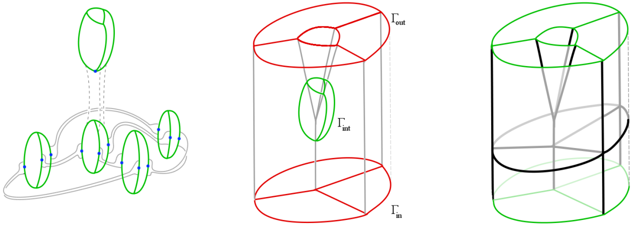
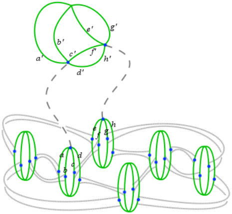


Resulting spin foam has precisely one *nontrivial* internal vertex

The power of the framework

Consider a more complicated example:



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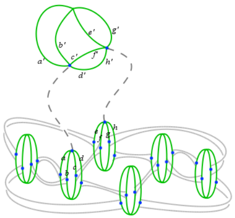
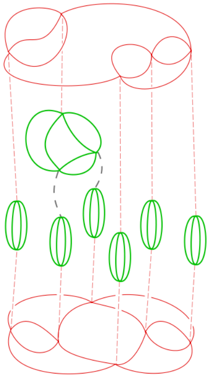




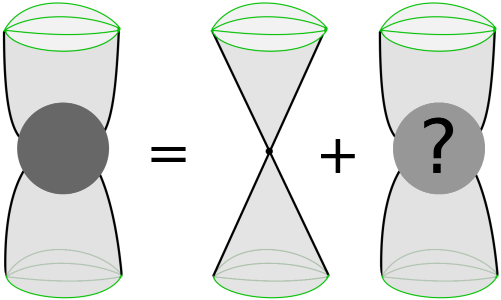
The power of our framework is that

- one can easily find the space of boundary states
- one can easily write down the amplitude

without need to imagine the underlying 2-complex.

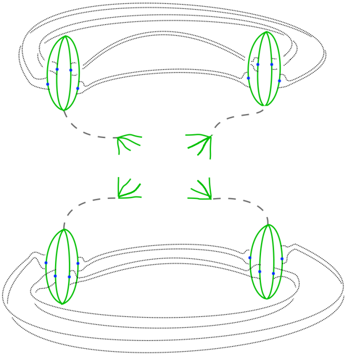
OSN-diagrams in use: Finding all spin foam applicable to Dipole Cosmology

Now we use OSN-diagrams framework to resolve the following problem:
We want to find spin foams with precisely one interaction vertex giving the boundary of the Dipole Cosmology spin foam:



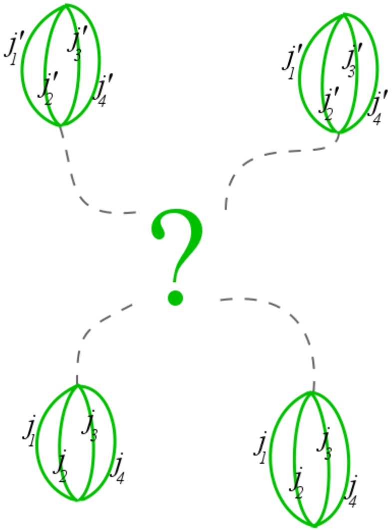
Tactics

1. We write the θ-like graphs for the boundary graph
2. Since the boundary is fixed, only 4 nodes are unrelated, all other nodes are related with something
3. Since there is one interaction vertex, there must be one more graph in the diagram. The interaction graph must have following properties:
   - It must have 4 nodes, each one being 4-valent.
   - If it has any extra nodes, they must be related with each other within the graph.



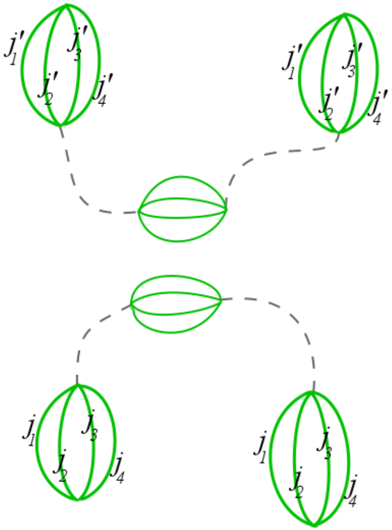
Result

First consider the graphs without any extra nodes



Result

The result is that each graph obtained from four 4-valent squids giving their hands to each other is a solution.



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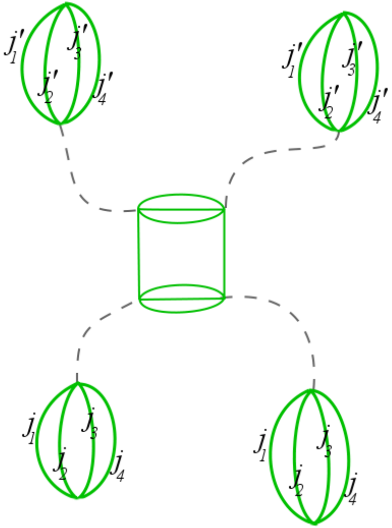

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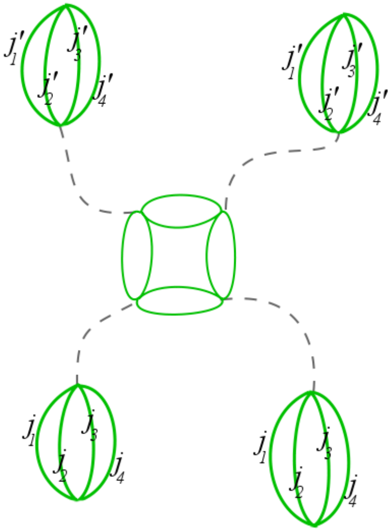

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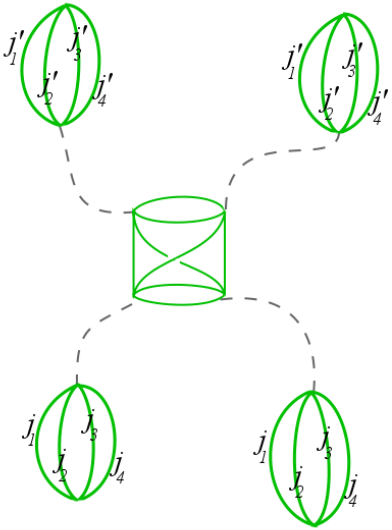

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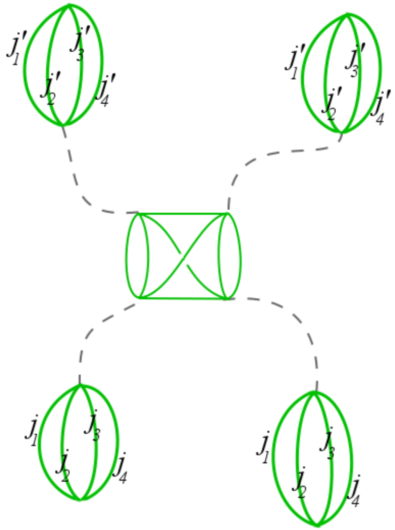

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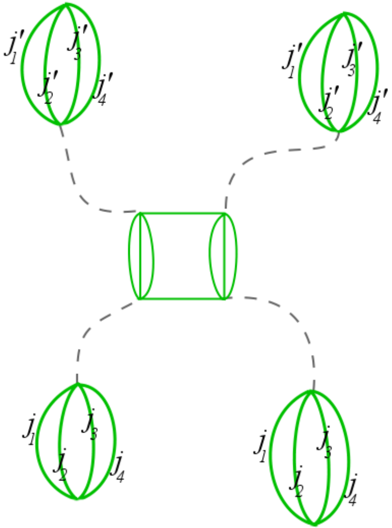

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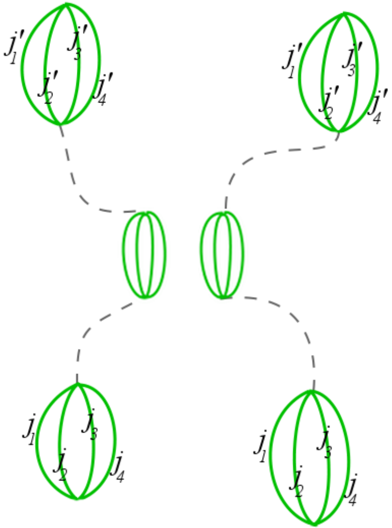

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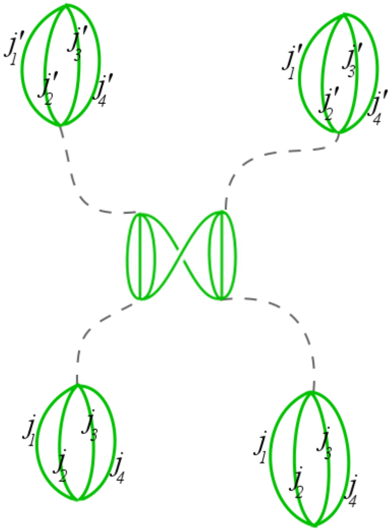

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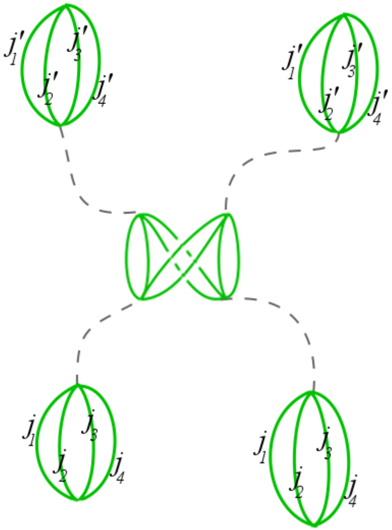

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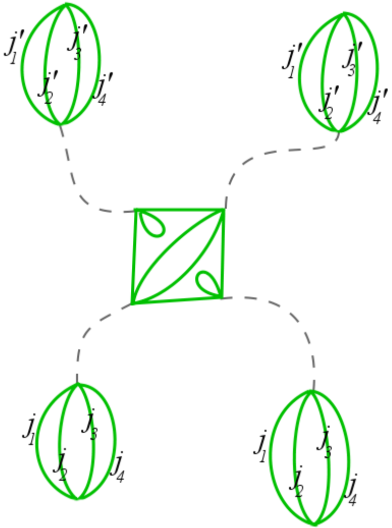

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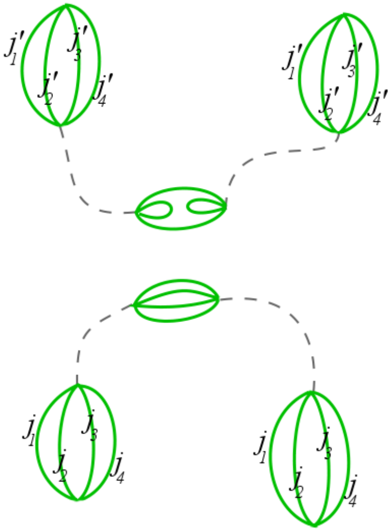

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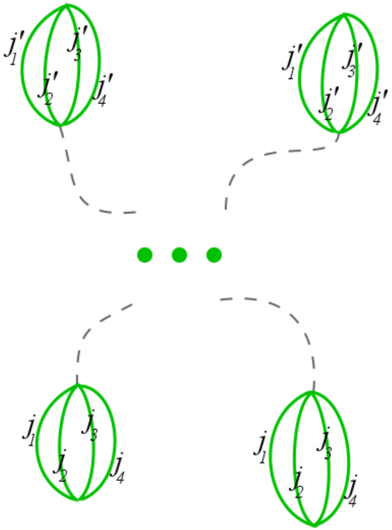
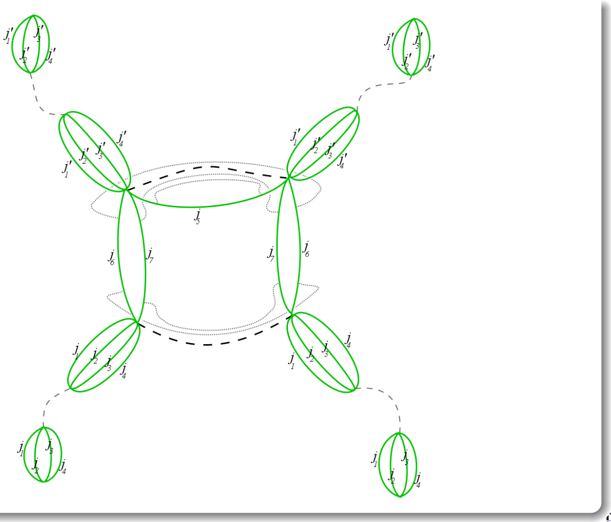


More complicated solutions

If allowed more nodes, all the extra nodes must be related and some internal js may appear (i.e. js that are not determined by the boundary):



General solution

The method presented above is not limited to the Dipole Cosmology example.
Given *any* boundary graph we can construct *all* OSN-diagrams corresponding to it

1. write down the thetas $\tilde{\theta}$ that reconstruct the boundary graph
2. write down the squids for each node of the boundary graph, and then
3. firstly consider all the ways the squids can give their hands to each other,
4. later consider all the ways the squids can be connected, also through other (arbitrarily complex) graphs, but all the added nodes must be related within themselves

Summary

- A notion of contractors being generalization of vertex amplitudes.
- A nice diagrammatic framework which allows to encode all the spin foam information in graphs equipped with some relations.
- A proposal for class of 2-complexes appropriate for spin foams
- A characterization of 1-vertex spin foams with boundary fixed to be the dipole graph

Further directions

- Explicit definition of common spin foam models in terms of Operator Spin Network Diagrams
  i.e. strict formulas for projectors and contractors for Euclidean and Lorentzian EPRL, ...
- Characterization of 2-complexes not belonging to the proposed class
- Imposing of cylindrical equivalence relations at the level of Operator Spin Network Diagrams
- Relation with GFT

- Calculation of amplitudes of the new dipole foams
- Issues of renormalization in case of internal js in Dipole Cosmology

Thank you for your attention!

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