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Backreaction of Hawking radiation in a quantum geometry

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Plan:

-In previous work we were able to quantize spherically symmetric vacuum gravity. The Hilbert space of states was found in closed form.

R. Gambini, JP, arXiv:1302.5265, R.Gambini, J. Olmedo, JP arXiv:1310.5996

-Later we considered a scalar field living on the quantum geometry and performed a Fock-like quantization for it. Hawking radiation with small corrections was found.

R. Gambini, JP, arXiv:1312.3595

-Today we would like to study the back reaction of the field on the geometry, assuming the formeris weak. For this we will compute the second order corrections to the metric as Dirac observables living on the Hilbert space that is a cross product of that of vacuum spherical gravity with the Fock-like Hilbert space of the scalar field. We will evaluate their expectation value.

Deriving the perturbative equations:

We follow standard perturbative quantization techniques. We assume we have a one parameter family of fields and that the field equations,

$$\frac{\delta S\left(\varphi^{A}\left(\epsilon\right)\right)}{\delta\varphi^{A}\left(\epsilon\right)}=0.$$

can be expanded in power series,

$$\left.\frac{\delta^{2} S\left(\varphi^{A}\left(\epsilon\right)\right)}{\delta\varphi^{A}\left(\epsilon\right)\delta\varphi^{B}\left(\epsilon\right)}\varphi^{B}\left(\epsilon\right)'\right|_{\epsilon=0}=0.$$

$$\left.\frac{\delta^{3} S\left(\varphi^{A}\left(\epsilon\right)\right)}{\delta\varphi^{A}\left(\epsilon\right)\delta\varphi^{B}\left(\epsilon\right)\delta\varphi^{C}\left(\epsilon\right)}\varphi^{B}\left(\epsilon\right)'\varphi^{C}\left(\epsilon\right)'\right|_{\epsilon=0}+\left.\frac{\delta^{2} S\left(\varphi^{A}\left(\epsilon\right)\right)}{\delta\varphi^{A}\left(\epsilon\right)\delta\varphi^{B}\left(\epsilon\right)}\varphi^{B}\left(\epsilon\right)''\right|_{\epsilon=0}=0.$$

and so on.

The action for the model:

We start from spherically symmetric gravity coupled to a scalar field (see our previous papers), the total Hamiltonian is,

$$H = \frac{N}{G}\left(-\frac{(1-\Lambda)E^\varphi}{2\sqrt{E^x}} - 2K_\varphi\sqrt{E^x}K_x - \frac{E^\varphi K_\varphi^2}{2\sqrt{E^x}} + \frac{\left((E^x)'\right)^2}{8\sqrt{E^x}E^\varphi} - \frac{\sqrt{E^x}(E^x)'(E^\varphi)'}{2(E^\varphi)^2} + \frac{\sqrt{E^x}(E^x)''}{2E^\varphi} + \frac{P_\phi^2}{4\sqrt{E^x}E^\varphi} + \frac{(E^x)^{3/2}(\phi')^2}{E^\varphi}\right) + N^r\left(\frac{-K_x(E^x)' + E^\varphi K_\varphi'}{G} + P_\phi\phi'\right).$$

Where:

$$g_{xx} = \frac{(E^\varphi)^2}{|E^x|}, \qquad g_{\theta\theta} = |E^x|,$$

$$K_{xx} = -\text{sign}(E^x)\frac{(E^\varphi)^2}{\sqrt{|E^x|}}\,\mathrm{K_x} \qquad K_{\theta\theta} = -\sqrt{|E^x|}\frac{A_\varphi}{2\gamma},$$

Redefining the lapse and shift leads to a Hamiltonian constraint with an Abelian algebra with itself.

$$N^r = N^r_{\text{old}} - \frac{2N_{\text{old}}K_\varphi\sqrt{E^x}}{(E^x)'},$$

$$N = N_{\text{old}}\frac{(E^x)'}{E^\varphi},$$

The resulting action is:

$$
\begin{aligned}
S = & \int dx \left[\frac{E^x(\epsilon)\dot{K}_x(\epsilon)}{G} + \frac{E^\varphi(\epsilon)\dot{K}_\varphi(\epsilon)}{G} + P_\phi(\epsilon)\dot{\phi}(\epsilon) \right. \\
& -\frac{1}{G}(\partial_x N(\epsilon)) \left[\frac{(\partial_x E^x(\epsilon))^2 \sqrt{E^x(\epsilon)}}{4(E^\varphi(\epsilon))^2} - \sqrt{E^x(\epsilon)} - K_\varphi^2(\epsilon)\sqrt{E^x(\epsilon)} + \sqrt{E^x(\epsilon)}\Lambda + 2GM \right] \\
& -N(\epsilon) \left[\frac{P_\phi^2(\epsilon)\partial_x E^x(\epsilon)}{4(E^\varphi(\epsilon))^2\sqrt{E^x(\epsilon)}} + \frac{(\partial_x E^x(\epsilon))(E^x(\epsilon))^{3/2}(\partial_x\phi(\epsilon))^2}{(E^\varphi(\epsilon))^2} - \frac{2K_\varphi(\epsilon)\sqrt{E^x(\epsilon)}P_\phi(\epsilon)(\partial_x\phi(\epsilon))}{(E^\varphi(\epsilon))^2} \right] \\
& \left. +N^r(\epsilon) \left[P_\phi(\epsilon)\partial_x\phi(\epsilon) + \frac{E^\varphi(\epsilon)\partial_x K_\varphi(\epsilon) - K_x(\epsilon)\partial_x E^x(\epsilon)}{G} \right] \right]
\end{aligned}
$$

We have added a 2D cosmological constant. We will need it when we renormalize the contribution for the scalar field. Since we will concentrate on the s-mode of the scalar field, its vacuum contribution does not generate a 4D cosmological constant term proportional to the spatial volume, but a 2D one.

We will choose a gauge where E^x is a function of x only and where K_φ vanishes (manifestly static gauge). We will also assume $\phi=\varepsilon\,\phi_{(1)}$, the scalar field is "weak" (it does not influence the metric at order zero nor one since the stress tensor is quadratic).

We also assume the initial data for the metric is zero at first order. Since the first order equation does not couple to the matter field, that means the first order perturbations of the metric vanish.

With the gauge fixing chosen the shift vanishes at all orders.

Zeroth order equations:

They imply that the zeroth order component of the lapse is a constant, we take it to be one. They also imply that E^φ is

And we recognize in it the Schwarzschild solution (with a cosmological constant from 2D)

$$E^{\varphi}_{(0)} = \frac{\partial_x E^x_0}{2\sqrt{1+\left(K^{(0)}_\varphi\right)^2 - \frac{2GM}{\sqrt{E^x_0}} - \Lambda}}.$$

First order equations:

As mentioned, the first order equations for the field, if one gives vanishing initial data just give zero contributions for the gravitational perturbations. For the fields the first order equations are,

$$\dot{\phi}_{(1)} - \frac{\partial_x \left(E_0^x\right) P_\phi^{(1)}}{4\left(E^\varphi_{(0)}\right)^2 \sqrt{E_0^x}} + \frac{K_\varphi^{(0)} \sqrt{E_0^x}\partial_x \phi_{(1)}}{E^\varphi_{(0)}} = 0$$

$$\dot{P}_\phi^{(1)} + \partial_x \left[\frac{\left(\partial_x E_0^x\right)\left(E_0^x\right)^{3/2}\partial_x \phi_{(1)}}{\left(E^\varphi_{(0)}\right)^2} - \frac{K_\varphi^{(0)}\sqrt{E_0^x} P_\phi^{(1)}}{E^\varphi_{(0)}}\right] = 0.$$

Which can be combined into the usual Zerilli-like second order (in the spatial and time derivatives) equation for the scalar field. This equation will provide the modes that are used to Fock quantize the field.

Second order equations:

The equations for the scalar field and its momentum take the same form as the first order ones.

The second order shift is set to zero by our gauge choice.

The are two non-trivial equations at second order, one determines $E^{\phi}_{(2)}$ as a quadrature of the field variables.

$$E^{\varphi}_{(2)} = -\frac{4\left(E^{\varphi}_{(0)}\right)^3 K_{\varphi}^{(0)} K_{\varphi}^{(2)}}{\left((E^x_0)'\right)^2} + \frac{2\left(E^{\varphi}_{(0)}\right)^3 \Lambda}{\left((E^x_0)'\right)^2}$$
$$+\frac{G\left(E^{\varphi}_{(0)}\right)^3}{\left((E^x_0)'\right)^2 \sqrt{E^x_0}} \int_x^{\infty} \left[\frac{-4\,(E^x_0)'\,(E^x_0)^2\,(\phi')^2 + 8K_{\varphi}^{(0)} E^x_0 P_{\phi}^{(1)}\left(\phi_{(1)}\right)' E^{\varphi}_{(0)} - (E^x_0)'\left(P_{\phi}^{(1)}\right)^2}{\left(E^{\varphi}_{(0)}\right)^2 \sqrt{E^x_0}}\right].$$

It should be emphasized that every quantity appearing in the quadrature are either well defined operators on the Fock space of the field or can be written as parameterized Dirac observables on the Hilbert space of zeroth order vacuum gravity. So we can straighforwardly “put a hat” on it.

The other equation determines the correction to the lapse, again as a quadrature and with all the quantities in the integrand either c-numbers of well defined parameterized Dirac observables on the physical Hilbert space of vacuum gravity, or on the Fock space of the scalar field,

$$N_{(2)} = \int_x^\infty dx \left(\frac{2\dot{K}_\varphi^{(2)} \left(E^\varphi_{(0)}\right)^3}{\left((E_0^x)'\right)^2 \sqrt{E_0^x}} + \frac{\left(P_\phi^{(1)}\right)^2 G}{(E_0^x)' E_0^x} + \frac{4E_0^x \left(\partial_x \phi_{(1)}\right)^2 G}{(E_0^x)'} - \frac{4K_\varphi^{(0)} P_\phi^{(1)} \phi'_{(1)} E^\varphi_{(0)} G}{\left((E_0^x)'\right)^2} \right)$$

Let us give a bit more details on the last point. For the scalar field we need to choose a vacuum. We will use the Unruh vacuum (though there is no obstruction to use other vacua). This will represent the Hawking radiation.
So the complete state is,

$$|\psi\rangle = |\tilde{g}, \vec{k}, M\rangle \otimes |\phi\rangle_{\text{Unruh}}.$$

For the quantization of the scalar field we must make explicit its equation, written in terms of second order derivatives,

$$\omega^2 R_\omega(r_*) + \frac{\partial^2 R_\omega(r_*)}{\partial r_*^2} - \frac{r_S}{r(r_*)^3}\left(1 - \frac{r_S}{r(r_*)}\right) R_\omega(r_*) = 0,$$

Where we assumed harmonic time dependence (ω is the frequency), r_* is the tortoise coordinate and r_S=2GM.

We can now use standard WKB techniques used in QFT in CST.

The only difference is that, since we are in a quantum space-time one has the dispersion relations typical on a lattice,

$$k_n = 2\pi n/((V - i_H)\Delta) \qquad \omega_n = \sqrt{\frac{2 - 2\cos(k_n\Delta)}{\Delta^2}}.$$

With V the number of vertices in the spin network, Δ the separation of the vertices, i_H the position of the last point before the horizon on the spin net. This also implies the existence of a maximum cutoff frequency 2/Δ.

To compute the term in the quadrature (essentially the stress energy tensor) we use the standard technique of computing the Green's function, (e.g. Candelas PRD 1980),

$$G_U(t, r_*, t', r'_*) = i\int_{-2/\Delta}^{2/\Delta} \frac{d\omega}{4\pi\omega} \frac{\exp(-i\omega(t-t'))}{r(r_*)\, r(r'_*)} \left[\frac{\overrightarrow{R}_\omega(r_*)\, \overrightarrow{R}^*_\omega(r'_*)}{1 - \exp(-4\pi r_S \omega)} + \Theta(\omega)\, \overleftarrow{R}_\omega(r_*)\, \overleftarrow{R}^*_\omega(r'_*)\right]$$

Everything is now explicit, so the integral can be computed.

We can evaluate the integrals for large values of ω. This is enough because we will focus on the terms that would be divergent in the continuum in this first approach The explicit form is:

$$G_B\left(t, r_*, t', r'_*\right)^{\Delta} = \frac{\left(-2i\pi^2 \ln(2) + 7i\zeta(3)\right)\left(r_* - r'_*\right)^2 - 4i\left(t - t'\right)^2 + 8\Delta\left(t - t'\right)}{32 r\left(r_*\right) r\left(r'_*\right) \pi \Delta^2}.$$

The stress energy tensor that appears in the quadrature is

$$T_{00} = \frac{r^2}{2}\left[\left(\frac{\partial \phi_{(1)}}{\partial t}\right)^2 + \left(\frac{\partial \phi_{(1)}}{\partial r_*}\right)^2\right].$$

And can be computed taking derivatives of G_B and taking the limit t->t' r_*->r_*'

$$\langle T_{00} \rangle = \frac{1}{16\Delta^2\pi} + \frac{2\pi^2 \ln(2) - 7\zeta(3)}{64\Delta^2\pi},$$

With the stress energy tensor under control we can go back to the second order equations of slides 8 and 9 and study them, first let us look at the equation for $E^{\phi}_{(2)}$,

$$\left\langle 4r_s^4\left(\phi'_{(1)}\right)^2+\left(P_\phi^{(1)}\right)^2\right\rangle-\frac{\partial}{\partial r}\left(\frac{r\Lambda_B}{4G_B}-\frac{r^3 E^{\varphi}_{(2)}}{\left(r^2-r_S r\right)^{3/2}G_B}\right)(r-r_S)\,r=0,$$

We have introduced subscripts “B” to emphasize the bare nature of Newton’s constant and the cosmological constant. Notice that we have distinguished between the Newton constant that plays the role of “coupling constant” from r_S (which also includes Newton’s constant) but that in the perturbative treatment really is a classical parameter of the zeroth order Hamiltonian. Normally such constants are not renormalized.

Substituting the expectation value we have,

$$\frac{\partial}{\partial r}\left(\frac{E^{\varphi}_{(2)}}{2\left(1-\frac{r_S}{r}\right)^{3/2}G_B}\right)+\frac{1}{8\Delta^2\pi\left(1-\frac{r_S}{r}\right)}+\frac{2\pi^2\ln(2)-7\zeta(3)}{32\Delta^2\pi\left(1-\frac{r_S}{r}\right)}-\frac{\Lambda_B}{4G_B}.$$

$\Delta_0^2 = \Delta^2\left(1-r_S/r\right)$ Is the geodesic distance squared.

We would like to absorb this quantity in the coupling constants. This forces us to choose a spin network state with a uniform spacing in terms of the geodesic distance.

Choosing,

$$\frac{\Lambda_B}{4G_B}=\frac{1}{8\Delta_0^2\pi}+\frac{2\pi^2\ln(2)-7\zeta(3)}{32\Delta_0^2\pi},$$

Keeping this in mind, let us look at the other equation, the one for the lapse $N_{(2)}$.

$$r\frac{\partial N_{(2)}}{\partial r}+\frac{8r^2\left\langle \dot{\phi}_{(1)}^2+\left(\phi'_{(1)}\right)^2\right\rangle G_B}{1-\frac{r_s}{r}}=0.$$

With the previous choices we get,

$$r\frac{\partial N_{(2)}}{\partial r}+\frac{G_B\left(4+2\pi^2\ln\left(2\right)-7\zeta\left(3\right)\right)}{32\Delta^2\pi\left(1-\frac{r_S}{r}\right)}=0.$$

This leads to a logarithmic divergence. This is reasonable, since we are dealing with an eternal black hole, there is an infinite amount of Hawking radiation between the horizon and infinity, so one would not expect the metric to be asymptotically flat.

This equation would lead us to renormalize Newton's constant as

$$G_R = G_B r_S^2 / \Delta_0^2.$$

And through a rescaling of the lapse one would end up with the renormalized Newton's constant in front of the lower order terms, as expected.

At this point, the limitations of the model get in the way. Because we have an eternal black hole with Hawking radiation ranging from the horizon to infinity, the radiation contributes a non-trivial amount to the ADM mass, therefore it cannot be treated as a perturbation.

The good news is we now have in place all the tools to treat a more realistic situation, like for instance where we consider as background the collapse of a shell. There the radiation can be treated as a perturbation. We expect to make progress on this model in the next few months.

Summary:

- We were able to formulate the back reaction of Hawking radiation living on a quantum space-time.
- We concentrated on the terms that in the continuum would have been divergent. Here they are finite but large, suggesting a finite renormalization is in order so the micro-structure does not influence macro variables.
- The cosmological constant is renormalized to zero and Newton's constant also is renormalized.
- We cannot complete the calculation for an eternal black hole, we expect to carry it out for a collapsing shell relatively soon.