Simplicity constraints and the immirzi parameter in discrete quantum gravity

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Summary

introduction and brief review

set up of discrete canonical Plebanski

reduction, part 1: simplicity constraints

reduction, part 2: gluing constraints

conclusions and outlook

Introduction and brief review

Plebanski action with Holst term:

$$\mathcal{S}_{Pleb,\mathcal{M}}[X,w,\phi] = \frac{1}{2}\int_{\mathcal{M}} \left[\mathrm{tr}(X \wedge F[w]) + \frac{1}{\gamma}\mathrm{tr}(\star X \wedge F[w]) + \frac{1}{2}\phi_{AB} \star X^A \wedge X^B \right]$$

SO(4) BF term | Holst term | Simplicity constraints

Barbero-Immirzi parameter

some definitions: $\mathrm{tr}(X_1\, X_2) = X_1{}^A\, \delta_{AB}\, X_2{}^B$

$$\star X^A = \epsilon^{AB}\, X^B$$

Introduction and brief review

Lagrange multiplier:

$$\phi^{AB} = \phi^{BA}$$

symmetric

$$\epsilon_{AB}\,\phi^{AB} = 0$$

traceless

Simplicity constraints:

$$\star X^A \wedge X^B = \frac{V}{4!}\epsilon^{AB}$$

where

$$V = \star X^A \wedge X^B\,\epsilon_{AB} = tr(X \wedge X)$$

Introduction and brief review

Non-degenerate solutions:

assuming non-degeneracy, that is: $V \neq 0$

$$X = \begin{cases} \pm(e \wedge e) \\ \pm \star (e \wedge e) \end{cases}$$

topological sector

gravitational sector

without Holst term

sectors become mixed

with Holst term

choosing the second sector of solutions

$$\mathcal{S}[e, w] = \frac{1}{2} \int_{\mathcal{M}} \left(\mathrm{tr}((e \wedge e) \wedge F[w]) + \frac{1}{\gamma} \mathrm{tr}(\star(e \wedge e) \wedge F[w]) \right)$$

solving half the equations of motion

trivializes the Holst term $\longrightarrow$ no effect on classical dynamics

Introduction and brief review

Spinfoam method:

consider a triangulated manifold $\mathcal{M} \longrightarrow \Delta_4$

$$X_{\mu\nu}{}^A \longrightarrow X_f$$

$$w_\mu{}^A \longrightarrow M_t = \mathcal{P}\exp\left(\int w\right)$$

start with a quantum theory of *BF* + *Holst* terms only

$$\boxed{\mathcal{Z}_\Delta = \int \mu_\Delta[X_f, M_t]\, e^{i\mathcal{S}_\Delta[X_f, M_t]}}$$

$$\mathcal{S}_\Delta[X_f, M_t] = \sum_{f\in\Delta} X_f\, M_f$$

advantage: well-defined measure

Introduction and brief review

Spinfoam method continued:

integrate bivectors explicitly

$$\mathcal{Z}_\Delta = \int \prod_{t\in\Delta} dM_t \; \delta(M_f)$$

re-expand in terms of representations of $\mathrm{SO}(4) \sim \mathrm{SU}(2) \otimes \mathrm{SU}(2)$

magnetic basis $|j_+, m_+ > \otimes |j_-, m_- >$

coherent state basis $|j_+, \vec{n}_+ > \otimes |j_-, \vec{n}_- >$

quantum bi-vector

reason: amplitude in localized chunks associated to tetrahedra and 4-simplices

making contact with boundary formalism

allows the imposition of simplicity constraints as operator constraints

Introduction and brief review

Spinfoam method continued:

impose simplicity constraints

remember: $$\star X^A \wedge X^B = \frac{V}{4!}\epsilon^{AB}$$

on the 4-simplices these split up as

diagonal simplicity	simplicity of bi-vector on each triangle
cross simplicity	pairwise simplicity for triangles sharing an edge
volume simplicity	pairwise simplicity for triangles sharing an vertex

turns out that we do not need volume simplicity

diagonal + cross on **BF amplitude** implies **volume**

Introduction and brief review

Spinfoam method continued:

strength of imposition

in the an operatorial sense on the representation basis states

first order path integral similar to a phase space path integral

one should check the properties of the constraints w.r.t. the BF symplectic structure

diagonal simplicity are 1st class

↓

impose strongly

cross simplicity constraints are 2nd class

$\gamma \neq \infty$ ↓

weaken the constraint imposition

Introduction and brief review

Spinfoam method continued:

Result

EPRL/FK spin foam models

[Engle, Pereira, Rovelli '08]
[Engle, Livine, Pereira, Rovelli '08]
[Freidel, Krasnov '08]
[Livine, Speziale '08]

wonderful property

boundary state space coincides with that of LQG restricted to a single graph

Introduction and brief review

Take-away point:

Immirzi parameter plays a non-trivial role

in both

spin foam models

and

loop models

Introduction and brief review

Some remaining questions:

spin foam quantization is not a conventional method of quantization

$\longrightarrow$ check that it gives gravity in the semi-classical limit

looking for gravitons, cosmological sectors etc.

[Rovelli '06 and subsequently]
[Bianchi, Rovelli, Vidotto '10]
[Ashtekar, Campiglia, Henderson '09]

$\longrightarrow$ check correspondence to Dirac quantization

Introduction and brief review

Re-examining the canonical theory:

[Buffenoir, Henneaux, Noui, Roche '04]

[Alexandrov, Buffenoir, Roche '07]

[Krasnov, Alexandrov '08]

continuum canonical analysis

$\longrightarrow$ not just primary but secondary simplicity constraints

canonical variables: $X_i = \epsilon_{0ijk}\, X_{jk}$ $\qquad w_j$

constraint system:

$$Primary = \mathrm{tr}(\star X_i\, X_j)$$

$$Secondary = f(X_i, w_j)$$

second class system

along with first class Gauss, 3-diffeo and Hamiltonian contraints

Introduction and brief review

explicit reduction results in an Immirzi parameter-free phase space

[Alexandrov, Buffenoir, Roche '07]

formal phase space path integral quantization

$$\mathcal{Z} = \int \mu_{\text{constraint hypersurface}} \, e^{i\mathcal{S}_{\text{constraint hypersurface}}}$$

kills the Immirzi parameter dependence in the path integral

[Alexandrov '08]

Introduction and brief review

The crux of the matter:

dichotomy between continuum canonical reasoning
and
spin foam reasoning

Introduction and brief review

Aim:

to shed light on this dichotomy by examining canonical theory on a discrete manifold

formulate discrete primary and secondary simplicity constraints

$\longrightarrow$ reduce and find resulting symplectic structure

Phase space of discrete manifold

3d hypersurface discretization: $\Sigma \longrightarrow \Delta_3$

	t	f	e
	labels tetrahedra	labels triangles	labels edges
BF phase space parameters		$X^i \to X_f$	$w_j \to M_f$
		bi-vectors	holonomies
		$\mathfrak{so}(4)$ valued	$\mathrm{SO}(4)$ valued

self-dual/anti-self-dual splitting

$$X_{f\pm} = \mathcal{P}_\pm X_f$$

$$M_{f\pm} = \mathcal{P}_\pm M_f$$

$$\mathcal{P}^{AB}_\pm = \frac{1}{2}(\delta^{AB} \pm \epsilon^{AB})$$

Phase space of discrete manifold

Phase space structure:

on *BF* + *Holst* phase space

$$\{X^A_{f\pm}, M^{BD}_{f\pm}\} = \frac{\gamma}{\gamma \pm 1} C^{ABC}_{\pm}\ M^{CD}_{f\pm}$$

$$\{M^{AB}_{f\pm}, M^{CD}_{f\pm}\} = 0$$

$$\{X^A_{f\pm}, X^B_{f\pm}\} = \frac{\gamma}{\gamma \pm 1} C^{ABC}_{\pm}\ X^C_{f\pm}$$

→ coming from discrete action

→ Jacobian identity

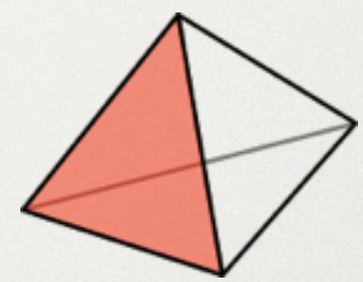
(replacing connection with holonomy)

C^{ABC} structure constants for $\mathfrak{so}(4)$

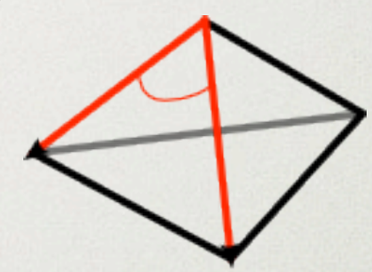
Phase space of discrete manifold

Collection of gauge-invariant quantities:

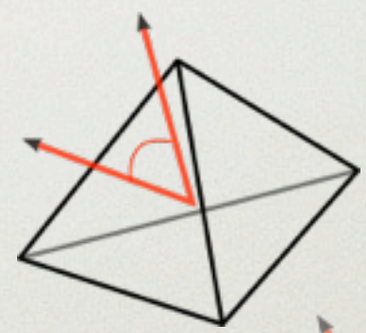
Areas A_f



2d-dihedral angles $\alpha_{ee'}$

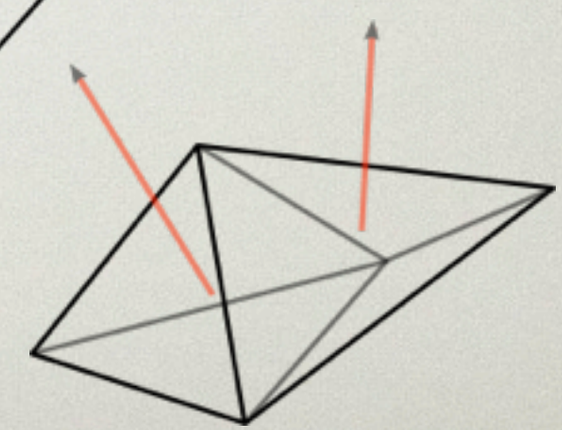


3d-dihedral angles ϕ_e



4d-dihedral angles $\theta_{f,e}$

there are three per triangle



Phase space of discrete manifold

Summary of constraints:

Gauss constraint	$\mathcal{G}_t = \sum_{f \subset t} X_f$	closure of tetrahedron
Primary simplicity	$D_f = A_{f+} - A_{f-}$	diagonal simplicity
	$C_e = \phi_{e+} - \phi_{e-}$	cross simplicity
Secondary simplicity	$E_{f,e} = \theta_{f,e+} - \theta_{f,e-}$	edge simplicity

self-dual geometry = anti-self-dual geometry

[Reisenberger '98]

Phase space of discrete manifold

Reconstructing the geometry

discrete primary and secondary simplicity

Theorem: [Dittrich, JR '08]

one can reconstruct consistent tetrad assignments for edges of the hypersurface

[Dittrich, JR '10]

\+ a consistent extrinsic curvature for the triangulation

for non-vanishing 3d-volume and non-parallel 4d normals for neighboring tetrahedra

Phase space of discrete manifold

Equivalence to discrete reality conditions

[Dittrich, JR '10]

in time gauge $X_{f\pm}$ becomes triad vectors $x_{f\pm}$

the diagonal, cross and edge simplicity take the form

$$x_{f+} = x_{f-}$$

$$e^{(-\Gamma_f[x])}\, m_{f+}\, e^{(-\Gamma_f[x])}\, m_{f-} = 1$$

$\Gamma_f[x]$ is the 3d spin connection constructed from the triads

discretization of continuum reality conditions

$$E_+ = E_-$$

$$A_+ + A_- = 2\Gamma[E]$$

[Alexandrov '06]

[Wieland '10]

Phase space of discrete manifold

Procedure:

solve Gauss constraints

Gauss constraints are first class

prevents any (time) gauge-fixing

$\longrightarrow$ solve by going to gauge-invariant phase space

Phase space of discrete manifold

Procedure:

find a basis for the gauge-invariant phase space

in each chiral sector pick

all areas

$A_{f\pm}$

$\phi_{e\pm}$

one per triangle

$\theta_{f,e\pm}$

Phase space of discrete manifold

Procedure:

find a basis for the constraint set

all diagonal D_f

C_e

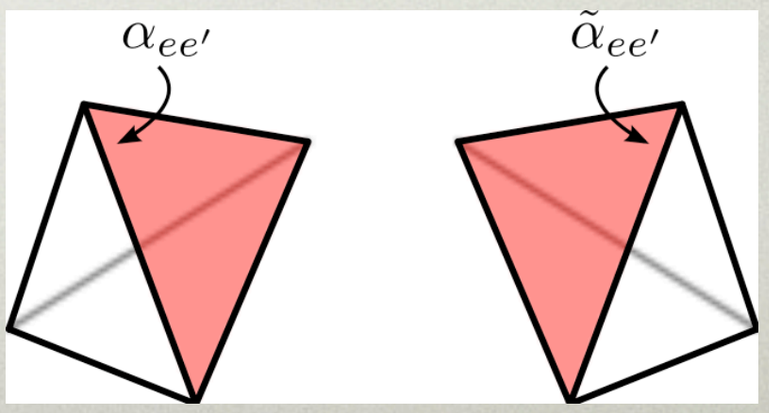
one edge per triangle $E_{f,e}$

BUT that does not exhaust this constraint set

in the sense that the 4d dihedral angles are still not well-defined

there exist relations among the θ and α

$$\theta_{f,e+} - \theta_{f,e'+} = \alpha_{ee'+} - \tilde{\alpha}_{ee'+}$$



Phase space of discrete manifold

Final description of constraints

$$D_f = A_{f+} - A_{f-}$$

$$C_e = \phi_{e+} - \phi_{e-}$$

$$E_{f,e} = \theta_{f,e+} - \theta_{f,e-}$$

relations between sectors

Set 1: simplicity constraints

$$D_f \qquad C_e \qquad E_{f,e}$$

Set 2: gluing constraints

$$G_{f,ee'} = \theta_{f,e+} - \theta_{f,e'+}$$

or

$$G_{ee'} = \alpha_{ee'+} - \tilde{\alpha}_{ee'+}$$

or

$$G_e = \phi_{e+} - f(A), \qquad K(\{A\})$$

$$L(\{\theta\})$$

relations within sector

Reduction, part 1: simplicity constraints

Procedure

compute Dirac matrix of constraints

$$\Delta_1 = \begin{array}{c} \begin{array}{ccc} D_f & C_e & E_{f,e} \end{array} \\ \begin{pmatrix} 0 & 0 & * \\ 0 & \frac{1}{\gamma}* & * \\ * & * & * \end{pmatrix} \end{array}$$

on the gravitational subsector

for $\gamma \to \infty$ $\{C_e, C_{e'}\}$ vanishes

cross simplicity forms first class subalgebra (without Holst term)

[Barrett, Crane '98]

[Baez, Barrett '00]

[Engle, Pereira '08]

Reduction, part 1: simplicity constraints

Procedure:

compute Dirac brackets

$$\{f,g\}_1 = \{f,g\} - \{f,\Phi_\alpha\}(\Delta^{-1})_1^{\alpha\beta}\{\Phi_\beta,g\}$$

Result: we have computed reduction for *arbitrary* triangulations

Reduction, part 1: simplicity constraints

Analysis

γ is still present

phase space variables A_{f+}, ϕ_{e+}, $\theta_{f,e+}$

reduced phase space

Loop Gravity

$$\{\phi_e, \phi_{e'}\}_1 = \gamma V_t \longrightarrow \{X^a_f, X^b_f\} = \gamma \epsilon^{abc} X^c_f$$

$$\{A_f, \theta_{f,e}\}_1 = 1 \qquad \{X^a_f, \widetilde{M}^{bd}_f\} = \gamma \epsilon^{abc} \widetilde{M}^{cd}_f$$

constructed from Ashtekar-Barbero connection

$$A^a_i = \Gamma^a_i + \gamma K^a_i$$

$$\widetilde{M} \sim \exp A \sim \exp \gamma\theta$$

agreement between our reduced phase space and (discrete) loop gravity phase space

Reduction, part 1: simplicity constraints

Analysis

Essentially, we have halved the SO(4) BF phase space down to an SU(2) BF phase space

in agreement with twisted geometries approach

Freidel, Speziale '10

phase space variables A_{f+}, ϕ_{e+}, $\theta_{f,e+}$

ϕ_{e+} are still independent which is not true for a geometric configuration

Thus: both LG and discrete Plebanski (at this stage) have genuinely larger phase spaces than Regge calculus

Reduction, part 2: gluing constraints

Procedure

compute Dirac matrix of constraints

$$\Delta_2 = \{G_e, G_{e'}\}_1$$

can compute for arbitrary triangulations **but** inversion is tricky

↓

special configuration: boundary of equilateral 4-simplex

compute Dirac brackets

$$\{A_f, \theta_f\}_2 = 1$$

$$\{\phi_e, \phi_{e'}\}_2 = 0$$

$$\{\theta_e, \theta_{e'}\}_2 = 0$$

Reduction, part 2: gluing constraints

Result:	Immirzi parameter has disappeared from the symplectic structure

Result:	first implementation of reality conditions on connection in a discrete setting

Reduction, part 2: gluing constraints

Analysis

Immirzi parameter is gone

we have reached Regge phase space ϕ_{e+} are no longer independent

[length Regge calculus: Regge '61]

[phase space: Bahr, Dittrich 09, Dittrich, Hoehn 09]

calculating phase space path integral, one would have no Immirzi parameter

γ-free spin foam?

Reduction, part 2: gluing constraints

Generalizing the analysis:

generic 4-simplex boundary

can re-express gluing as non-local constraints on the area $\tilde{G}_e = \cos\phi_e - f_e(\{A\})$

constraint hypersurface : length Regge calculus

generic triangulations

gluing constraints not captured completely by $\tilde{G}_e = \cos\phi_e - f_e(\{A\})$

constraint hypersurface : area Regge calculus [Barrett '94]

remainder captured by a set conjugate constraints $K(\{A\})$ $L(\{\theta\})$

constraint hypersurface : length Regge calculus

Reduction, part 2: gluing constraints

Degenerate configurations $V_{t\pm} = 0$ degenerate 3-volume $\sin\theta_e = 0$ parallel normals

$$D_f = A_{f+} - A_{f-}$$

$$C_e = \phi_{e+} - \phi_{e-}$$

$$E_{f,e} = \theta_{f,e+} - \theta_{f,e-}$$

non-degenerate configurations

Set 1: simplicity constraints

$D_f \qquad C_e \qquad E_{f,e}$

Set 2: gluing constraints

$$G_{ee'} = \alpha_{ee'+} - \tilde{\alpha}_{ee'+}$$

or

$$G_e = \phi_{e+} - f(A), \qquad K(\{A\}) \quad L(\{\theta\})$$

30d hypersurface **4-simplex example** 20d hypersurface

'configuration space analysis' of 4-simplex: obtain exactly 5 additional degrees of freedom if 4d volume is zero

arise in asymptotic analysis
(vector geometries or SU(2) BF configurations)

[Conrady, Freidel '08]

[Barrett, Fairbairn, Hellmann '08]

Outlook

Statement and question

Immirzi parameter is no longer present after a complete reduction by simplicity *and* gluing

should we be looking for γ-free spin foam models?

[Alexandrov '08, '10]

Outlook

Things to do

irregular constraint systems require more careful analysis

should deal with degenerate points separately

reduction in gauge variant variables

closer connection to continuum analysis

generalize reduction of gluing constraints to arbitrary triangulations

understand geometrical nature of constraints $K(\{A\})$ $L(\{\theta\})$

quantize the model