Simplicity constraints and the Immirzi parameter in discrete quantum gravity

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Summary

introduction and brief review

set up of discrete canonical Plebanski

reduction, part 1: simplicity constraints

reduction, part 2: gluing constraints

conclusions and outlook
Introduction and brief review

**Plebanski action with Holst term:**

\[
S_{Pleb,\mathcal{M}}[X, w, \phi] = \frac{1}{2} \int_\mathcal{M} \left[ \text{tr}(X \wedge F[w]) + \frac{1}{\gamma} \text{tr}(\star X \wedge F[w]) + \frac{1}{2} \phi_{AB} \star X^A \wedge X^B \right]
\]

- SO(4) BF term
- Holst term
- Simplicity constraints

Barbero-Immirzi parameter

Some definitions:

\[
\text{tr}(X_1 \, X_2) = X_1^A \, \delta_{AB} \, X_2^B
\]

\[
\star X^A = \epsilon^{AB} \, X^B
\]
Introduction and brief review

**Lagrange multiplier:**

\[ \phi^{AB} = \phi^{BA} \]

symmetric

\[ \epsilon_{AB} \phi^{AB} = 0 \]

traceless

**Simplicity constraints:**

\[ \star X^A \wedge X^B = \frac{V}{4!} \epsilon^{AB} \]

where

\[ V = \star X^A \wedge X^B \epsilon_{AB} = tr(X \wedge X) \]
Introduction and brief review

Non-degenerate solutions: assuming non-degeneracy, that is: $V \neq 0$

$$X = \{ \pm (e \wedge e), \pm \star (e \wedge e) \}$$

- topological sector
- gravitational sector

sectors become mixed

without Holst term

with Holst term

choosing the second sector of solutions

$$S[e, w] = \frac{1}{2} \int_{\mathcal{M}} \left( \text{tr}((e \wedge e) \wedge F[w]) + \frac{1}{\gamma} \text{tr}(\star(e \wedge e) \wedge F[w]) \right)$$

solving half the equations of motion

trivializes the Holst term

no effect on classical dynamics

Dienstag, 12. April 2011
Introduction and brief review

**Spinfoam method:**

consider a triangulated manifold

\[ \mathcal{M} \longrightarrow \Delta_4 \]

\[ X_{\mu\nu}^A \longrightarrow X_f \]

\[ w_\mu^A \longrightarrow M_t = \mathcal{P} \exp \left( \int w \right) \]

start with a quantum theory of BF + Holst terms only

\[ Z_\Delta = \int \mu_\Delta[X_f, M_t] e^{i S_\Delta[X_f, M_t]} \]

\[ S_\Delta[X_f, M_t] = \sum_{f \in \Delta} X_f M_f \]

**advantage:** well-defined measure
Introduction and brief review

*Spinfoam method continued:* 

\[ Z_{\Delta} = \int \prod_{t \in \Delta} dM_t \delta(M_f) \]

re-expand in terms of representations of \( SO(4) \sim SU(2) \otimes SU(2) \)

- magnetic basis
  \[ |j_+, m_+ > \otimes |j_-, m_- > \]

- coherent state basis
  \[ |j_+, \vec{n}_+ > \otimes |j_-, \vec{n}_- > \]

**reason:** amplitude in localized chunks associated to tetrahedra and 4-simplices making contact with boundary formalism allows the imposition of simplicity constraints as operator constraints
Spin foam method continued:

impose simplicity constraints

**remember:** \[ \star X^A \wedge X^B = \frac{V}{4!} \epsilon^{AB} \]

on the 4-simplices these split up as

- diagonal simplicity: simplicity of bi-vector on each triangle
- cross simplicity: pairwise simplicity for triangles sharing an edge
- volume simplicity: pairwise simplicity for triangles sharing an vertex

turns out that we do not need volume simplicity

**diagonal + cross** on **BF amplitude** implies **volume**
Introduction and brief review

**Spinfoam method continued:**

- strength of imposition in the an operatorial sense on the representation basis states
- first order path integral similar to a phase space path integral
  - one should check the properties of the constraints w.r.t. the BF symplectic structure
- diagonal simplicity are 1st class
  - impose strongly
- cross simplicity constraints are 2nd class
  - $\gamma \neq \infty$
  - weaken the constraint imposition
**Spinfoam method continued:**

- **Result**

EPRL/FK spin foam models

- **wonderful property**

boundary state space coincides with that of LQG restricted to a single graph

[Engle, Pereira, Rovelli '08]
[Engle, Livine, Pereira, Rovelli '08]
[Freidel, Krasnov '08]
[Livine, Speziale '08]
Introduction and brief review

Take-away point:

Immirzi parameter plays a non-trivial role

in both

spin foam models

and

loop models
Some remaining questions:

- Spin foam quantization is not a conventional method of quantization.
  - Check that it gives gravity in the semi-classical limit.
  - Looking for gravitons, cosmological sectors etc. [Rovelli ’06 and subsequently]
  - [Bianchi, Rovelli, Vidotto ’10]
  - [Ashtekar, Campiglia, Henderson ’09]

- Check correspondence to Dirac quantization.
Re-examining the canonical theory:

continuum canonical analysis

not just primary but secondary simplicity constraints

canonical variables:

\[ X_i = \epsilon_{0ijk} \ X_{jk} \quad w_j \]

constraint system:

Primary = \text{tr}(\star X_i \ X_j)

Secondary = f(X_i, w_j)

second class system

along with first class Gauss, 3-diffeo and Hamiltonian contraints
Introduction and brief review

explicit reduction results in an Immirzi parameter-free phase space

formal phase space path integral quantization

\[ Z = \int \mu_{\text{constraint hypersurface}} e^{i S_{\text{constraint hypersurface}}} \]

kills the Immirzi parameter dependence in the path integral

[Alexandrov, Buffenoir, Roche '07]

[Alexandrov '08]
Introduction and brief review

*The crux of the matter:*

- Dichotomy between continuum canonical reasoning and spin foam reasoning
Introduction and brief review

**Aim:**

- to shed light on this dichotomy by examining canonical theory on a discrete manifold

- formulate discrete primary and secondary simplicity constraints
  
  → reduce and find resulting symplectic structure
Phase space of discrete manifold

3d hypersurface discretization: \[ \Sigma \rightarrow \Delta_3 \]

- \( t \): labels tetrahedra
- \( f \): labels triangles
- \( e \): labels edges

BF phase space parameters

- \( X^i \rightarrow X_f \)
- \( w_j \rightarrow M_f \)

- \( \text{bi-vectors} \)
- \( \text{holonomies} \)

- \( \mathfrak{so}(4) \) valued
- \( \text{SO}(4) \) valued

Self-dual/anti-self-dual splitting

- \( X_f^\pm = P^\pm X_f \)
- \( M_f^\pm = P^\pm M_f \)

\[ P_{AB}^{\pm} = \frac{1}{2} (\delta_{AB} \pm \epsilon_{AB}) \]
Phase space of discrete manifold

Phase space structure:

on $BF + Holst$ phase space

\[
\{X^A_{f\pm}, M^{BD}_{f\pm}\} = \frac{\gamma}{\gamma \pm 1} C^{ABC}_{\pm} M^{CD}_{f\pm}
\]

\[
\{M^{AB}_{f\pm}, M^{CD}_{f\pm}\} = 0
\]

\[
\{X^A_{f\pm}, X^B_{f\pm}\} = \frac{\gamma}{\gamma \pm 1} C^{ABC}_{\pm} X^C_{f\pm}
\]

$C^{ABC}_{\pm}$ structure constants for $so(4)$

coming from discrete action

Jacobian identity
(replacing connection with holonomy)
Phase space of discrete manifold

Collection of gauge-invariant quantities:

- Areas: $A_f$
- 2d-dihedral angles: $\alpha_{ee'}$
- 3d-dihedral angles: $\phi_e$
- 4d-dihedral angles: $\theta_{f,e}$

there are three per triangle
Phase space of discrete manifold

**Summary of constraints:**

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Equation</th>
<th>Simplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss constraint</td>
<td>( G_t = \sum_{f \subset t} X_f )</td>
<td>closure of tetrahedron</td>
</tr>
<tr>
<td>Primary simplicity</td>
<td>( D_f = A_{f+} - A_{f-} )</td>
<td>diagonal simplicity</td>
</tr>
<tr>
<td></td>
<td>( C_e = \phi_{e+} - \phi_{e-} )</td>
<td>cross simplicity</td>
</tr>
<tr>
<td>Secondary simplicity</td>
<td>( E_{f,e} = \theta_{f,e+} - \theta_{f,e-} )</td>
<td>edge simplicity</td>
</tr>
</tbody>
</table>

**self-dual geometry = anti-self-dual geometry**

[Reisenberger ‘98]
Phase space of discrete manifold

Reconstructing the geometry

discrete primary and secondary simplicity

Theorem: [Dittrich, JR '08]

one can reconstruct consistent tetrad assignments for edges of the hypersurface

+ a consistent extrinsic curvature for the triangulation

[Dittrich, JR '10]

for non-vanishing 3d-volume and non-parallel 4d normals for neighboring tetrahedra
Phase space of discrete manifold

*Equivalence to discrete reality conditions*  

[Dittrich, JR ’10]

\[ X_f^{\pm} \] becomes **triad** vectors \[ x_f^{\pm} \]

the diagonal, cross and edge simplicity take the form

\[
\begin{align*}
x_f^+ &= x_f^- \\
e^{(-\Gamma_f[x])} m_f^+ e^{(-\Gamma_f[x])} m_f^- &= 1
\end{align*}
\]

\[ \Gamma_f[x] \] is the 3d spin connection constructed from the triads

discretization of **continuum** reality conditions

\[
\begin{align*}
E_+ &= E_- \\
A_+ + A_- &= 2\Gamma[E]
\end{align*}
\]

[Alexandrov ’06]

[Wieland ’10]
Phase space of discrete manifold

Procedure:

Gauss constraints are first class

solve by going to gauge-invariant phase space

solve Gauss constraints

prevents any (time) gauge-fixing
Phase space of discrete manifold

Procedure:

find a basis for the gauge-invariant phase space

in each chiral sector pick all areas one per triangle

\[ A_f^\pm \quad \phi_e^\pm \quad \theta_{f,e}^\pm \]
Phase space of discrete manifold

*Procedure:*

find a basis for the constraint set

```
D_f  C_e  E_{f,e}
```

all diagonal\[ \text{one edge per triangle}\]

**BUT** that does not exhaust this constraint set

in the sense that the 4d dihedral angles are still not well-defined

there exist relations among the $\theta$ and $\alpha$

$$\theta_{f,e^+} - \theta_{f,e^+} = \alpha_{ee^+} - \tilde{\alpha}_{ee^+}$$
Phase space of discrete manifold

**Final description of constraints**

Set 1: simplicity constraints

\[ D_f, C_e, E_{f,e} \]

Set 2: gluing constraints

\[ G_{f,ee'} = \theta_{f,e+} - \theta_{f,e'+} \]

or

\[ G_{ee'} = \alpha_{ee'+} - \tilde{\alpha}_{ee'+} \]

or

\[ G_e = \phi_{e+} - f(A), \quad K(\{A\}) \]

\[ L(\{\theta\}) \]

relations between sectors

\[ D_f = A_{f+} - A_{f-} \]

\[ C_e = \phi_{e+} - \phi_{e-} \]

\[ E_{f,e} = \theta_{f,e+} - \theta_{f,e-} \]

relations within sector
Reduction, part 1: simplicity constraints

**Procedure**

compute Dirac matrix of constraints

\[
\Delta_1 = \begin{pmatrix}
D_f & C_e & E_{f,e} \\
0 & 0 & * \\
0 & \frac{1}{\gamma} & * \\
* & * & * \\
\end{pmatrix}
\]

on the gravitational subsector

for \( \gamma \to \infty \) \( \{C_e, C_{e'}\} \) vanishes

cross simplicity forms first class subalgebra (without Holst term)

[Barrett, Crane '98]
[Baez, Barrett '00]
[Engle, Pereira '08]
Reduction, part 1: simplicity constraints

Procedure:

\[
\{f, g\}_1 = \{f, g\} - \{f, \Phi_\alpha\}(\Delta^{-1})_1^{\alpha\beta}\{\Phi_\beta, g\}
\]

Result: we have computed reduction for arbitrary triangulations
Reduction, part 1: simplicity constraints

**Analysis**
\[ \gamma \text{ is still present} \]
\[ \text{phase space variables } A_{f+}, \phi_{e+}, \theta_{f,e+} \]

Reduced phase space
\[ \{ \phi_e, \phi'_e \}_1 = \gamma V_t \]
\[ \{ A_f, \theta_{f,e} \}_1 = 1 \]

Loop Gravity
\[ \{ X^a_f, X^b_f \} = \gamma \epsilon^{abc} X^c_f \]
\[ \{ X^a_f, \tilde{M}^{bd}_f \} = \gamma \epsilon^{abc} \tilde{M}^{cd}_f \]

constructed from Ashtekar-Barbero connection
\[ A_i^a = \Gamma_i^a + \gamma K_i^a \]
\[ \tilde{M} \sim \exp A \sim \exp \gamma \theta \]

Agreement between our reduced phase space and (discrete) loop gravity phase space
Reduction, part 1: simplicity constraints

Analysis

Essentially, we have halved the SO(4) BF phase space down to an SU(2) BF phase space in agreement with twisted geometries approach

\[ \phi_{e+}, \theta_{f,e+} \]

\( \phi_{e+} \) are still independent which is not true for a geometric configuration

Thus: both LG and discrete Plebanski (at this stage) have genuinely larger phase spaces than Regge calculus

Freidel, Speziale '10

Dienstag, 12. April 2011
Reduction, part 2: gluing constraints

Procedure

compute Dirac matrix of constraints

\[ \Delta_2 = \{G_e, G'_e\}_1 \]

can compute for arbitrary triangulations \textbf{but} inversion is tricky

special configuration: boundary of equilateral 4-simplex

compute Dirac brackets

\[ \{A_f, \theta_f\}_2 = 1 \]

\[ \{\phi_e, \phi'_e\}_2 = 0 \]

\[ \{\theta_e, \theta'_e\}_2 = 0 \]
Reduction, part 2: gluing constraints

Result: Immirzi parameter has disappeared from the symplectic structure

Result: first implementation of reality conditions on connection in a discrete setting
Reduction, part 2: gluing constraints

**Analysis**

Immirzi parameter is gone

we have reached Regge phase space $\phi_{e+}$ are no longer independent

calculating phase space path integral, one would have no Immirzi parameter

$\gamma$-free spin foam?

[length Regge calculus: Regge '61]
[phase space: Bahr, Dittrich 09, Dittrich, Hoehn 09]
Reduction, part 2: gluing constraints

*Generalizing the analysis:*

**generic 4-simplex boundary**

\[ \bar{G}_e = \cos \phi_e - f_e(\{A\}) \]

Can re-express gluing as non-local constraints on the area

**constraint hypersurface**: length Regge calculus

**generic triangulations**

Gluing constraints not captured completely by

\[ \bar{G}_e = \cos \phi_e - f_e(\{A\}) \]

**constraint hypersurface**: area Regge calculus

Remainder captured by a set conjugate constraints

\[ K(\{A\}) \]
\[ L(\{\theta\}) \]

**constraint hypersurface**: length Regge calculus
Reduction, part 2: gluing constraints

Degenerate configurations

\[ V_{t \pm} = 0 \quad \text{degenerate 3-volume} \quad \sin \theta_e = 0 \quad \text{parallel normals} \]

Set 1: simplicity constraints

\[ D_f = A_f^+ - A_f^- \]
\[ C_e = \phi_e^+ - \phi_e^- \]
\[ E_{f,e} = \theta_{f,e}^+ - \theta_{f,e}^- \]

Set 2: gluing constraints

\[ G_{ee'} = \alpha_{ee'}^+ - \tilde{\alpha}_{ee'}^+ \]

or

\[ G_e = \phi_e^+ - f(A), \quad K(\{A\}) \]
\[ L(\{\theta\}) \]

non-degenerate configurations

30d hypersurface

4-simplex example

20d hypersurface

'configuration space analysis' of 4-simplex: obtain exactly 5 additional degrees of freedom if 4d volume is zero

arise in asymptotic analysis

(vector geometries or SU(2) BF configurations)

[Conrady, Freidel '08]
[Barrett, Fairbairn, Hellmann '08]
Outlook

Statement and question

Immirzi parameter is no longer present after a complete reduction by simplicity \textit{and} gluing should we be looking for $\gamma$-free spin foam models? 

[Alexandrov '08, '10]
Outlook

Things to do

- irregular constraint systems require more careful analysis
  should deal with degenerate points separately

- reduction in gauge variant variables
  closer connection to continuum analysis

- generalize reduction of gluing constraints to arbitrary triangulations

- understand geometrical nature of constraints
  \[ K(\{A\}) \]
  \[ L(\{\theta\}) \]

- quantize the model