2+1D Loop Quantum Gravity on the Edge
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We are interested in the discretization of gauge theories, and gravity in particular, such that gauge invariance is preserved. Doing this for gravity is a major unsolved problem.

Our work is based on the procedure used in [Freidel, Geiller, Ziprick 1110.4833] and later developed in [Dupuis, Freidel, Girelli 1701.02439], which separates discretization into two steps:

- **Subdivision**, or decomposition into subsystems.
- **Truncation**, or coarse graining of the subsystems.

We combine this with the formalism described in [Donnelly, Freidel 1601.04744] and [Geiller 1703.04748] (see also [Rovelli 1308.5599]), where the presence of boundaries in gauge theories introduces:

- New degrees of freedom (edge modes), which may be used to dress observables and render them gauge invariant.
- New boundary symmetries, which transform the edge modes and control the gluing map between subsystems.
Summary of results

- In this work [Freidel, Girelli, Shoshany 1811.04360] we rigorously discretize 2+1 gravity (with zero cosmological constant), while keeping track of the boundaries of the subsystems and the corresponding edge modes, symmetries and charges.

- As will be shown in detail below, we find relations between:
  - The continuous phase space of piecewise flat and torsionless geometries,
  - The discrete spin network phase space of loop gravity,
  - The phase space of a collection of particle-like curvature and torsion defects.
Flat 2+1D gravity ($\Lambda = 0$)

- Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. A possible choice is $G = \text{SU}(2)$ and $\mathfrak{g} = \text{su}(2)$. **Bold font** denotes algebra elements.
- Let $M$ be a 2+1-dimensional manifold and let $\Sigma$ be a 2-dimensional spatial manifold such that $M = \Sigma \times \mathbb{R}$.
- We define the following geometric variables:
  - $A$: $\mathfrak{g}$-valued connection 1-form,
  - $E$: $\mathfrak{g}^*$-valued frame field 1-form,
  - $F \equiv dA + \frac{1}{2} [A, A]$: $\mathfrak{g}$-valued curvature 2-form,
  - $T \equiv d_A E \equiv dE + [A, E]$: $\mathfrak{g}^*$-valued torsion 2-form.
- Let the dot product denote the Killing form: $E \cdot F \equiv E^i \wedge F_i$. The action and equations of motion are:
  \[ S = \int_M E \cdot F, \quad F = 0, \quad T = 0. \]
- The symplectic potential is:
  \[ \Theta = - \int_\Sigma E \cdot \delta A. \]
Consider a generalized Euclidean gauge transformation

\[
\begin{align*}
A & \mapsto g^{-1}Ag + g^{-1}dg, \\
E & \mapsto g^{-1}(E + d_A z)g,
\end{align*}
\]

where the rotation \( g \) is a \( G \)-valued 0-form, and the translation \( z \) is a \( g^* \)-valued 0-form.

On-shell, \( S \) is invariant, but \( \Theta \) transforms with a boundary term:

\[
\Theta \mapsto \Theta - \int_{\partial \Sigma} (z \cdot \delta A - (E + d_A z) \cdot \Delta g),
\]

where \( \Delta g \equiv \delta gg^{-1} \) is a shorthand for the Maurer-Cartan form on field space.

\( A \) and \( E \) can be dressed by adding new boundary degrees of freedom (edge modes) in order to make them (and thus \( \Theta \)) invariant. A new boundary symmetry algebra is introduced, which transforms the edge modes.
A map of Waterloo

Represents the spatial manifold $\Sigma$. 
Decomposition into subsystems

The blue graph represents the spin network $\Gamma$. 
We would like to keep only the relevant gauge-invariant observables from each subsystem.

Assume that the only way to probe the geometry (curvature and torsion) is using holonomies along loops of the spin network. Then, as far as observables are concerned, the following are equivalent:
More precisely:

- Let $\gamma_v$ be a loop in the spin network.
- Let $\mathfrak{h}, \mathfrak{h}^*$ be the Cartan subalgebras of $\mathfrak{g}, \mathfrak{g}^*$.
- Define Cartan elements $M_v \in \mathfrak{h}$ and $S_v \in \mathfrak{h}^*$. They encode all the gauge-invariant information we can obtain about the geometry inside the loop.
- We will see below that $M_v$ and $S_v$ are the charges of the edge mode symmetries.
- $M_v$ and $S_v$ have the same value whether the geometry inside the loop is continuous, concentrated only at one point, or anything in between.
The truncated geometry

The disks represent $M_v, S_v$ for each loop.
The spin network $\Gamma$ is composed of loops $\gamma_v$, links $e^*$ and nodes $c^*$. 
To each blue link $e^*$ we assign a red dual edge $e$. 
Δ is composed of cells $c$, edges $e$ and vertices $v$. 
Everything together

The spin network $\Gamma$ and the cellular decomposition $\Delta$. 
We have a spin network graph $\Gamma$ dual to a cellular decomposition $\Delta$ such that:

- Each link $e^*$ of $\Gamma$ is dual to an edge $e$ of $\Delta$,
- Each node $c^*$ of $\Gamma$ is dual to a cell $c$ of $\Delta$,
- Each vertex $v$ of $\Delta$ is regularized by an infinitesimal disk $v^*$ and dual to a loop $\gamma_v$.

The cells describe a *piecewise flat and torsionless geometry*.

- The geometry is completely flat and torsionless everywhere on the interior of each cell.
- The curvature and torsion are concentrated at the vertices $v$, are distributional:
  \[ F \propto \delta(v), \quad T \propto \delta(v), \]
  and depend on the Cartan elements $M_v, S_v$ as will be shown below.
The holonomy variables inside the cells

- Inside each cell \( c \), we define:
  
  - \( h_c (x) \): a dressed rotational holonomy from the node \( c^* \) (at the center of \( c \)) to a point \( x \in c \).
  
  - \( y_c (x) \): a dressed translational holonomy from the node \( c^* \) (at the center of \( c \)) to a point \( x \in c \).

- Then \( A \) and \( E \) are given inside \( c \) by:

\[
A \big|_c = h_c^{-1} \, dh_c, \quad E \big|_c = h_c^{-1} \, dy_c \, h_c.
\]

- By construction, \( F = T = 0 \) inside \( c \).

- The relation for \( A \) in terms of \( h_c \) may be inverted to find:

\[
h_c (x) = h_c (c^*) \, \exp \int_{c^*}^x A.
\]

\( h_c (c^*) \), the value of the dressed holonomy \( h_c \) at the node \( c^* \), conveys extra information that cannot be obtained from the connection alone. Note that in general \( h_c (c^*) \neq 1 \).

- Similarly, in general \( y_c (c^*) \neq 0 \).
The holonomy variables inside the cells
The holonomy variables inside the disks

- Inside each disk $v^*$, we analogously define:
  - $h_v(x)$: a dressed rotational holonomy from the vertex $v$ (at the center of $v^*$) to a point $x \in v^*$.
  - $y_v(x)$: a dressed translational holonomy from the vertex $v$ (at the center of $v^*$) to a point $x \in v^*$.
  - In general, $h_v(v) \neq 1$ and $y_v(v) \neq 0$.

- Then $A$ and $E$ are given inside $v^*$ by:

$$
A|_{v^*} = h_v^{-1} dh_v + h_v^{-1} M_v h_v \, d\phi_v,
$$

$$
E|_{v^*} = h_v^{-1} dy_v h_v + h_v^{-1} (S_v + [M_v, y_v]) h_v \, d\phi_v,
$$

where $\phi_v$ is the angular coordinate on the disk.

- By construction, $F = p_v \delta(v)$ and $T = j_v \delta(v)$ inside $v^*$, where

$$
p_v \equiv h_v^{-1} M_v h_v, \quad j_v \equiv h_v^{-1} (S_v + [M_v, y_v]) h_v.
$$

- This means that $F = T = 0$ everywhere inside $v^*$. Note that $v^*$ is a punctured disk, so $v$ itself is not inside $v^*$, it is on its boundary!
The continuity conditions on the cells

- A and E must be continuous when moving from a cell \( c \) to an adjacent cell \( c' \). This is obtained by introducing
  - \( h_{cc'} = h_{c'c}^{-1} \): a constant \( G \) element,
  - \( y_c^{c'} = -h_{cc'}y_c^{c}h_{c'c} \): a constant \( g^* \) element.
- Then we have, for \( x \) on the boundary between the cells:

\[
\begin{align*}
h_{c'}(x) &= h_{c'c}h_c(x), \\
y_{c'}(x) &= h_{c'c} \left( y_c(x) - y_{c'}^c \right) h_{cc'}.
\end{align*}
\]
The continuity conditions on the disks

- Similarly, for a cell $c$ and a disk $v^*$ we introduce:
  - $h_{vc} = h_{cv}^{-1}$: a constant $G$ element,
  - $y^c_v = -h_{vc}y^v_c h_{cv}$: a constant $g^*$ element.

- Then for $x$ on the boundary between the cell and the disk:

$$h_c(x) = h_{cv} e^{M_v \phi_v(x)} h_v(x),$$
$$y_c(x) = h_{cv} \left( e^{M_v \phi_v(x)} (y_v(x) + S_v \phi_v(x)) e^{-M_v \phi_v(x)} - y^c_v \right) h_{vc}.$$
The right action

- The dressed holonomies $h_c, y_c, h_v, y_v$ may be acted upon by two commuting actions.
- The right action, with 0-form parameters $g(x)$ and $z(x)$, is:
  
  $h_c \mapsto h_c g, \quad y_c \mapsto y_c + h_c z h_c^{-1},$
  
  $h_v \mapsto h_v g, \quad y_v \mapsto y_v + h_v z h_v^{-1}.$

  It imposes the gauge transformation we have seen before:

  $A \mapsto g^{-1} A g + g^{-1} d g, \quad E \mapsto g^{-1} (E + d_A z) g.$
The left action

Let $H$ be the Cartan subgroup of $G$ and $\mathfrak{h}$ the Cartan subalgebra. Define constant rotation parameters $g_c \in G$, $g_v \in H$ and constant translation parameters $z_c \in \mathfrak{g}^*$, $z_v \in \mathfrak{h}^*$ for each cell $c$ and disk $v^*$.

The left action is:

$$h_c \mapsto g_c h_c, \quad y_c \mapsto z_c + g_c y_c g_c^{-1},$$

$$h_v \mapsto g_v h_v, \quad y_v \mapsto z_v + g_v y_v g_v^{-1}.$$ 

It leaves $A$ and $E$ invariant. It represents a new boundary symmetry for the edge mode degrees of freedom. On the vertices, it is generated by the charges $M_v$ and $S_v$. (Note that $g_c, z_c$ commute with $M_v, S_v$.)

Since the left and right action commute, the charges $M_v$ and $S_v$ are gauge-invariant.
The discretized symplectic potential

- We decompose $\Theta$ into a sum over cells and disks:

$$\Theta = \sum_c \Theta_c + \sum_{\nu} \Theta_{\nu^*}, \quad \Theta_c \equiv -\int_c E \cdot \delta A, \quad \Theta_{\nu^*} \equiv -\int_{\nu^*} E \cdot \delta A.$$
The boundary of the cells

- The cell $c$ does not contain the disks surrounding it. Thus the boundary $\partial c$ of each cell is composed of:
  - Edges, labeled $(cc')$ where $c'$ is an adjacent cell. Note that $(cc') = (c'c)^{-1}$.
  - Arcs, labeled $(cv)$ where $v$ is an adjacent disk.
The boundary of the disks

- Each disk $v^*$ is punctured and so does not contain the vertex $v$. Thus the boundary $\partial v^*$ is composed of:
  - The vertex $v$,
  - Arcs, labeled $(vc)$ where $c$ is an adjacent cell. Note that $(cv) = (vc)^{-1}$.

In conclusion, we may rearrange the sums as:

$$
\Theta = \sum_c \Theta_c + \sum_v \Theta_{v^*} = \sum_{cc'} \Theta_{cc'} + \sum_{vc} \Theta_{vc} + \sum_v \Theta_v.
$$
The edge term

- On each cell, we plug in \( A = h_c^{-1} dh_c \) and \( E = h_c^{-1} dy_c h_c \) and get:

\[
\Theta_c = \int_{\partial c} dy_c \cdot \Delta h_c,
\]

where \( \Delta h_c \equiv \delta h_c h_c^{-1} \) is a shorthand for the Maurer-Cartan form on field space.

- After rearranging the sum, each edge \((cc')\) has two contributions, one from each of the two cells \(c, c'\) sharing the edge, with opposite orientation since \((cc') = (c'c)^{-1}\):

\[
\Theta_{cc'} = \int_{(cc')} (dy_c \cdot \Delta h_c - dy_{c'} \cdot \Delta h_{c'}) .
\]

- Using the continuity conditions, this simplifies to

\[
\Theta_{cc'} = \Delta h_c \cdot \int_{(cc')} dy_c,
\]

where \( \Delta h_{c'} \equiv \delta_{cc'} h_{c'c} \) is a constant.
The holonomy-flux algebra

- Defining the flux along the edge,

\[ X^c_{c'} \equiv \int_{(cc')} \, dy_c, \]

we get

\[ \Theta_{cc'} = \Delta h^c_{c'} \cdot X^c_{c'}. \]

- We see that we have obtained the familiar spin network phase space \( T^* G \),

\[ \Theta_{cc'} = \Delta h^c_{c'} \cdot X^c_{c'} \equiv \text{Tr} \left( \delta h_{e^*} h_{e^*}^{-1} X_e \right), \]

where:

- \( h_{e^*} \equiv h_{cc'} \) is the holonomy along the link \( e^* \) connecting the node \( c^* \) to the node \( c'^* \),
- \( X_e \equiv X^c_{c'} \) is the flux along the edge \( e \) (dual to the link \( e^* \)) which is the boundary between the cells \( c \) and \( c' \).
The arc term

Similarly, on the arcs we obtain

$$\Theta_{vc} = \Delta h^c_v \cdot X^c_v,$$

where

$$\Delta h^c_v \equiv \delta h_{vc} h_{cv}, \quad X^c_v \equiv \int_{(vc)} d \left( e^{M_v \phi_v} y_v e^{-M_v \phi_v} + S_v \phi_v \right).$$

This is again the spin network phase space $T^* G$, where now

- $h_{vc}$ is the holonomy from the vertex $v$ to the node $c^*$,
- $X^c_v$ is the flux along the arc $(vc)$.
The vertex term

- On the vertices, we obtain after some calculations

\[ \Theta_v = X_v \cdot \delta M_v - \left( S_v + [M_v, X_v] \right) \cdot \Delta h_v (v), \]

where the “vertex flux” \( X_v \) depends on \( y_v (v) \).

- If we define the “position” \( q_v \) and “momentum” \( p_v \):

\[ q_v \equiv h_v^{-1} X_v h_v \in g, \quad p_v \equiv h_v^{-1} M_v h_v \in g^*, \]

we can write

\[ \Theta_v = q_v \cdot \delta p_v - S_v \cdot \Delta h_v (v). \]

This may be interpreted as the phase space of a relativistic particle with mass \( M_v \) and spin \( S_v \). The phase space variables contain the edge modes \( h_v (v) \) and \( y_v (v) \).

- Since each edge is shared by two cells, and each arc is shared by a cell and a disk, the edge mode contributions from each side of the edge or arc cancel. However, the edge modes on the vertices (the corners of the cells) have nothing to cancel with.
In conclusion, the full discretized symplectic potential is:

\[ \Theta = \sum_{(cc')} \Delta h_{c'}^c \cdot X_{c'}^c - \sum_{(vc)} \Delta h_v^v \cdot X_v^v + \]

\[ + \sum_v \left( X_v \cdot \delta M_v - (S_v + [M_v, X_v]) \cdot \Delta h_v(v) \right). \]

- First term: Spin network phase space \( T^*G \) per link \((cc')\).
- Second term: Spin network phase space \( T^*G \) per arc \((vc)\).
- Third term: Particle-like defects. These degrees of freedom are the edge modes.
The Gauss constraint

- In the continuum, the Gauss constraint imposes $\mathbf{T} = 0$ everywhere except the vertices.
- In the discrete theory, we have one constraint on each cell and disk. On each cell $c$: 
  $$\sum_{c' \ni c} \mathbf{x}_{c'} = \sum_{v \ni c} \mathbf{x}_v.$$
The Gauss constraint

- On each disk $v^*$:

$$\sum_{c \in v} x^c_v = S_v + e^{M_v} y_v (v) e^{-M_v} - y_v (v).$$

The Gauss constraint generates the rotational part of the symmetry transformations corresponding to the left action. Recall that this is the edge mode symmetry which leaves $A$ and $E$ invariant.

The constraint is satisfied identically in our construction.
The curvature constraint

- In the continuum, the curvature constraint imposes \( F = 0 \) everywhere except the vertices.
- In the discrete theory, we have for each vertex \( v \):
  \[
  h_{c_1 c_2} \cdots h_{c_N c_1} = h_{c_1 v} e^{M_v} h_{vc_1}.
  \]
- It generates the translational part of the symmetry corresponding to the left action, and is also satisfied identically.
Shrinking the disks

- Using the curvature constraint, one can choose a particular transformation parameter which considerably simplifies things:
  - \( X_v \) becomes simply the edge mode \( y_v(v) \),
  - The vertex Gauss constraint becomes simply \( \sum_{c \ni v} X^c_v = S_v \).

- If we now assume that \( S_v = 0 \) for all \( v \), that is, there are no torsion defects, then we may shrink the disks to points.
  - The length of all arcs becomes zero, and we can take \( X^c_v = 0 \) for all \( v \) and \( c \). Then the cell Gauss constraint simplifies to \( \sum_{c' \ni c} X^c_{c'} = 0 \), which is the correct constraint for loop gravity. The vertex potential becomes that of a spinless particle: \( \Theta_v = q_v \cdot \delta p_v \).
  - Therefore, the torsionless case corresponds to the usual loop gravity phase space, along with a collection of curvature defects:

\[
\Theta = \sum_{(cc')} \Delta h^c_{c'} \cdot X^c_{c'} + \sum_v q_v \cdot \delta p_v.
\]
We can now also “freeze” the edge modes.

Recall that the “position” $q_v$ and “momentum” $p_v$ are:

$$q_v = h_v^{-1} (v) y_v (v) h_v (v), \quad p_v = h_v^{-1} (v) M_v h_v (v).$$

We interpret $y_v (v)$ as the position in the rest frame and $M_v$ as the rest mass, while $h_v (v)$ acts like a Lorentz transformation.

By setting $y_v (v) = 0$ and $h_v (v) = 1$, we are “undressing” the holonomies. Then $q_v = 0$ and $\Delta h_v (v) = 0$, so

$$\Theta_v = q_v \cdot \delta p_v - S_v \cdot \Delta h_v (v) = 0.$$

The vertex (particle-like) potential has vanished.

In conclusion, the loop gravity phase space is a special case of the full phase space, where the geometry is torsionless and the edge modes are frozen:

$$\Theta = \sum_{(cc')} \Delta h_{c'}^c \cdot \chi_{c'}^c, \quad \sum_{c' \ni c} \chi_{c'}^c = 0, \quad \prod_i h_{c_i c_{i+1}} = e^{M_v}.$$
Shrinking the cells

- Instead of shrinking the disks, we may shrink the cells (or equivalently, expand the disks).
  - The boundary of each disk is identified with the spin network loop that contained it.
  - The edges \( (cc') \) now have zero length, and thus the fluxes \( X_c^{c'} \equiv \int_{(cc')} dy_c \) vanish.
  - Therefore, the spin network symplectic potential \( \Theta_{cc'} = \Delta h_c^{c'} \cdot X_c^{c'} \)
    vanishes, and we are left with
    \[
    \Theta = - \sum_{(vc)} \Delta h_v^c \cdot X_v^c + \sum_v (q_v \cdot \delta p_v - S_v \cdot \Delta h_v).
    \]
  - If we furthermore assume \( S_v = 0 \), then we can again take \( X_c^v = 0 \)
    for all \( v \) and \( c \), and get
    \[
    \Theta = \sum_v q_v \cdot \delta p_v.
    \]
  - In conclusion, in the spinless case, the phase space may be reduced
    to a collection of particle-like curvature defects.
Summary

- We have found a relation between:
  - The continuous phase space of piecewise flat and torsionless geometries,
  - The discrete spin network phase space of loop gravity coupled to a collection of particle-like curvature and torsion defects.
- In the torsionless case, we further found a relation between:
  - The loop gravity phase space alone,
  - The phase space of particle-like curvature defects.
Future Plans

- The defect picture in our formalism suggests a new perspective for addressing the problem of the continuum limit, possibly related to that of [Delcamp, Dittrich, Riello 1607.08881].

- In 2+1D, our formalism can be extended to include a “dual polarization”, in which the holonomies and fluxes switch places. (Paper to appear soon.)

- The most challenging task is to generalize our formalism to the physically relevant case of 3+1D.
  - The cells will now be 3-dimensional, and the spin network loops will encircle the edges of the cells.
  - This is work in progress, but we have already been able to show that piecewise flat and torsionless geometries in 3+1D satisfy the Gauss, vector and scalar constraints of loop quantum gravity.

- Thank you for listening!