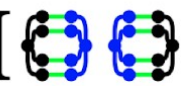
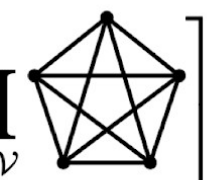
Asymptotic analysis of spin-foams with time-like faces in a new parametrization

José Diogo Simão,
Theoretical Physics Institute, FSU Jena

$$Z(\Delta^*) = \sum_{j\to\mathcal{F}} \sum_{\iota\to\mathcal{E}} \int \mathrm{d}\chi_i \Delta_{\chi_i} \left[\prod_{f\in\mathcal{F}} j_f^2(\gamma^2+1)\right] \left[\prod_{e\in\mathcal{E}}\right]$$



$$\left[\prod_{v\in\mathcal{V}}\right]$$



In collaboration with Sebastian Steinhaus

ArXiv 2106.15635

Summary of results and goal

(Engle et al. '07; Conrady, Hnibyda '10)

- Our paper focuses on the asymptotic behaviour of EPRL-CH spin-foams with space-like as well as time-like faces. Its main results are:

  → A Minkowski theorem for 3d polyhedra in Minkowski space-time;

  → An alternative parametrization for the states associated to time-like faces;

  → The spin-foam amplitude for time-like faces is dominated by geometric configurations, but also by non-geometric ones;

  → ~~The "Cosine Problem" is absent for particular types of spin-foams;~~

  (thanks to H. Liu for claryfing this point)

- The goal for this talk:

> To show that the EPRL-CH model with geometric boundary data does not (exclusively) asymptote to Regge theory, suggesting the model needs further constraining.
>
> (problem lies in time-like faces)

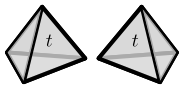
The roadmap

The EPRL-CH spin-foam model in a nutshell

Representation theory of the special linear group

Asymptotic analysis

Posing the asymptotic problem

New parameterisation for 

Geometry from critical point conditions

Conclusions and discussion

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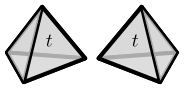
New parameterisation for 

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The EPRL-CH model in a nutshell

- GR = BF + constraints

$$(\, g = \eta^{IJ}\theta_I \otimes \theta_J \,)$$

$$S_T = \int_M F^{IJ} \wedge \left(\star + \frac{1}{\gamma}\right) \theta_I \wedge \theta_J \qquad \int_M \mathrm{Tr}\; B \wedge F\,, \quad B = (\star + \gamma^{-1})\, \theta \wedge \theta$$

"Simplicity constraints"

- The general idea: (Engle et al. '07; Freidel, Krasnov '07; Conrady, Hnibyda '10)

Discretize BF → Quantize → Apply constraints at quantum level

The EPRL-CH model in a nutshell

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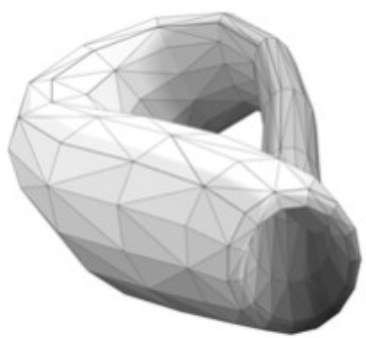
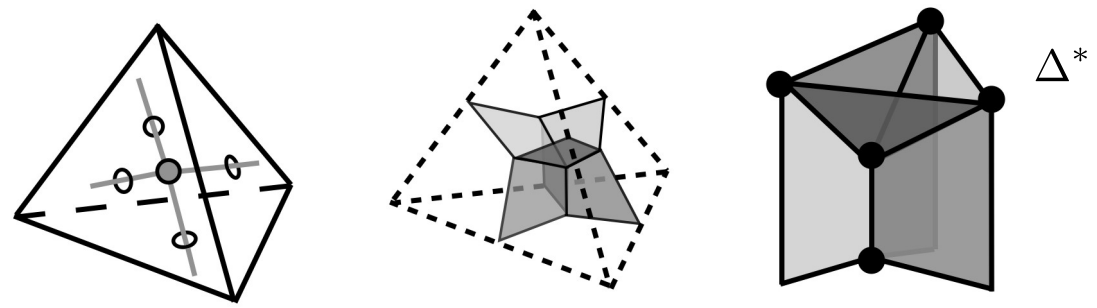
Discretize BF → Quantize → Apply constraints at quantum level

Cellular decomposition of M



Δ

Dual 2d cell complex



Δ^*

The EPRL-CH model in a nutshell

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$$S_T = \int_M F^{IJ} \wedge \left(\star + \frac{1}{\gamma}\right) \theta_I \wedge \theta_J \qquad \int_M \mathrm{Tr}\; B \wedge F\,, \quad B = (\star + \gamma^{-1})\, \theta \wedge \theta$$

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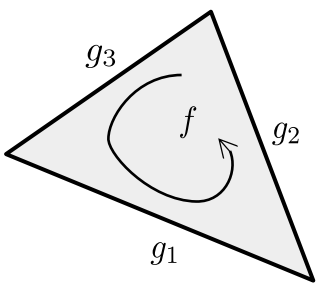

$$Z(M) \sim \int \mathcal{D}A\mathcal{D}B\; e^{i\int_M \mathrm{Tr}\; B\wedge F} \sim \int \mathcal{D}A\; \delta(F)$$

$$Z(\Delta^*) = \int_{\mathrm{SL}(2,\mathbb{C})} \prod_{e\in\mathcal{E}} \mathrm{d}g_e \prod_{f\in\mathcal{F}} \delta\left(\prod_{e\in\partial f} g_e\right)$$

The EPRL-CH model in a nutshell

- GR = BF + constraints

$$S_T = \int_M F^{IJ} \wedge \left(\star + \frac{1}{\gamma}\right) \theta_I \wedge \theta_J \qquad \int_M \mathrm{Tr}\; B \wedge F\,, \quad B = (\star + \gamma^{-1})\, \theta \wedge \theta$$

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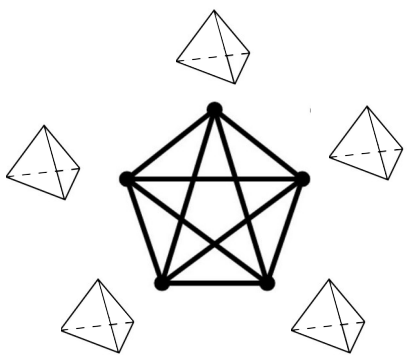
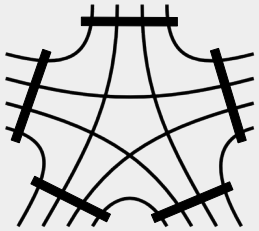


$$Z(\Delta^*) = \sum_{\{\rho\}\to\{f\}} \prod_{f\in\mathcal{F}} \dim(\rho) \prod_{v\in\mathcal{V}}$$



SL(2,C) unitary irreducible reps.

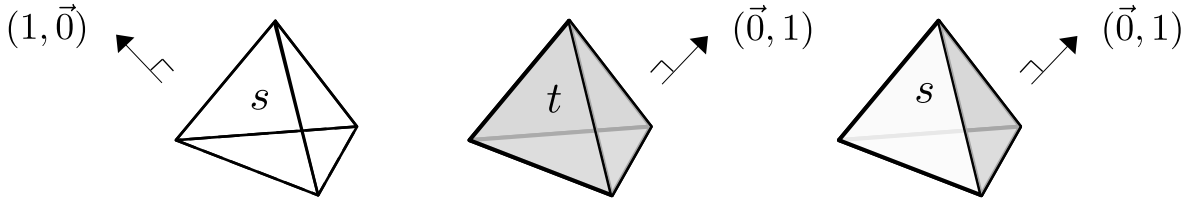
$$\Big| = \rho(g)$$

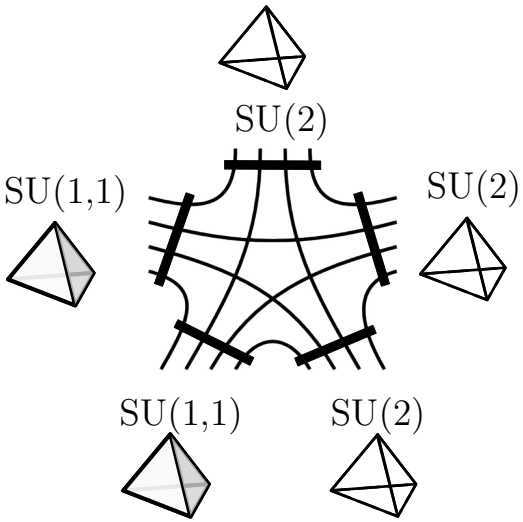
$$\Big|_{\cdots} = \int \mathrm{d}g\, \rho_1(g) \ldots \rho_n(g)$$

The EPRL-CH model in a nutshell

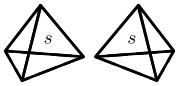
Discretize BF → Quantize → Apply constraints at quantum level

- Discretized constraints require choice of causal character of each  ;



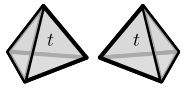


- Causal character specifies a representation of an $\mathrm{SL}(2,\mathbb{C})$ subgroup:

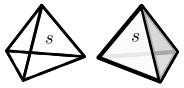


Achronal, SU(2) principal series (Engle et al. '07)

Orthochronal, SU(1,1) discrete series (Conrady, Hnibyda '10)



Parachronal, SU(1,1) continuous series (Conrady, Hnibyda '10)



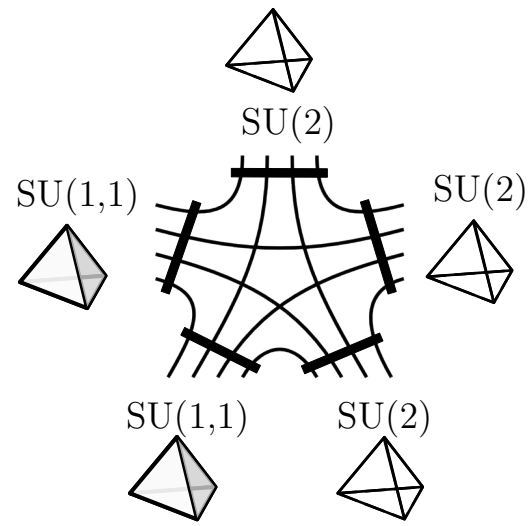
Heterochronal, SU(1,1) discrete + SU(2) principal series (Conrady, Hnibyda '10)

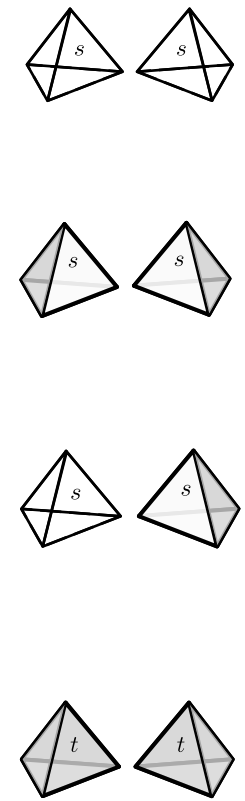
The EPRL-CH model in a nutshell

- Constraints single out a set $n = f(p)$ of admissible representations $\rho^{(p,n)}$, which depend on the causal character every region is assumed to have.

$$Z(\Delta^*) = \sum_{\{\rho\}\to\{f\}} \prod_{f\in\mathcal{F}} \dim(\rho) \prod_{v\in\mathcal{V}}$$

Constrained!





- There is no classical geometry at the quantum level: only representation theory and combinatorics (and a weak notion of "causality").

The roadmap

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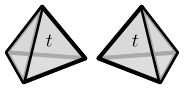
Posing the asymptotic problem

New parameterisation for 

Geometry from critical point conditions

Conclusions and discussion

Representation theory of the special linear group

- Representations of SL(2,C) are constructed on the space $\mathcal{D}_\chi$ of homogeneous functions of complex degree (n_1, n_2),

$$F : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$$
$$F(\alpha z_1, \alpha z_2) = \alpha^{n_1 - 1} \bar{\alpha}^{n_2 - 1} F(z_1, z_2),\ \alpha \in \mathbb{C}\,,$$

$$D^\chi : \mathrm{SL}(2, \mathbb{C}) \to U(\mathcal{D}_\chi)$$
$$D^\chi(g) F(\mathbf{z}) = F(g^T \mathbf{z})\,.$$

- We focus on the *principal series*, for which $n_1 = \overline{n}_2$, and label the reps. by $\chi = (n, \rho)$, where

$$n_1 = (-n + i\rho)/2,\ n_2 = (n + i\rho)/2\,, \quad n \in \mathbb{Z}\,,\ \rho \in \mathbb{R}$$

- The principal series is unitary with respect to the inner product

$$\langle F_1, F_2 \rangle = \int_{\mathbb{C}P} \bar{F}_1(\mathbf{z}) F_2(\mathbf{z})\, \omega\,,$$
$$\omega = \frac{i}{2}(z_2 \mathrm{d}z_1 - z_1 \mathrm{d}z_2) \wedge (\bar{z}_2 \mathrm{d}\bar{z}_1 - \bar{z}_1 \mathrm{d}\bar{z}_2)\,.$$

Representation theory of the special linear group

- Due to homogeneity, the functions are uniquely defined by their values on either the sphere S^3 or the hyperboloids $H^3_\pm = \{\mathbf{z} \in \mathbb{C}^2 \mid |z_1|^2 - |z_2|^2 = \tau\,, \tau = \pm 1\}$; this can be used to construct an SU(2) or SU(1,1) basis for the principal series.

- There exists a Plancherel theorem relating the principal series and *covariant* square-integrable functions on SU(1,1):

$$f(\gamma v) = e^{i\omega n \tau} f(v)$$

$$\gamma = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix}$$

covarinace

$$L^2(\mathrm{SU}(1,1))_{\mathrm{cov}} \oplus L^2(\mathrm{SU}(1,1))_{\mathrm{cov}} \simeq \mathcal{D}_\chi\,,$$

(Ruehl 1970)

$$\sum_\tau \int_{\mathrm{SU}(1,1)} |f_\tau(v)|^2 \,\mathrm{d}v = \int_{\mathbb{C}P} |F(z_1, z_2)|^2 \,\omega\,.$$

- A second Plancherel theorem relates the square integrable functions and the unitary irreducible representations of SU(1,1):

$$\bigoplus_k \mathcal{D}^+_k \bigoplus_k \mathcal{D}^-_k \bigoplus_\epsilon \int^\oplus \mathrm{d}s\, \mathcal{C}^\epsilon_s \simeq L^2(\mathrm{SU}(1,1))\,,$$

(Harish-Chandra 1952)

$$\sum_{mm'} \left[\underbrace{\sum_\epsilon \int_0^\infty \mathrm{d}s\, |\psi^{s,\epsilon}_{mm'}|^2}_{\text{Continuous series}} + \underbrace{\sum_k (|\psi^{k+}_{mm'}|^2 + |\psi^{k-}_{mm'}|^2)}_{\text{Discrete series}} \right] = \int_{\mathrm{SU}(1,1)} |f(v)|^2 \,\mathrm{d}v\,.$$

Representation theory of the special linear group

- A similar argument applies to SU(2). One then has the *canonical basis* for $\mathrm{SL}(2,\mathbb{C})$

$$\mathbb{1}_{(n,\rho)} = \sum_{j=\frac{n}{2}}^{\infty} \sum_{m=-j}^{m=j} |\chi; j\, m\rangle \langle \chi; j\, m| \,,$$

$$F^{\chi}_{j,m}(\mathbf{z}) = \sqrt{2j+1}\,(|z_1|^2 + |z_2|^2)^{i\rho/2-1} \underbrace{D^{j}_{\frac{n}{2}\,m}(u(\mathbf{z}))}_{\text{SU(2) rep. matrices}} \,,$$

as well as the SU(1,1) *pseudo-basis*

$$\mathbb{1}_{(n,\rho)} = \sum_{\tau} \Bigg[\underbrace{\int_0^{\infty} \mathrm{d}s \sum_{\substack{\pm m=\epsilon \\ \epsilon-\frac{n}{2}\in\mathbb{Z}}} |\chi,\tau; s,\epsilon,m\rangle \langle \chi,\tau; s,\epsilon,m|}_{\text{Continuous series}} + \underbrace{\sum_{k-\frac{n}{2}\in\mathbb{Z}} \sum_{m=\tau k}^{\tau\infty} |\chi,\tau; k,m\rangle \langle \chi,\tau; k,m|}_{\text{Discrete series}} \Bigg] \,,$$

$$F^{\chi,\tau}_{s,\epsilon,m}(\mathbf{z}) = \sqrt{\mu_\epsilon(s)}\, \Theta\left(\tau(|z_1|^2 - |z_2|^2)\right) \left(\tau(|z_1|^2 - |z_2|^2)\right)^{i\rho/2-1} D^{s,\epsilon}_{\frac{\tau n}{2},m}(v_\tau(\mathbf{z})) \,,$$

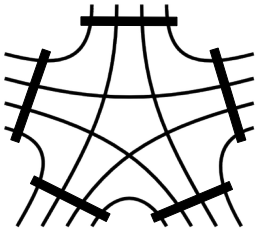
$$F^{\chi,\tau}_{k,m}(\mathbf{z}) = \sqrt{2k-1}\, \Theta\left(\tau(|z_1|^2 - |z_2|^2)\right) \left(\tau(|z_1|^2 - |z_2|^2)\right)^{i\rho/2-1} D^{k,\tau}_{\frac{\tau n}{2},m}(v_\tau(\mathbf{z})) \,.$$

SU(1,1) rep. matrices

Representation theory of the special linear group

- States of the theory live in spaces of unitary irreps. of SU(2) and SU(1,1), labeled by spins $\{|j\,m\rangle\,, |k,m\rangle\,, |s\,m\rangle\}$;

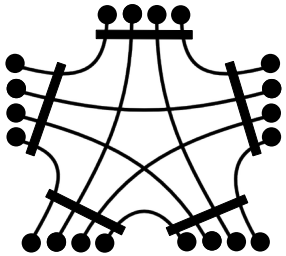
Vertex amplitude

$\prod_{v\in\mathcal{V}}$ 

$= D^{\chi}(g)$

$$\mathbb{1}_{\chi} = \sum_{j\,m} |j\,m\rangle\,\langle j\,m|$$

$$\mathbb{1}_{\chi} = \sum_{k\,m} |k\,m\rangle\,\langle k\,m| + \sum_{m}\int \mathrm{d}s\;|s\,m\rangle\,\langle s\,m|$$



$= \langle j\,m|\,D^{\chi}(g)\,|j'\,m'\rangle$

- This defines a vertex amplitude, which in more abstract notation can be written as

$$A_v = \int_{\mathrm{SL}(2,\mathbb{C})} \prod_{a=1}^{n} \mathrm{d}g_a\,\delta(g_n) \prod_{b<a} \langle D^{\chi}(g_a)\Psi_{ab}, D^{\chi}(g_b)\Psi_{ba}\rangle\ ,$$

and this expression may be generalized to other combinatorics.

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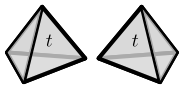
New parameterisation for 

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- We are interested in studying the spin-foam for large spin values.
- Restrict to vertex amplitude, take uniform scaling $\{j,\, k,\, s\} \to \{\Lambda j,\, \Lambda k,\, \Lambda s\}$, and rewrite:

$$\int_{\mathrm{SL}(2,\mathbb{C})} \prod_{a=1}^{n} \mathrm{d}g_a \delta(g_n) \prod_{b<a} \langle D^{\chi}(g_a)\Psi_{ab}, D^{\chi}(g_b)\Psi_{ba}\rangle := \int_{\mathrm{SL}(2,\mathbb{C})} \left[\prod_i \mathrm{d}g_i\right] f(\{g_i\}) e^{i\Lambda S(\{g_i\})}$$

Fix boundary data and analyze the critical points:

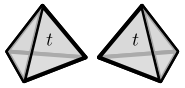
$$\begin{cases} \delta_{\mathbf{z}_{ab}} S_{ab}(\mathbf{z}_{ab}, g_a, g_b) = 0,\ \forall\, a, b\,, \\ \sum_{b>a} \delta_{g_a} S_{ab}(\mathbf{z}_{ab}, g_a, g_b) = 0,\ \forall\, a\,, \end{cases} \qquad \Re\, S_{ab}(\mathbf{z}_{ab}, g_a, g_b) = 0,\ \forall\, a, b\,.$$

(Barret et al. '09)

(Kamiński et al. '17)

(Liu, Han '18)

- The full expression involves many inner products between boundary states. In the following we will discuss the (relatively more involved) *parachronal interfaces.*

This requires explicitly computing $\overline{D^{\chi}(g_a)\Psi_{ab}} \cdot D^{\chi}(g_b)\Psi_{ba}$ **for** 

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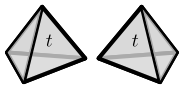
New parameterisation for 

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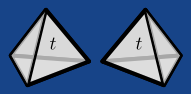
New parametrization for 

- Ψ_{ab} is an SL(2,C) state induced from an SU(1,1) state in the *continuous series.*

- The CH extension makes use of states in a generalized eigenbasis of the K_1 generator:

$$F = (J_3, K_1, K_2)\,, \qquad F^2\,|j,\,\lambda,\,\sigma\rangle = j(j+1)\,|j,\,\lambda,\,\sigma\rangle\ , \quad j = -\tfrac{1}{2} + is\,, \quad s \in \mathbb{R}_0^+$$

$$K_1\,|j,\,\lambda,\,\sigma\rangle = \lambda\,|j,\,\lambda,\,\sigma\rangle\ , \quad \lambda \in \mathbb{R}, \qquad P\,|j,\,\lambda,\,\sigma\rangle = (-1)^\sigma\,|j,\,\lambda,\,\sigma\rangle\ .$$

(Lindblad, Nagel 1970)

- The most useful form of these states in asymptotic analysis is as *coherent states:*

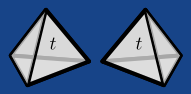
$$\mathbb{1}_{s,\epsilon} = \mu_\epsilon(s) \int_{\mathrm{SU}(1,1)} \mathrm{d}g\; D^{s,\epsilon}(g)\,|j,\hat{\lambda},\hat{\sigma}\rangle\,\langle j,\hat{\lambda},\hat{\sigma}|\,D^{s,\epsilon\,\dagger}(g)\,.$$

For some choice of $\hat{\lambda},\hat{\sigma}$

- The Conrady-Hnybida prescription is to nulify $\langle \Delta F^2\rangle = \langle F^2\rangle - \langle F_i\rangle\,\langle F^i\rangle$, finding

(Conrady, Hnibyda '10) $$\hat{\lambda} = \sqrt{s^2 + \frac{1}{4}}, \ \hat{\sigma} = 1\,.$$

Compare with $m = j$ in the space-like EPRL case

New parametrization for 

- The K_1 generator actually admits an eigenbasis with constant imaginary part, and we can prove completeness and orthogonality relations:

$$\sum_\sigma \int_{\mathbb{R}+i\alpha} \mathrm{d}\lambda \, |j,\, \lambda,\, \sigma\rangle \, \langle j,\, \overline{\lambda},\, \sigma| = \mathbb{1}\,, \qquad \langle j,\, \overline{\lambda},\, \sigma | j,\, \lambda',\, \sigma\rangle = \delta(\lambda - \lambda')\,.$$

- Conjugation of λ is essential to preserve self-adjointness of K_1 in the continuous series space $\mathcal{C}_s^\epsilon$.

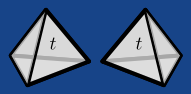
$$\begin{aligned}
\langle K_1\phi, \psi\rangle_{\mathcal{C}_s^\epsilon} &= \sum_\sigma \int_{\mathbb{R}+i\alpha} \mathrm{d}\lambda \; \overline{K_1 f_{\overline{\lambda}}(\phi)} f_\lambda(\psi) \\
&= \sum_\sigma \int_{\mathbb{R}+i\alpha} \mathrm{d}\lambda \; \lambda \; \overline{f_{\overline{\lambda}}(\phi)} f_\lambda(\psi) \\
&= \langle \phi, K_1\psi\rangle_{\mathcal{C}_s^\epsilon} \,.
\end{aligned}$$

f_λ is the distribution associated to $|j, \lambda, \sigma\rangle$

- The equation $\langle \Delta F^2\rangle = -s^2 - \dfrac{1}{4} + |\lambda|^2 = 0$ then admits a circle of solutions, and we choose

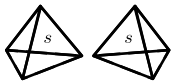
$$\hat{\lambda} = \frac{i}{2} - s = ij, \quad \hat{\sigma} = 1$$

Compare with $m = j$ in the space-like EPRL case

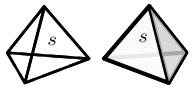
New parametrization for 

- The expression for Ψ_{ab} under this choice then simplifies into polynomials. The CH choice did not allow for an explicit expression. (Liu, Han. '18)
- Although much simpler than before, the states are still qualititvely more involved than those associated to other types of interfaces.

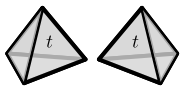
terms of the spins, as outlined in section 2. The states then read



$$\Psi^{j}_{ab}(\mathbf{z}) = \sqrt{2j+1}\,\langle z,\, z\rangle_{\mathbb{1}}^{j(i\gamma-1)-1}\,\langle z,\, +_{ab}\rangle_{\mathbb{1}}^{2j}\,, \tag{2.23}$$



$$\Psi^{k,\tau}_{ab}(\mathbf{z}) = \sqrt{2k-1}\,\Theta\left(\tau\,\langle z,\, z\rangle_{\sigma_3}\right)\left(\tau\,\langle z,\, z\rangle_{\sigma_3}\right)^{k(i\gamma+1)-1}\left(\tau\,\langle \tau_{ab},\, z\rangle_{\sigma_3}\right)^{-2k}\,, \tag{2.24}$$



$$\Psi^{s,\epsilon,\tau}_{ab}(\mathbf{z}) = \sqrt{\mu_\epsilon(s)}\Theta\left(\tau\,\langle z,z\rangle_{\sigma_3}\right)\left(\tau\,\langle z,z\rangle_{\sigma_3}\right)^{-2is-\frac{1}{2}} A^{j}_{\tau\frac{n}{2},1}\,\langle z,l^-_{ab}\rangle_{\sigma_3}^{(i+\gamma)s-\frac{1}{2}}\,\langle l^-_{ab},z\rangle_{\sigma_3}^{(i-\gamma)s-\frac{1}{2}}\,, \tag{2.25}$$

$$\tilde{\Psi}^{s,\epsilon,\tau}_{ab}(\mathbf{z}) = \sqrt{\mu_\epsilon(s)}\Theta\left(\tau\,\langle z,z\rangle_{\sigma_3}\right)\left(\tau\,\langle z,z\rangle_{\sigma_3}\right)^{-2is-\frac{1}{2}} A^{j}_{\tau\frac{n}{2},0}\,\langle z,l^-_{ab}\rangle_{\sigma_3}^{(i+\gamma)s-\frac{1}{2}}\,\langle l^-_{ab},z\rangle_{\sigma_3}^{(i-\gamma)s-\frac{1}{2}}\,\cdot$$
$$\cdot\,\frac{1}{2s}\left[\left((i+\gamma)s-\frac{1}{2}\right)\frac{\langle z,l^+_{ab}\rangle_{\sigma_3}}{\langle z,l^-_{ab}\rangle_{\sigma_3}} - \left((i-\gamma)s-\frac{1}{2}\right)\frac{\langle l^+_{ab},z\rangle_{\sigma_3}}{\langle l^-_{ab},z\rangle_{\sigma_3}}\right]\,, \tag{2.26}$$

where the general notation $|\cdot_{ab}\rangle$ stands for a rotated state $|h^{T\,-1}_{ab}\cdot\rangle$. We differentiate between

(JDS, Steinhaus '21)

The roadmap

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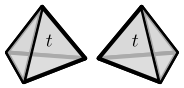
Representation theory of the special linear group

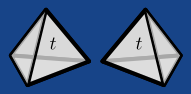
Asymptotic analysis

Posing the asymptotic problem

New parameterisation for 

Geometry from critical point conditions

Conclusions and discussion

Critical points for 

- Recall the main goal. The vertex amplitude reads

$$A_v = \int_{\mathrm{SL}(2,\mathbb{C})} \prod_{a=1}^{n} \mathrm{d}g_a \delta(g_n) \prod_{b<a} \langle D^\chi(g_a)\Psi_{ab}, D^\chi(g_b)\Psi_{ba}\rangle \ ,$$

which we now write as

$$A_v = \int_{\mathrm{SL}(2,\mathbb{C})} \prod_{a=1}^{n} \mathrm{d}g_a \delta(g_n) \prod_{a<b} \int_{\mathbb{C}P} \omega(\mathbf{z}_{ab}) f_{ab}(\mathbf{z}_{ab}, g_a, g_b)\, e^{\Lambda S_{ab}(\mathbf{z}_{ab}, g_a, g_b)} \ .$$

- The parachronal action can then be shown to reduce to

$$S_{ab} = (i+\gamma) s_{ab} \ln \frac{\langle l^-_{ba}, g_b^T z\rangle_{\sigma_3}}{\langle l^-_{ab}, g_a^T z\rangle_{\sigma_3}} + (i-\gamma) s_{ab} \ln \frac{\langle g_b^T z, l^-_{ba}\rangle_{\sigma_3}}{\langle g_a^T z, l^-_{ab}\rangle_{\sigma_3}} \ ,$$

where:

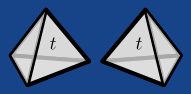
z is a spinor **integration variable**

g_a is a group **integration variable**

s_{ab} is a spin determined by the **data**

h_{ab} is a SU(1,1) element determined by the **data**

$$l^\pm_{ab} = \frac{1}{\sqrt{2}} h_{ab} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \ , \quad \langle u, v\rangle_{\sigma_3} = u^\dagger \sigma_3 v \ .$$

Critical points for 

- Action is purely imaginary, and expanding

$$|g_a^T z\rangle = \alpha_{ab}\left(|l_{ab}^+\rangle + \beta_{ab}\,|l_{ab}^-\rangle\right)\,,$$

we obtain variations

$$\delta_{\mathbf{z}_{ab}} S_{ab} = \left(\frac{\langle l_{ba}^-|\,\sigma_3 g_b^T}{\alpha_{ba}} - \frac{\langle l_{ab}^-|\,\sigma_3 g_a^T}{\alpha_{ab}}\right)\delta\mathbf{z}_{ab}\,,$$

$$\delta_{g_a} S_{ab} = -(i+\gamma)s_{ab}\,i\epsilon_l\left(\langle l_{ab}^-, H^l l_{ab}^+\rangle_{\sigma_3} + \beta_{ab}\,\langle l_{ab}^-, H^l l_{ab}^-\rangle_{\sigma_3}\right)\,, \qquad \begin{aligned} g_a(\epsilon) &= g_a e^{i\epsilon_l H^l} \\ H &= (\vec{F}, i\vec{F})\,, \end{aligned}$$

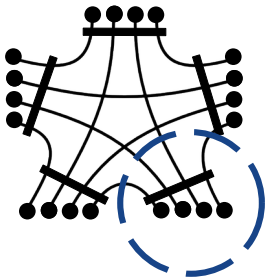
which define the critical points together with

$$\begin{cases} \delta_{\mathbf{z}_{ab}} S_{ab}(\mathbf{z}_{ab}, g_a, g_b) = 0,\ \forall\, a, b\,, \\ \sum_{b>a} \delta_{g_a} S_{ab}(\mathbf{z}_{ab}, g_a, g_b) = 0,\ \forall\, a\,. \end{cases}$$

Bivector constraints

Closure conditions

Geometry from closure conditions



$$\sum_{b:\,\mathrm{t.l.}\,ab} s_{ab}\left(\langle l^-_{ab}, \vec{F} l^+_{ab}\rangle_{\sigma_3} + \beta_{ab}\,\langle l^-_{ab}, \vec{F} l^-_{ab}\rangle_{\sigma_3}\right) = 0\,, \qquad \vec{F} = (J_3, K_1, K_2)\,,$$

$$l^\pm_{ab} = h_{ab}\cdot l^\pm\,, \quad \underbrace{h_{ab}}_{\text{Boundary data}} \in \mathrm{SU}(1,1)\,, \qquad l^\pm = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} \pm \begin{pmatrix}0\\1\end{pmatrix}\right].$$

- The states $l^\pm$ are eigenstates of the K_1 generator of SU(1,1), and

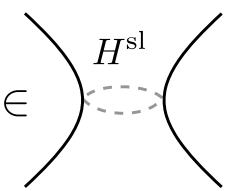
$$\langle l^\pm, \vec{F}\, l^\pm\rangle = \mp\frac{i}{2}\,(0,1,0) \;\in \overbrace{H^{\mathrm{sl}}}^{\text{Unit space-like vector}}.$$

- The adjoint action on the algebra $\mathfrak{su}(1,1) \simeq \mathbb{R}^{2,1}$ is an isometry, so it generates the space-like hyperboloid via

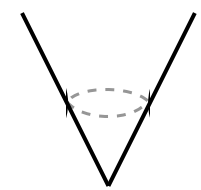
$$\begin{aligned}\langle l^\pm \,|\, h^{-1}\, F_i h\cdot l^\pm\rangle\, \hat{e}^i &= \langle l^\pm \,|\, h^\dagger \sigma_3\, F_i h\, \cdot l^\mp\rangle\, \hat{e}^i \\ &= \pm\frac{i}{2}\,(-\sinh t,\, -\cosh t\cos\phi,\, \cosh t\sin\phi)\,.\end{aligned}$$

- Closure condition includes vectors

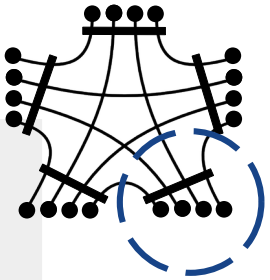
$$\vec{m}_{ab} = 2i\,\langle l^+_{ab}, \vec{F} l^-_{ab}\rangle \;\in$$



and $\quad \tilde{m}_{ab} = 2\,\langle l^-_{ab}, \vec{F} l^-_{ab}\rangle \;\in$



Geometry from closure conditions



- After separating the real and imaginary parts, we find

$$\forall \text{ t.l. } a\,, \sum_{b:\,\text{t.l.}ab} s_{ab}\,(\vec{m}_{ab} + \Im\beta_{ab}\tilde{m}_{ab}) = 0\,, \qquad \forall \text{ t.l. } a\,, \sum_{b:\,\text{t.l.}ab} s_{ab}\Re\beta_{ab}\tilde{m}_{ab} = 0\,.$$

- The real part $\Re\beta_{ab}$ can be made to vanish under a choice of the τ parameters, but the constraint $\Im\beta_{ab} = 0$ is not dynamically imposed.

- We have Minkowski's theorem in $\mathbb{R}^{2,1}$:

Theorem 4 (Minkowski's theorem). *Let $\{v_f, A_f\}$ be a set consisting of unit vectors v_f in either $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$ and positive numbers A_f. Suppose such vectors are not all co-planar, no vector is light-like, and the following holds*

$$\sum_f v_f A_f = 0\,. \tag{D.7}$$

Then there exists a unique convex polyhedron in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$, respectively, such that A_f is the area of its face f and v_f points orthogonally outward to the face.

(JDS, Steinhaus '21)

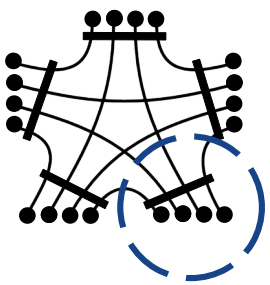
Geometry from closure conditions

- After separating the real and imaginary parts, we find

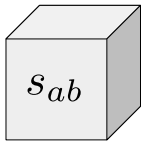
$$\forall \text{ t.l. } a\,, \sum_{b:\,\text{t.l.}ab} s_{ab}\,(\vec{m}_{ab} + \Im\beta_{ab}\tilde{m}_{ab}) = 0\,, \qquad \forall \text{ t.l. } a\,, \sum_{b:\,\text{t.l.}ab} s_{ab}\Re\beta_{ab}\tilde{m}_{ab} = 0\,.$$

- The real part $\Re\beta_{ab}$ can be made to vanish under a choice of the τ parameters, but the constraint $\Im\beta_{ab} = 0$ is not dynamically imposed.

- For the particular case in which $\Im\beta_{ab} = 0$, the closure condition uniquely determines a time-like polyhedron with areas given by the spins s_{ab}.



$$\forall \text{ t.l. } a\,, \sum_{b:\,\text{t.l.}ab} s_{ab}\,\vec{m}_{ab} = 0$$



Call these *geometrical configurations*
$\Im\beta_{ab} = 0$

Geometry from bivector constraints

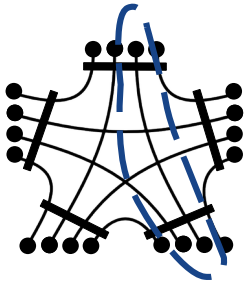
- Recall the bivector constraints associated to a time-like interface:

$$\frac{\langle l^-_{ba}|\,\sigma_3 g_b^T}{\alpha_{ba}} - \frac{\langle l^-_{ab}|\,\sigma_3 g_a^T}{\alpha_{ab}} = 0\,, \qquad |g_a^T z\rangle = \alpha_{ab}\left(|l^+_{ab}\rangle + \beta_{ab}\,|l^-_{ab}\rangle\right)\,.$$

- For geometrical configurations, one may show this is equivalent to

$$m^{\mu\nu}_{ab}\, g_a J_{\mu\nu} g_a^{-1} = m^{\mu\nu}_{ba}\, g_b J_{\mu\nu} g_b^{-1}$$

were $J_{\mu\nu}$ are the SL(2,C) generators, and $m^{\mu\nu} = \star\left[(\xi\vec{m},0)\wedge(0,\vec{1})\right]^{\mu\nu}$.



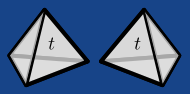
- Considering the canonical spin homomorphism

$$\pi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(3,1)$$
$$g \mapsto \frac{1}{2}\mathrm{Tr}\left(g\sigma_\mu g^\dagger \sigma_\nu\right)\hat{e}^\mu \otimes \hat{e}^\nu\,,$$

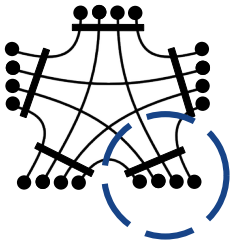
one finds the (actual) bivector constraints

$$\pi(\overline{g}_a)^{\wedge 2}\left((\vec{0},1)\wedge(\xi\vec{m}_{ab},0)\right) = \pi(\overline{g}_b)^{\wedge 2}\left((\vec{0},1)\wedge(\xi\vec{m}_{ba},0)\right)\,.$$

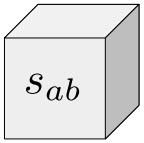
ξ is a fixed 3d rotation matrix

Dominant configurations for 

- Recall closure:



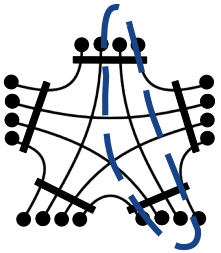
$$\forall \text{ t.l. } a\,, \sum_{b:\,\text{t.l.}ab} s_{ab}\,\xi\vec{m}_{ab} = 0$$



- Embedd 3d polyhedra in 4d Minkowski:

$$P_a^t \subset \mathbb{R}^{2,1} \hookrightarrow \hat{P}_a^t \subset \mathbb{R}^{3,1}$$
$$\text{s.t. } \hat{P}_a^t \perp \pi(\overline{g}_a)(\vec{0},1)\,,\ \hat{P}_{ab}^t \perp \pi(\overline{g}_a)(\xi\vec{m}_{ab},0) \text{ if } P_{ab}^t \perp \vec{m}_{ab}$$

- Bivector constraints then determine a gluing of faces of polyhedra according to the prescribed combinatorics:



$$\pi(\overline{g}_a)^{\wedge 2}\left((\vec{0},1)\wedge(\xi\vec{m}_{ab},0)\right) = \pi(\overline{g}_b)^{\wedge 2}\left((\vec{0},1)\wedge(\xi\vec{m}_{ba},0)\right)$$

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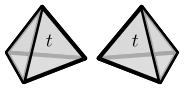
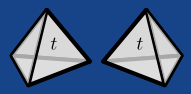
New parameterisation for 

Geometry from critical point conditions

Conclusions and discussion

Semiclassical behaviour of 

- The amplitude for parachronal interfaces,

$$A_v = \int_{\mathrm{SL}(2,\mathbb{C})} \prod_{a=1}^{n} g_a \delta(g_n) \prod_{a<b} \int_{\mathbb{C}P} \omega(\mathbf{z}_{ab}) f_{ab}(\mathbf{z}_{ab}, g_a, g_b)\, e^{\Lambda S_{ab}(\mathbf{z}_{ab}, g_a, g_b)} \,,$$

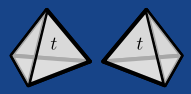
  is dominated when $\Im\beta_{ab} = 0$ by gluings of 3d Minkowski polyhedra into 4d polyhedral complexes. The gluing respects the prescribed combinatorics, and the 2d areas are given by the spins.

- One can evaluate the action at these geometrical configurations, finding the Regge action

$$S_v = i\gamma \left[\sum_{ab}^{\text{para.}} s_{ab}\, \theta_{N_a^{(\vec{0},1)}, N_b^{(\vec{0},1)}} \right] + i \left[\sum_{ab}^{\text{para.}} s_{ab}\, \ln \left| \frac{\alpha_{ba}}{\alpha_{ab}} \right|^2 \right] ,$$

  where $\theta_{N_a^{(\vec{0},1)}, N_b^{(\vec{0},1)}}$ is the dihedral angle between the normals to two 3d polyhedra.

Non-geometrical configurations ($\Im\beta_{ab} \neq 0$) are not sub-leading. Not all dominant configurations recover Regge geometry for Regge data, suggesting the model needs further constraining.

Semiclassical behaviour of 

- The function that precedes the exponential reads

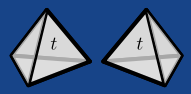
$$f_{ab}(\mathbf{z}_{ab}, g_a, g_b) = \frac{N^j_{\epsilon,\tau_{ab},\tau_{ba}}}{2} \Theta(\tau_{ab}\,\Re\beta_{ab})\Theta(\tau_{ba}\,\Re\beta_{ba})\,|\alpha_{ab}\alpha_{ba}|^{-2}\,\cdot$$
$$\cdot\,(\tau_{ab}\tau_{ba}\,\Re\beta_{ab}\,\Re\beta_{ba})^{-\frac{1}{2}}\left[(2s_{ab}-i)\,\Im\beta_{ab} - 2\gamma s_{ab}\,\Re\beta_{ab}\right]\,,$$

  and it shares the same branch structure as the approximation found in (Liu, Han. '18).

- The branch points coincide with critical points at geometrical configurations, and the function is indefinite at those points.

- No explicit expression (afaik) for asymptotics of multidimensional integrands where critical and branch points coalesce

> We do not have an explicit asymptotic formula for parachronal amplitudes; this needs further research!

Semiclassical behaviour of 

- It has been pointed out in the literature that the implementation of the simplicity constraints for parachronal interfaces could be problematic. Here we reproduce an observation from Conrady:

  The simplicity constraints for SU(1,1) read (Conrady, '10)

$$\vec{C} = \begin{pmatrix} J_3 \\ K_1 \\ K_2 \end{pmatrix} + \frac{1}{\gamma} \begin{pmatrix} K_3 \\ -J_1 \\ -J_2 \end{pmatrix} .$$

  Just as in the original EPRL case, $\vec{C}\,|\psi\rangle = 0$ cannot be imposed directly, since the algebra of constraints does not close. The usual prescription is to construct the master constraint

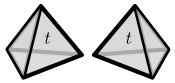
$$M = C_i \eta^{ij} C_j \,, \quad \eta_{ij} = \mathrm{diag}(1, -1, -1) \,,$$

  and demand $M\,|\psi\rangle = 0$. This condition is equivalent to the usual CH restrictions on the spins.

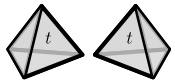
- Classically, it already is the case that $M = 0 \not\Rightarrow \vec{C} = 0$.

- This method agrees with the "weak imposition" procedure, but the above considerations suggest that for both methods simplicity might fail.

Summary and outlook

- 1) An **alternative parametrization** for  can be obtained, allowing for a closed-form expression for the associated action.

- 2) There exists a **Minkowski theorem** for 3d polyhedra in Minkowski space-time, allowing an extension of the analysis to non-simplicial foams.

- 3) The amplitude of  interfaces is **not exclusively dominated by geometrical configurations** for Regge boundary data.

- 4) We do not yet have a good control on the **explicit asymptotic formula** of parachronal amplitudes.

> One should understand if and how the non-geometrical configurations relate to the failing of simplicity. This may clarify how to further constrain the CH extension.

Thank you for your attention!