Flux Coherent States

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Work in collaboration with

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Papers:

1. Oriti, Pereira, LS JPA 2012 1110.5885
2. Oriti, Pereira, LS CQG 2012 1202.0526
3. Pittelli, LS to appear soon

Overview

Long term: General problem of recovering semiclassical gravity once a quantum/emergent gravity model has been proposed.

Many different approaches: Semiclassical states/large N/composite fields/phase transitions/hydrodynamic limit will connect the microscopic d.o.f. with the macroscopic ones (e.g. classical phase space of GR+constraints)

Short term: coherent states in LQG (not just per se, but also for their impact on GFT)

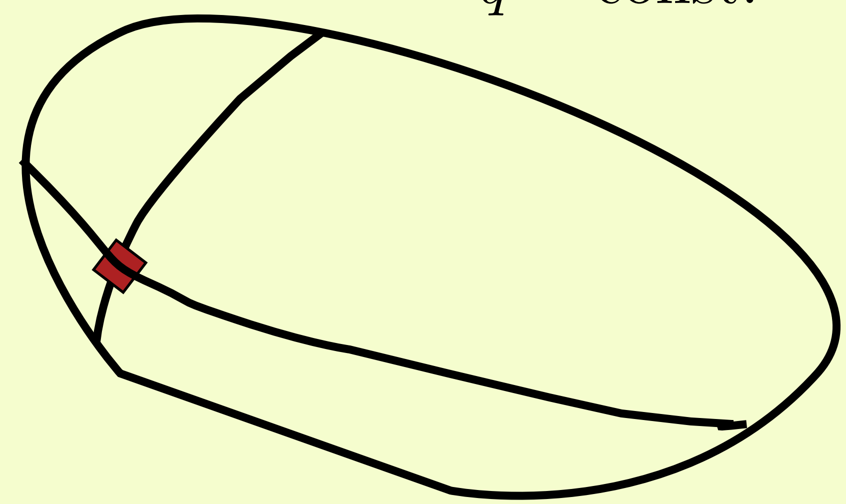
1st part: a proposal for new coherent states (with insights from NC Fourier transform)[+epsilon: elucidate the relations with HK coherent states.]

2nd part: coherent states for collective/large scale observables.

Disclaimer: several proposals for coherent states. I will focus only on one (heat kernel). Generalizations might be possible.

Intro: coherent states

Coherent state as the closest possible approximation to a phase space point.



$$\Delta p \simeq \Delta q \simeq \hbar^{1/2}$$

One possible approach to access the semiclassical limit of a quantum theory is to look for coherent states.

Caveat: discussion is at the kinematical level, but indeed dynamics is essential: one has to check that the CS are evolving nicely.

Part I

Gaussian as heat kernel on the line

Gaussian: coherent state peaked on a point of the classical phase space

$$|(x_0, p_0)_t\rangle \longrightarrow \langle x|(x_0, p_0)_t\rangle = \frac{1}{t^{1/2}} \exp\left(\frac{(x - x_0 - ip_0)^2}{4t}\right)$$

Creation annihilation operators

The scale controls the magnitude of the fluctuations.

$$\hat{a}(t) = t\hat{x} + \frac{i}{2\hbar t}\hat{p} \qquad \hat{a}|(0,0)_t\rangle = 0 \qquad \hat{a}|z\rangle = z|z\rangle \qquad z = \frac{x_0 + ip_0}{t}$$

Notice that one can define the real gaussian as the solution to the heat equation on the real line.

$$|z\rangle_t = \exp(z\hat{a}^\dagger(t))|(0,0)_t\rangle = |(x_0, p_0)_t\rangle$$

Analytic continuation

Peak on phase space

Translation in the direction of the momentum $\exp\left(-ip_0\hat{x}\right)$

Classical discrete phase space

Classical phase space parametrized (at fixed 3-dimensional slice) by triads and connections

$$\{e^i_a(x), A^b_j(y)\} = \gamma\delta^b_a\delta^i_j\delta^3(x-y)$$

To construct quantum states the strategy is to pass from continuum structures to configurations with support on graphs.

From now on, we forget about the continuum phase space, and we will be focussing on the description of geometry in terms of decorated graphs (ideally, a sampling of the continuum geometry, more on this later...)

We will consider a fixed graph, suitably constructed, whose edges are decorated with SU(2) elements containing information about the connection and su(2) elements associated to integrated fluxes

The embedding is fixed too. One has to consider the case of dynamical graphs and dependence on the embedding.

Classical discrete phase space/2

Triads and connections are stored into smeared quantities (fluxes & holonomies): parallel transports on the edge of the (embedded) graph and the flux as certain integral over plaquettes of the dual triangulation

$$E^i_{e,x_0} := tr\left(\frac{i\sigma^i}{2}\left[\int_{S_e} h(x_0 \to x) * E(x) h^{-1}(x_0 \to x)\right]\right)$$

$$A^a_i(x) \to h(\gamma) = \mathcal{P}\exp\left(-i\int_\gamma A^a T_a\right)$$

Semiclassical states that we are seeking will store information about these discrete data specified on a given graph.

This is a limitation of the discussion, further work is needed (ongoing project e.g. in GFT)

Quantization

Classical phase space coordinates lead to the holonomy-flux algebra

$$[\hat{E}^i_e, \hat{h}_{e'}] = i\hbar(8\pi G_N \gamma)\delta_{ee'} R^i \triangleright h_e$$

$$\left[\hat{E}^i_e, \hat{E}^j_{e'}\right] = i\hbar(8\pi G_N \gamma)\epsilon^{ij}{}_k \delta_{ee'} \hat{E}^k_e$$

Note that h and fluxes are not a canonical basis as the familiar p&q's of the particle on a line.

Important: In the rest of the discussion I will redefine the fluxes by scaling them with an area, so that they are dimensionless.

Even more important: we want to compare quantum discrete states to classical discrete geometries. The reconstruction of a continuum (intrinsic and extrinsic) geometry is an additional step. The uncertainties in determining the corresponding continuum geometry will involve both classical and quantum effects.

Thiemann & collaborators 2001

Coherent States for LQG

Kinematical states constructed out of graphs, with L links and V vertices

The wavefunction will be a function of L variables (normally group elements), one for each link of the graph.

Furthermore, gauge invariance is imposed by means of projection (through integrations on each vertex of the graph)

$$\mathcal{L}^2\left((\mathrm{SU}(2))^L/(\mathrm{SU}(2))^V\right)$$

Already a lot is known on coherent states for LQG, at least at the kinematical level (Hamiltonian constraint still pending)

Applied in various contexts (LQG, spinfoam boundary states etc.)

Hall's states & analytic continuation

Hall's states: constructed with the smearing of the Dirac delta on Lie group with the heat kernel, and then analytically continue the peak group element.

Solve the heat equation, with initial condition given by the Dirac delta

$$\partial_t K(t,g) = \triangle_G K(t,g) \qquad K(t,g) = \sum_j d_j e^{-tC_j} \chi_j(h)$$

$$\psi_g^t(h) = K^t(hg^{-1}) \to \psi_G^t(h) = K^t(hG^{-1}), \qquad G \in \mathrm{SL}(2,\mathbb{C})$$

These states are structurally very similar to gaussians.

Their peakedness properties are controlled by the heat kernel time parameter.

Properties were discussed in depth in the series of papers by Thiemann and collaborators

Flux representation of LQG

Freidel & Majid
Joung, Mourad, Noui

All the previous work has been based on the construction of wavefunctions as functions on (several copies of) SU(2)

The possibility of having a Fourier transform between SU(2) and its Lie algebra gives us the opportunity to work with wavefunctions constructed on several copies of su(2).

Possibility given by properly defined plane waves

$$e_g(x) = \exp\left(i\xi^i(g)x^j\delta_{ij}\right)$$

$$\xi^i(g) = -\frac{i}{2}\mathrm{Tr}(|g|\sigma^i)$$

Flux representation of LQG, in which everything is expressed in terms of functions on su(2) endowed with a star product

Baratin, Dittrich, Oriti, Tambornino

$$e_g(x) \star_x e_h(x) := e_{gh}(x)$$

D. Oriti, R. Pereira & L.S. 2011

New (?) states

Heat kernel proposal: define the coherent states edgewise with heat kernels by analytically continue the peak of the kernel from a group element of SU(2) to a group element of SL(2,C).

Reason: relations between SU(2), T*SU(2), SL(2C)

What about a state constructed multiplying the heat kernel with a plane wave, instead?

$$\Psi^t_{(h_0,x_0)}(h) = K^t(hh_0^{-1})e_h(-x_0)$$

Reason: multiplication by a plane wave can be seen as a translation

$$\mathcal{F}\left(\Psi^t_{(h_0,x_0)}\right)(x) = \mathcal{F}\left(K^t_{(h_0)}\right)(x - x_0)$$

Properties (List)

	Hall cont'd	Plane wave
Holonomy expec.value*	correct + O(t)	correct + O(t)
Flux expec. value	correct + O(t)	Exact
Fluctuations	O(t)	O(t)
Resolution of identity	Yes	Yes
Overlap properties	Yes	Yes
Annihilation Op.	Yes	???

In summary

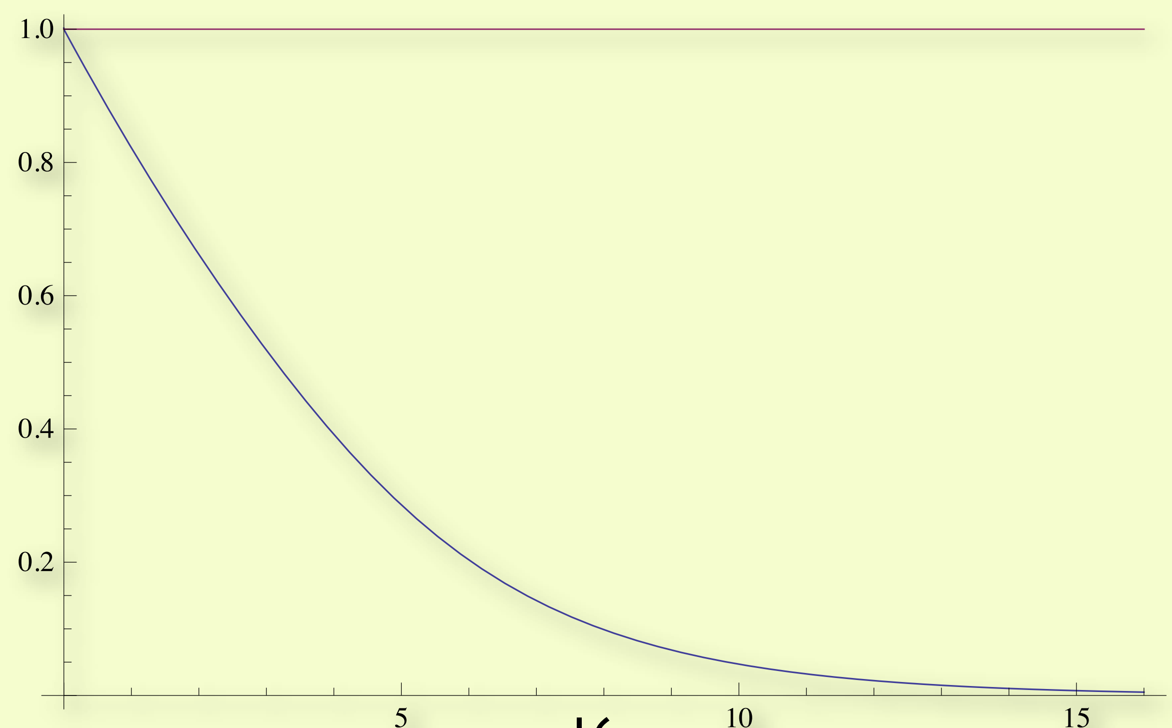
- Message: with a bit of work one might construct adapted coherent states such that their characteristics are optimal w.r.t. a given choice of variables parametrizing the classical phase space (at the discrete level)
- However, so far, the state is specified on a given graph and edge per edge, i.e. still in a microscopic sense
- This corresponds to the specific choice of parametrization of the discrete phase space
- Other choices have been considered in the literature

Comments

$$\psi(h) = \frac{1}{||K^t||} K^t(hg^\dagger)$$

The expectation value of the holonomy operator is a bit off the peak for the expectation value. Consider a state which is peaked (i.e. the modulus of the wavefunction is maximum) on an SU(2) element

$$\langle D^{1/2}_{ab}(h) \rangle = F(t) D^{1/2}_{ab}(g) \qquad F(t) \approx 1 - 0.19t + 0.01t^2$$



Comment cont'd

The situation can be ameliorated with the introduction of operators, functions of the holonomy, that correct this feature at least for peaks close enough to the identity element of SU(2).

$$\xi^i(g) = -\frac{i}{2}\mathrm{Tr}(|g|\sigma^i)$$

$$\varphi^i(g) = \xi^i(g) f(\xi^2(g))$$

$$\left[\hat{E}^i, \hat{\varphi}^j\right] \approx i\delta^{ij}\mathbb{I} + O(\varphi^2)$$

$$\left\langle \varphi^i(h) \right\rangle = \varphi(g) + O(\varphi(g)^2)$$

This result is independent from t, but depends on the position of the peak. However, for holonomies close to the identity (small curvature), it is ok. The function f can be computed

Relation to the HK?

It is clear that these states are closely related to the HK. However, we were not able to make the relationship as clear as it is possible.

In Fourier space it is manifest, but given the presence of the star product it is not obvious to understand what the operator is (even though it should be some generalization of the Laplacian)

The expectation is correct.

$$\mathcal{F}\left(\Psi^t_{(h_0,x_0)}\right)(x) = \mathcal{F}\left(K^t_{(h_0)}\right)(x - x_0) \longrightarrow \exp_{\star_{x-x_0}}\left(-t(x-x_0)^2_\star\right)$$

$$\partial_t \exp_{\star_{x-x_0}}\left(-t(x-x_0)^2_\star\right) = -(x-x_0)^2_\star \star_{x-x_0} \exp_{\star_{x-x_0}}\left(-t(x-x_0)^2_\star\right)$$

$$\delta_{ij}x^i \star_x x^j \rightarrow \triangle \qquad \delta_{ij}(x-x_0)^i \star_{x-x_0} (x-x_0)^j \rightarrow ?$$

Answer: easy

These new coherent states are in fact heat kernels in disguise, with a modified laplacian replacing the standard one. One easily sees this with an analogy with a U(1) local gauge invariance.

$$\begin{tikzcd} v^j(h, x_0) = -\frac{i}{2}\mathrm{Tr}\left(\sigma^j |h| \sigma_k\right) x_0^k \arrow[r] & \mathcal{E}^i := E^i - v^i_{x_0}(h) \arrow[d] \\ |(h_0, x_0)\rangle_t = e^{t\mathcal{D}_{x_0}} |h_0\rangle & \mathcal{D}_{x_0} = -\delta_{ij}\mathcal{E}^i\mathcal{E}^j \arrow[l] \end{tikzcd}$$

This can be seen using the following property

$$E^i U_{x_0} = U_{x_0} \mathcal{E}^i_{-x_0}$$

There is also an annihilation operator

$$\hat{a} = e^{t\mathcal{D}_{x_0}} h e^{-t\mathcal{D}_{x_0}}$$

Properties (Updated List)

	Hall cont'd	Plane wave
Holonomy expec.value*	correct + O(t)	correct + O(t)
Flux expec. value	correct + O(t)	???
Fluctuations	O(t)	O(t)
Resolution of identity	Yes	Yes
Overlap properties	Yes	Yes
Annihilation Op.	Yes	YES

Part II

Sahlmann, Thiemann, Winkler 2001

Tensor product states

Choose a geometry (intrinsic and extrinsic)

Limitation to nice slices (low curvature)

Choose a graph and a given embedding (e.g. Poissonian sprinkling, method proposed by Bombelli, Corichi and Winkler)

Decorate the graph with the labels inferred from the embedding into the geometry

Construct a factorized state (before gauge invariance)

Optimize the various free parameters to minimize fluctuations

Based on a "microscopic sampling" of geometry

The problem(s)

When you have a simple plane wave, you do not sample it locally, but rather you specify a global charge.

Need coarse grained quantities referring to more global quantities (analogues of center of mass quantities, etc.)

When interactions are present, factorized states might be problematic.

We do not want to be semiclassical at the scale of the single edge of the graph

Example: fluid dynamics does not come from a bunch of semiclassical particles (rather, one assumes the existence of certain equilibrium conditions allowing for a coarse grained description of the full quantum density matrix in terms of fluid equations)

Advantage of LQG: already working with geometrical degrees of freedom (might be that the situation is more complicated, and that the metric degrees of freedom should appear as bound states in some pregeometric theory)

Sahlmann, Thiemann, Winkler   D. Oriti, R. Pereira, LS 2012.

Statistical properties of factorized states

Clearly, if we have an observable defined on each edge

$$\left\langle \otimes_e \psi_e \left| \sum_e \hat{O}_e \right| \otimes_e \psi_e \right\rangle = \sum_e \langle O_e \rangle$$

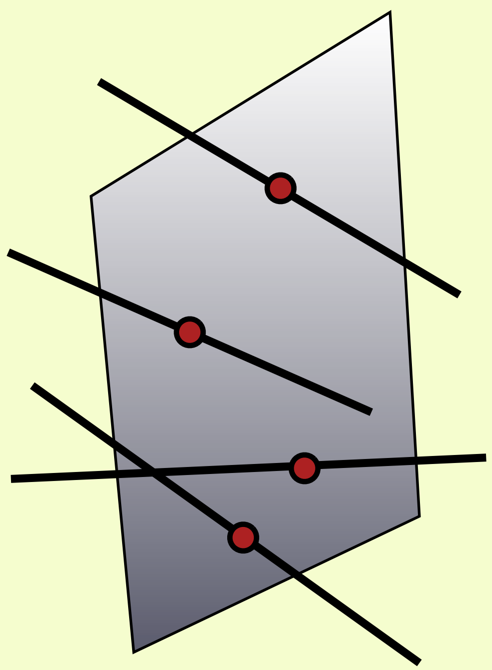
and

$$\Delta \left(\sum_e \hat{O}_e \right) = \sum_e \Delta O_e$$

While mean values have no problem, the fluctuations grow with the size of the system we are considering (in this sense they are not optimized)

Example: fluxes

Just to fix the ideas, let us assume that we want to consider the flux across a surface S intersecting several times the graph, on different links



We might need to construct a **non-factorized** state adapted to this observable.

$$\hat{E}^i(S) = \sum_{e_I \cap S \neq \emptyset} \hat{E}_{e_I}$$

and to the conjugate variable

$$\left[\hat{E}^i(S), \hat{\Phi}^j(S)\right] \approx i\delta^{ij}\mathbb{I}$$

Conjugate variable

The holonomy operator has the wrong commutation relation with fluxes to represent some form of canonically conjugate variable (the integrated connection would be one)

There is one approximate "canonically conjugated variable"

$$\varphi^i(g) = \xi^i(g) f(\xi^2(g))$$

$$\left[\hat{E}^i, \hat{\varphi}^j\right] \approx i\delta^{ij}\mathbb{I} + O(\varphi^2)$$

(and hence the nice slicing requirement!)

One possible choice of canonically conjugate variable is

$$\Phi^i(S) = \sum_{e_I \cap S \neq \emptyset} c_I \hat{\varphi}(h_{e_I}) \qquad c_I \sim 1/N$$

Intensive quantity
kind of average magnetic flux
(Electric flux was extensive)

Clearly there are many ambiguities: one has to find a complete parametrization of the discrete classical phase space in terms of extended surfaces/objects (analogy: phase space of a many particle system)

Wavefunction

All the quantities we are manipulating are gauge variant: they must be then trasported to the same reference point, along a give system of paths, when we want to combine them and construct consistently a state

For instance, in our simple example

$$\Psi^t(\{x\}) = \underbrace{\psi^t_{(h_0(S),E_0(S))}}_{\text{Expect. value of the collective conj. variable; Expect. value of the total flux across the surface S}} \left(\sum_{e_I \cap S \neq \emptyset} x_I \right) \otimes \psi_{\text{rest}}$$

Expect. value of the
collective conj. variable

Expect. value of the
total flux across the surface S

Consequence: while the observables associated to the single edges might get large quantum fluctuations, the collective one has (fixed) minimal fluctuations (independently from the number of microscopic edges)

More general

Consider a graph with N links (before gauge invariance is imposed), each link colored with a flux that is computed with respect to a unique reference point.

A GL(N) transformation (most general invertible linear transformation) maps the state sampled through the isolated links to a state sampled through links group together in various ways.

One state is not factorized with respect to the degrees of freedom/ coordinates with which the other state is constructed.

Bottomline: even when working with a fixed graph, one can use non-factorized states designed to store the information about extended portions of the geometry from the very beginning.

At the classical level is obviously the same, provided that an invertible map between micro and macro coordinates exists.

At the quantum level, statistical properties (e.g. fluctuations) can be adapted to the situation.

Where is the RG?

Scenario depicted so far: given the algebra of operators, dictated by the classical theory (modulo ordering ambiguities), you try to construct the wavefunctions (i.e. the Hilbert space of physical states)

Where is the RG?

If we use extended/collective variables, scales are appearing in the states! Not just the scale associated to the edges (naively the Planck scale) but rather scales associated to the extension of the geometrical structures used to define the states themselves.

Remark: there are indeed several scales. The curvature(s) scales, the scale of the observables to be optimized, and the sampling scale used to construct the scale (statistical in nature)

<u>More to be understood, here.</u>

Discussion

- Relevance: new boundary states for spinfoams/GFT
- Mean field approach to GFT (work in progress with A. Pittelli)

D. Oriti & LS 2010

- Shortcomings: staircase problems
- Limitation/1: need to understand how to fully parametrize the classical (discrete) phase space in terms of "extended" variables
- Limitation/2: single graph (even though a random one). LQG includes all the possible graphs. There a suitable weighting/equilibrium condition has to be introduced (GFT?).
- Dynamics: missing, but crucial to decide how to construct meaningful states (GFT again?)

Thank you