Loop quantum cosmos are never singular

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Outline

- Motivation
- Features of Effective Dynamics
- Classical Singularities in Cosmology: Complete Classification
- Illustration of singularities and their resolution via a Model
- General analysis of singularities
- Summary and further directions

Motivation

- Singularities are the boundaries of spacetime in the classical theory. Some common characteristics: inextendibility of geodesics, spacetime curvature diverges, all objects crushed to zero volume.
  – Interestingly singularities were first discovered in simplest models. Were considered as artifacts of highly symmetric models. Powerful singularity theorems of Penrose, Hawking and Geroch proved that singularities are generic features of classical spacetime.
- Singularities of GR are windows of new physics in quantum gravity. Point to the limitations of the classical theory. A quantum theory of spacetime is expected to cure all spacelike singularities.
- Using the technology of LQG, various symmetric models have been studied (e.g. LQC). Key prediction is a bounce at Planck scale and absence of big bang/crunch singularities. Results confirmed in a variety of models (including some inhomogeneous situations).
  – Is singularity resolution an artifact of symmetry reduction?
  – Or is there a non-singularity theorem for quantum geometry?
- Questions we are interested in: Is the singularity resolution generic in isotropic, flat, homoegenous LQC? Does LQC resolve all spacelike singularities(including future singularities)? Any counter examples (work of Cailleteau, Cardoso, Vandersloot & Wands (08))?
- Caveats: Assume effective spacetime picture with arbitrary matter. Exclude higher order quantum effects resulting from state dependent properties.

LQC: Homogeneous and Isotropic setting

Spatial homogeneity and isotropy: Manifold is $\mathbb{R}^3$. Introduce a fiducial flat metric and fiducial triad $\mathring{e}_i^a$ and co-triad $\mathring{\omega}_a^i$.
Symmetries $\Rightarrow$

$$A_a^i = c\,\mathring{V}^{-1/3}\mathring{\omega}_a^i, \quad E_i^a = p\,\mathring{V}^{-2/3}\,(\det\mathring{\omega})\,\mathring{e}_i^a$$

Basic variables: c and p satisfy $\{c,p\} = 8\pi G\gamma/3$.

Roberston-Walker metric:

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)$$

– Relation to scale factor:
$|p| = a^2$ (two possible orientations for the triad)
$c = \gamma\dot{a}$ (on the space of physical solutions of GR).

Elementary variables
– Holonomies: $h_k(\mu) = \cos(\mu c/2)\mathbb{I} + 2\sin(\mu c/2)\tau_k$, $\mu \in (-\infty,\infty)$.

Elements of form $\exp(i\mu c/2)$ – generate algebra of almost periodic functions

Hilbert space: $\mathcal{H}_{\rm kin} = L^2(\mathbb{R}_{\rm B}, d\mu_{\rm B})$

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Hamiltonian Constraint (flat model)

$$C_{\rm grav} = -\int_{\mathcal{V}} d^3x \, N \, \varepsilon_{ijk} \, F^i_{ab} \, (E^{aj} E^{bk} / \sqrt{|\det E|})$$

Procedure: Express $C_{\rm grav}$ in terms of elementary variables and their Poisson brackets

– Classical identity of the phase space:[a]

$$\varepsilon_{ijk} (E^{aj} E^{bk} / \sqrt{|\det E|}) \longrightarrow \mathrm{Tr}(h_k^{(\mu)} \{h_k^{(\mu)-1}, V\} \tau_i)$$

(Peak tied to the fiducial volume of the cell introduced to define symplectic structure)

– Express field strength in terms of holonomies and quantize.

Leads to two types of quantum modifications:

(i) Curvature modifications from field strength. Solely responsible for bounce at $\rho = \rho_{\rm crit} \sim 0.41 \rho_{\rm Pl}$.

(ii) Inverse triad corrections (also for the matter part). Not tied to any curvature scale in the flat model. Lack of predictive power.

[a] Thiemann (98)

To extract physics:

– Isolate a 'time' variable. Find physical Hilbert space, inner product and suitable (Dirac) observables. Construct semi-classical states at late 'times' and evolve them. Compare with classical trajectory. Extract predictions.

Quantization performed rigourously for various models[a b c d e f g]
Robustness of key features (i.e. bounce for generic states, supremum on energy density) proved using an exactly solvable model (sLQC)[h]

Interestingly, LQC admits an effective spacetime description. Effective dynamics an excellent approximation to quantum dynamics for states which are semi-classical at late times. Obtained using geometric formulation of QM. Hilbert space regarded as phase space which has a bundle structure: base space is classical phase space and states with same $(\langle \hat{q}^i \rangle, \langle \hat{p}_i \rangle)$ constitute fibers. Using semi-classical states find horizontal sections preserved by the quantum Hamiltonian to a desired accuracy[i].

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[a] Massless Scalar in Flat Universe: Ashtekar, Pawlowski, PS (06)

[b] Closed Universe: Ashtekar, Pawlowski, PS, Vandersloot (06); Szulc, Kaminski, Lewandowski (06)

[c] Anti-deSitter Universe: Bentivegna, Pawlowski (08)

[d] deSitter Universe: Ashtekar, Pawlowski (08)

[e] Bianchi-I Model: Ashtekar, Wilson-Ewing (08); Martin-Benito, Mena-Marugan, Pawlowski (08); Szulc (08)

[f] Gowdy Models: Martin-Bento, Garay, Mena Marugan (08)

[g] Massive Scalar (Inflationary potential): Ashtekar, Pawlowski, PS (08)

[h] Ashtekar, Corichi, PS (08)

[i] Taveras (08); PhD thesis of Willis (04)

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Effective Dynamics

Obtained from effective Hamiltonian (improved dynamics)

$$\mathcal{H}_{\rm eff} = -\frac{3}{8\pi G\gamma^2}\,\frac{\sin^2(\lambda\beta)}{\lambda^2}V + \mathcal{H}_{\rm matt}$$

where $\{\beta, V\} = 4\pi G\gamma$, $\beta := c/|p|^{1/2}$ and $V = |p|^{3/2}$.

Modified dynamical equations are obtained using Hamilton's equations of motion.

Modified Friedman equation:

$$H^2 = \frac{8\pi G}{3}\,\rho\left(1 - \frac{\rho}{\rho_{\rm crit}}\right), \quad \rho_{\rm crit} = \frac{\sqrt{3}}{(32\pi^2\gamma^3 G^2\hbar)} = 0.41\rho_{\rm Pl}$$

Modified Raichaudhuri equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,\rho\left(1 - 4\frac{\rho}{\rho_{\rm crit}}\right) - 4\pi G\,P\left(1 - 2\frac{\rho}{\rho_{\rm crit}}\right)$$

Quantum geomtery only affects gravitational part, hence matter part yields the unmodified conservation law: $\dot{\rho} = 3H(\rho + P)$. (Hence no change in equation of state: $w = P/\rho$).

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Some features of new physics

- Modified Friedman equation implies that $0 \leq \rho \leq \rho_{\rm crit}$. Hubble rate has an upper bound:

$$|H|_{\rm max} = \left(\frac{1}{\sqrt{3}\, 16\pi G\hbar\gamma^3}\right)^{1/2}$$

- When $\rho = \rho_{\rm crit}$, Hubble vanishes and the fate of the universe is determined by the Raichaudhuri equation:

$$\frac{\ddot{a}}{a} = 4\pi G(1+w)\rho_{\rm crit}$$

  For matter with $w > -1$ that is which satisfies null energy condition (NEC) loop quantum dynamics predicts a bounce and for matter with $w < -1$ universe undergoes a recollapse.

- There exists a critical value of scale factor isolated from the values at which energy density diverges classically. As an example for $w =$ constant, integrating conservation law:

$$a_{\rm crit} = a_o \left(\rho_{\rm crit}/\rho_o\right)^{3(1+w)} > 0$$

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Some features of new physics

- Spacetime curvature:

$$R = 6\left(H^2 + \frac{\ddot{a}}{a}\right) = 8\pi G\rho\left(1 - 3w + 2\frac{\rho}{\rho_{\rm crit}}(1+3w)\right)$$

It is bounded above in effective spacetime unless $w = P/\rho \to \infty$. That is when pressure diverges with energy density going to zero. Seems very pathological case but may arise in some scenarios.
– For matter satisfying NEC, maximum is reached for massless scalar:
$|R| = 48\pi G\,\rho_{\rm crit}$.

- When $\rho \ll \rho_{\rm crit}$, dynamics approximates GR:

$$H^2 = \frac{8\pi G}{3}\rho, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

and

$$R = 8\pi G(\rho - 3P)$$

Numerical simulations for various models show that GR is a good approximation already when $\rho \approx 0.01\rho_{\rm Planck}$.

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Types of Singularities (complete classification)

- Big Bang/Big Crunch: The scale factor vanishes in finite time. Energy density blows up. All of the curvature invariants diverge. Inevitable fate of the matter satisfying null energy condition (except Λ): $\rho + P \geq 0$.
- Type I singularities (Big Rip): Common in a class of dark energy models. Occur at a finite time. The scale factor, energy density and pressure diverge. All curvature invariants diverge. Dominant energy condition is violated: $|\rho \pm P| \geq 0$. The universe rips apart!
- Type II singularities (Sudden): Discovered by Barrow and co-workers. Occur at finite value of scale factor. As the singularity is approached the energy density vanishes but pressure diverges, causing divergence in spacetime curvature. Energy conditions may be violated only near the singularity.
- Type III singularities: Singularity occurs at a finite value of scale factor. Energy density, pressure and curvature invariants diverge.
- Type IV Singularity: Occurs at a finite value of scale factor. Energy density and pressure remain finite. Curvature invariants are bounded. However, curvature derivatives diverge.

Various works on future singularities

- Sami, PS & Tsujikawa (2006): Showed that the Type I big rip singularity occurring in certain phantom models $(w < -1)$ can be successfully resolved in LQC.
- Samart & Gumjudpai; Naskar & Ward (2007): Phantom dynamics with exponential potentials. Big rip avoided.
- Wu & Zhang; Fu, Yu & Wu (2008): Interacting phantom models. Type I singularity resolved.
- Kamenshchik, Kiefer, Sandhofer (2007): Some quantization aspects of Type II singularity studied.
- Cailleteau, Cardoso, Vandersloot & Wands (2008): Unbounded exponential potential in phantom model. A Type I singularity exists in classical theory. Quantum geometry converts it to a Type II singularity. Quantum geometry does not prevent divergence of spacetime curvature. Artifact of a particular potential/model? Unphysical matter? Breakdown of effective equations?

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Modeling singularities

Let us consider a general matter model based on the ansatz: [a]

$$P = -\rho - f(\rho), \quad f(\rho) = \frac{AB\rho^{2\alpha-1}}{A\rho^{\alpha-1} + B}$$

Using the conservation law (which is unmodified also in LQC):

$$\rho = \left(-\frac{A}{B} \pm \left(\frac{A^2}{B^2} - 6(\alpha - 1)A\ln\left(\frac{a}{a_o}\right)\right)\right)^{1/(1-\alpha)}$$

We probe the dynamics and properties of curvature invariant (R) using modified Friedman, Raichaudhuri equations and

$$\dot{R} = 6(\ddot{H} + 4H\dot{H})$$

Due to symmetries of the spacetime, investigating the behavior Ricci scalar is sufficient to understand all higher order invaraints.

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[a] Nojiri, Odintsov, Tsujikawa (2005).

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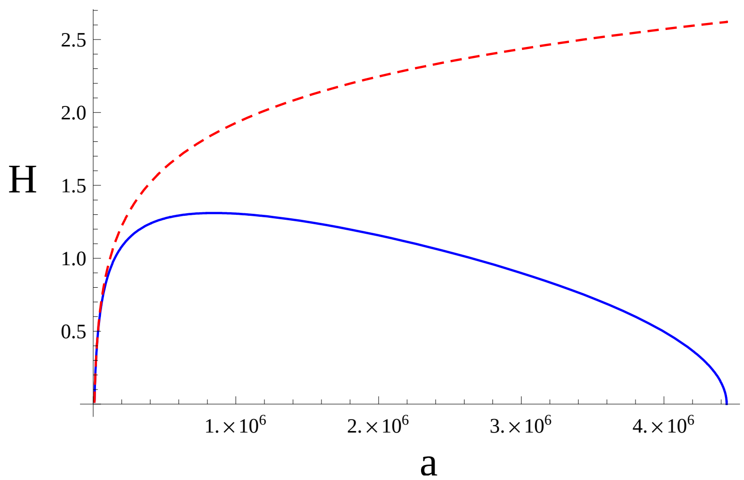
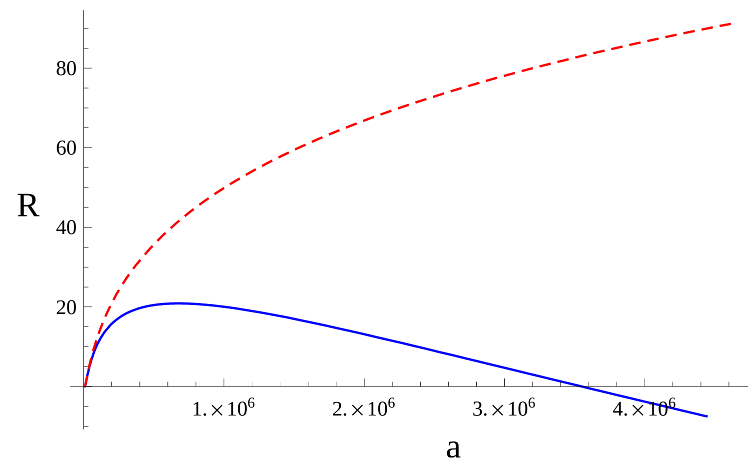
Type I Singularity and its Resolution

Occurs when parameters: $3/4 < \alpha < 1$, $A > 0$.

Classically the model has no initial singularity but only a future singularity where scale factor diverges along with $\rho, H, |R| \to \infty$ in finite time.





Due to upper bound on ρ in LQC, energy density can not diverge. Universe recollapses when $\rho = \rho_{\rm crit}$. Classical big rip is generically avoided.

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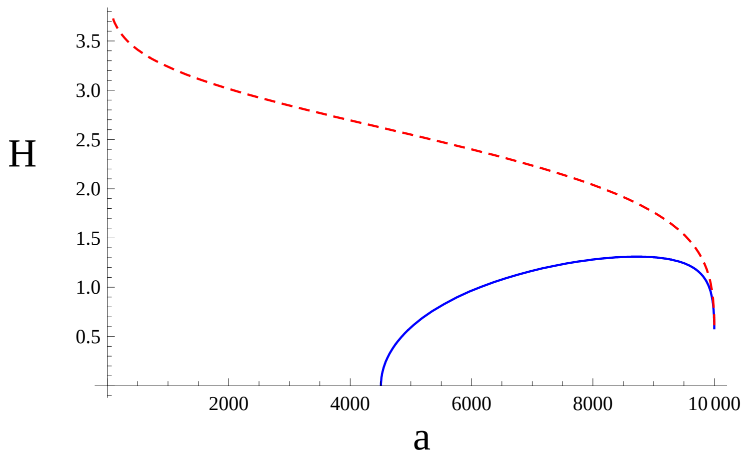
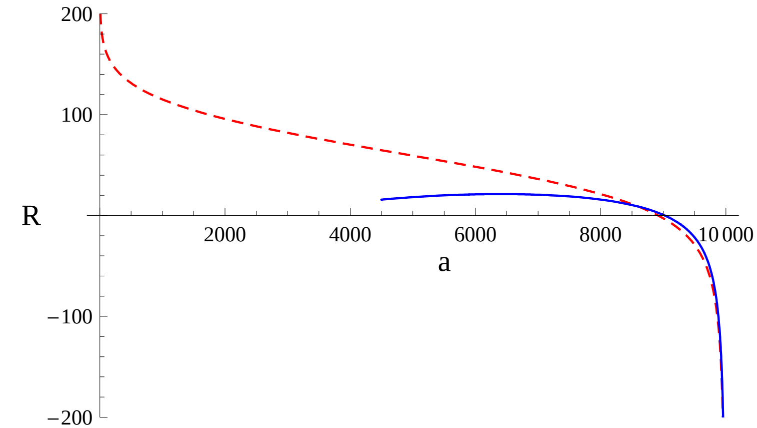
Type II Singularity and its Non-Resolution

A necessary condition to have a sudden singularity is $A/B < 0$.





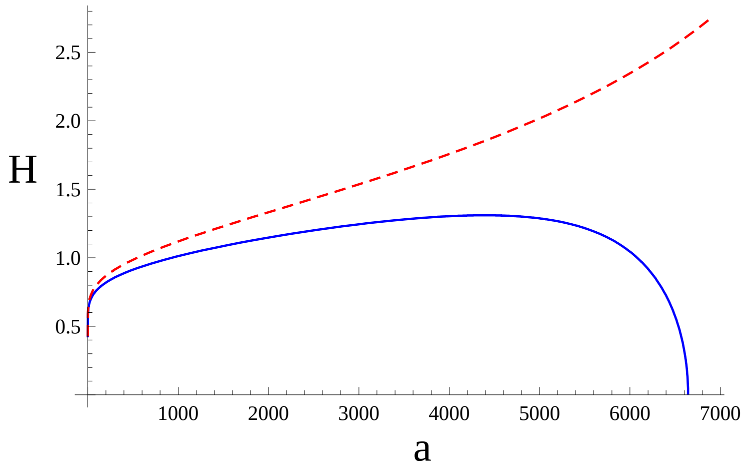
LQC resolves the initial (final) big bang (crunch) singularity. However, the curvature grows unbounded both in classical theory and LQC when the sudden singularity is approached.

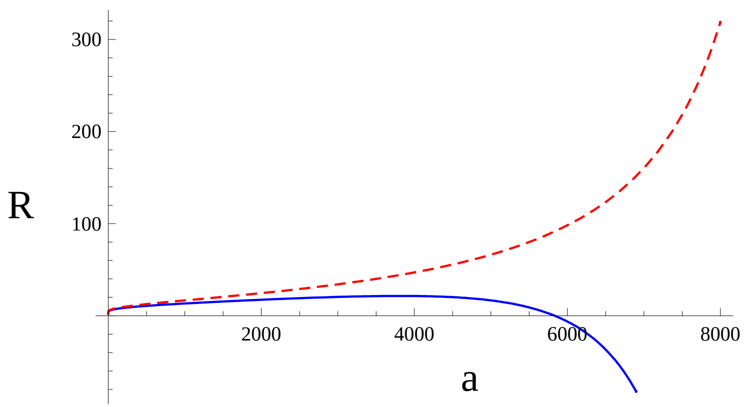
Quantum geometry does not control divergence of curvature. A generic feature of Type II models in LQC. (Generalization of conclusion by Cailleteau, Cardoso, Vandersloot & Wands (2008)).

Type III Singularity and its Resolution

Parameter: $\alpha > 1$.

In the classical theory energy density, pressure, Hubble, Ricci, ... diverge as $a \to a_o$ in finite time.





All curvature invariants and Hubble are bounded in LQC.
Type III singularity is resolved.

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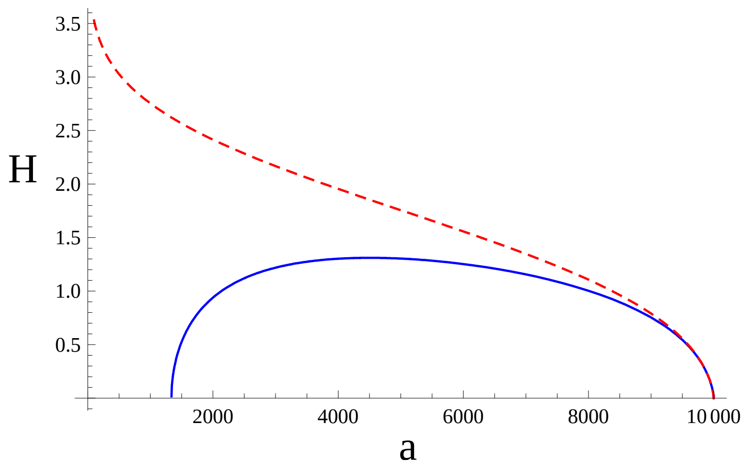
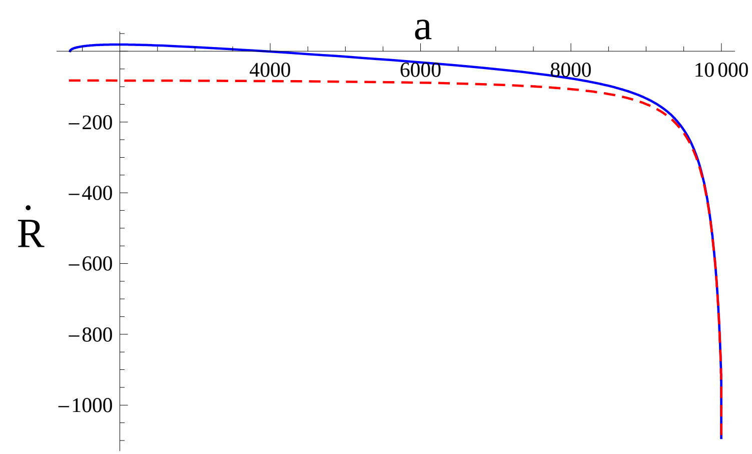
Type IV Singularity and its Non-Resolution

Parameter: $0 < \alpha < 1/2$. Given a value, it determines the order of dervative of R which diverges.





The Hubble and Ricci curvature are bounded and finite at the Type IV extremal event in both classical theory and LQC. However, derivative of R diverges. All curvature invariants are finite.

LQC does not resolve this curvature derivative singularity.

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– LQC resolves singularities in all those cases where $\rho \to \infty$. The scale factor at which divergence occurs is excluded from the effective spacetime. This includes Big Bang/Crunch, Type I and Type III singularities.

– In Type II singularities, $\rho \to 0$ and $P \to \pm\infty$ causing divergence of spacetime curvature at a finite value of scale factor. Since dynamical equations approximate those of classical GR when $\rho \ll \rho_{\rm crit}$, it is not surprising that quantum geometry does not resolve Type II singularities.

– Type IV singularities are not curvature divergent singularities. Only derivatives of curvature diverge. Since these also occur when $\rho \to 0$, quantum geometric effects play no role to resolve the Type IV singularities.

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When is a singularity really a singularity?

– Singularity is the boundary of the spacetime which can be reached by an observer in a finite time but beyond which spacetime can not be extended.

– Hawking, Penrose, Geroch Incompleteness Theorems: Existence of geodesics which can not be continued to arbitrary values of the affine parameter.

"A spacetime is singularity free if it is geodesically complete." *Sagredo* [a]

– For the Robertson-Walker metric, geodesic equations:

$$a^2 r' = \chi, \quad t'^{\,2} = \epsilon + \chi^2/a^2 \quad \chi = \mathrm{constt}$$

Also $t'' = -H\,(t'^{\,2} - \epsilon)$; $\epsilon = 0$ (unity) for null (particle) geodsics.

Geodesic equations break down only when $a \to 0$ and/or $H \to \pm\infty$.

[a] What is a singularity in GR? (Geroch, Ann. Phys. 48, 526 (1968))

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Nature of Singularities

What happens at or near the singularity? What does an in-falling observer experience? How strong are the effects of curvature?

In GR, there exist scenarios where curvature diverges yet there is no real singularity!

Ellis & Schmidt: A singularity is strong if in falling objects or observer (along with his/her apparatus) experience infinite tidal forces and are thus destroyed irrespective of their physical characteristics. Otherwise the singularity is weak.
– Weak singularities allow passage of strong detectors. Examples are shell crossing singularities in gravitational collapse scenarios. Tidal forces are finite and there is no physical singularity.

– Tipler's criteria: In the FRW case, singularity is strong if the volume vanishes (or diverges). The following integral diverges as $\tau \rightarrow \tau_e$:

$$\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' R_{ab} u^a u^b$$

– Krolak's criteria: Weaker than that of Tipler. For $\tau \rightarrow \tau_e$ the singularity is strong if following integral diverges:

$$\int_0^{\tau} d\tau' R_{ab} u^a u^b$$

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Big Bang/Crunch, Type I and Type III singularities have divergence in Hubble and/or vanishing scale factor. Geodesic equations break down and strength integrals diverge. These are events

– beyond which geodesics can not be extended

– which are strong curvature singularities

Type II and Type IV singularities occur at finite value of scale factor with Hubble finite. Geodesic equations do not break down and both Tipler and Krolak's integrals are finite (Fernandez-Jambrina, Lazkoz (2006)). These are events

– beyond which geodesics can be extended

– which are weak curvature singularities
(Note that in general, classical spacetime is still geodesically incomplete because there may exist a strong singularity in the past evolution).

Results from a general analysis

- Recall that in LQC, Hubble is bounded above and the scale factor never vanishes. For all Hubble divergent singularities there exists a critical value of scale factor at which bounce or recollapse takes place. All the singular values of scale factor are excluded from the effective spacetime. Dynamical equations never break down.

  – Quantum geometry binds the extrinsic curvature

  – No strong curvature singularities in LQC. Tipler and Krolak integrals are finite.

  – Geodesic equations do not break down in scenarios where they break down classically.

- Evolution near Type II and Type IV singularities mimics the one in classical theory. The Hubble is bounded and singularity occurs at a finite scale factor. Geodesics can be extended beyond these events. Quantum geometry simply ignores weak curvature singularities. Does not over kill!

- Combining all results: Geodesics can be extended to arbitrary values of the affine parameter. The effective spacetime in LQC turns out to be geodesically complete.

Summary

- Assuming that the effective equations are valid for a general matter model we analyzed in detail nature and strength of all possible cosmological singularities and behavior of geodesics in the $k = 0$ isotropic and homogeneous model with homogeneous matter. (No restrictions using energy conditions).
- The issue of (traditional) past (or future) singularity has been investigated in various models. Big Bang and Big Crunch are resolved. Bounce is generic in various situations (at an effective level).
- All the singularities which involve divergence of energy density (or Hubble) are resolved. The underlying reason is the existence of upper bound on the energy density of the matter in LQC which translates in to an upper bound for the trace of extrinsic curvature.
- There exist extremal events (type II and type IV singularities) in GR where the divergence in curvature occurs only because of pressure. Quantum geometry plays little role in such cases and divergence in curvature will not be controlled.
- Strong curvature singularities may be completely eliminated or tamed to weak ones. (An example is the work of Cailleteau, Cardoso, Vandersloot & Wands).
- Effective spacetime in LQC is always geodesically complete, irrespective of the choice of matter.

Further directions

- Can these results be extended to richer spacetimes? Extension to curved spaces $(k = \pm 1)$ and Bianchi-I anisotropic model is straightforward. Extension to Gowdy models should also be possible.

- Need to study the consequences of higher order state dependent corrections to the effective Hamiltonian.

- Significant constraints on quantization ambiguities. Lessons for full theory. Generic singularity resolution tied to a careful quantization of Hamiltonian constraint. In the isotropic case, any other known scheme which is to the improved dynamics of LQC does not guarantee geodesic completeness.

- What are the lessons for more general cases? We should be careful in posing the singularity resolution question. Important aspect of singularity resolution is geodesic completeness. Curvature boundedness may be misleading in some situations. Examples of Type II and IV extremal events show that quantum geometry may not always bind the spacetime curvature (and yet there be no singularity).