Strong Gravity and the BKL Conjecture

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The Conjecture (Belinskii, Khalatnikov, Lifshitz) (1971)

Near to a singularity spatially separated points decouple, and the role of most forms of matter is negligible.

Problems

- Does not appear geometric
- Mathematical implications vague
- Only conjecture - no analytic proof.

Motivation for applying the BKL conjecture

- The Einstein field equations are hard to solve:
  - Similar to homogenous or isotropic cosmology - attempt to solve the EFEs in a simple case
- Numerical evidence supports the BKL conjecture:
  - Recently a lot of numerical work has been done (Garfinkle, Uggla, Elst, Ellis, Wainwright, Curtis, Moncrief, Berger...) providing support for the BKL conjecture.
Why Ashtekar Variables

Why Use Ashtekar Variables?

- Simple formulation
- Basic Variables $\tilde{E}_i^a, A_i^a$ - no inverses.
  - ADM Variables need $q^{ab}$ and $q_{ab}$. Not well defined at singularity.
- Candidate for quantum theory
- Singularity Resolution

Our work is done in self dual variables - re-evaluation in real variables to come later.

Singularity Resolution

- Aim is to resolve a large class of cosmological singularities.
- Similar position to first finding singularities
  - First specific singularities found
  - Disagreement about how generic singularities are
  - Singularity proven
**Ashtekar Variables**

**A brief recap of Ashtekar variables with units**

We work with a densitized triad $\tilde{E}$ and connection $A$. Early letters (a,b,c...) denote spatial indices, later (i,j,k...) internal.

\begin{align*}
\tilde{E}^a_i \tilde{E}^b_i &= \sqrt{q}q^{ab} \\
A^a_i &= \frac{1}{G} (\Gamma^i_a - iK^i_a) \\
\tilde{E}^a_i K^i_b &= qK^a_b \\
\{A^a_i(x), \tilde{E}^b_j(y)\} &= i\delta^b_a \delta^i_j \delta^3(x - y) \\
\tilde{E}^a_i &\sim q_{ab} \sim 1 \\
A^a_i &\sim \frac{M}{LT}
\end{align*}
The Kasner Singularity

Bianchi I Solution

The Kasner space-time takes the form:

\[ ds^2 = -d\tau^2 + \tau^2 p_1 \, dx^2 + \tau^2 p_2 \, dy^2 + \tau^2 p_3 \, dz^2 \]  \hspace{1cm} (7)

\[ p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2 \]  \hspace{1cm} (8)

\[ p_i \in \mathbb{R} \]  \hspace{1cm} (9)

- One degree of freedom
- Homogeneous vacuum solution
- Unstable in perturbations
- Adding scalar field (stiff fluid) gives rise to a stable sector
Strong Gravity as a Toy Model

The Strong Coupling Limit

We examine the limit $G \to \infty$ whilst keeping $\tilde{E}^a_i$ and $A^i_j$ fixed.

Relationship with BKL Conjecture

Consider the derivative operator

$$D_a h_i = \partial_a h_i + \frac{G}{c} \epsilon^{k}_{ij} A^i_a h_k$$

(10)

- From this $\partial_a \to 0$ is equivalent to $G \to \infty$.
- The strong coupling limit forms a toy model of the BKL conjecture - we ignore spatial derivatives.
- Not the entire picture - generally we see spatial derivatives suppressed rather than ignored.
Constraints in Strong Gravity

Full Constraints

\[ \tilde{S} = 2c \tilde{E}_i^a \tilde{E}_j^b \partial_{[a} A_{b]} k \epsilon^{ijk} + 2G \tilde{E}_i^a \tilde{E}_j^b A_{[i} A_{b]} \]  
(11)

\[ V_a = c \tilde{E}_i^b (2 \partial_{[a} A_{b]} i + G A_{a}^j A_{b}^k \epsilon^i_{jk}) \]  
(12)

\[ G_i = c \partial_a \tilde{E}_i^a + G \epsilon_{ijk} A_{a}^j \tilde{E}^{ak} \]  
(13)

Reduced Constraints

\[ \tilde{S} = 2G \tilde{E}_i^a \tilde{E}_j^b A_{[i} A_{b]} \]  
(14)

\[ V_a = G \tilde{E}_i^b A_{a}^j A_{b}^k \epsilon^i_{jk} \]  
(15)

\[ G_i = G \epsilon_{ijk} A_{a}^j \tilde{E}^{ak} \]  
(16)

\[ \{ \tilde{S}, V_a \} \approx 0 \quad \{ \tilde{S}, G_i \} \approx 0 \quad \{ G_i, V_a \} \approx 0 \]  
(17)
Simpler Variables

New Variable $\tilde{M}_i^j$

We can introduce a new variable to simplify things: $\tilde{M}_i^j = \tilde{E}_a^i A_i^j$

\[
\{ \tilde{M}_i^j(x), \tilde{M}_k^l(y) \} = i(\tilde{M}_i^l \delta_k^j + \tilde{M}_k^i \delta_l^j) \delta^3(x - y) \quad (18)
\]

\[
\{ \tilde{E}_k^a(x), \tilde{M}_i^j(y) \} = i\tilde{E}_i^a \delta_k^j \delta^3(x - y) \quad (19)
\]

Constraints in terms of $\tilde{M}_i^j$

\[
\tilde{S} = G((\tilde{M}_i^j)^2 - \tilde{M}_i^j \tilde{M}_j^i) \quad (20)
\]

\[
= G(Tr(\tilde{M})^2 - Tr(\tilde{M}^2)) \quad (21)
\]

\[
G_i = G\epsilon_{ijk} M^{kj} \quad (22)
\]
Equations of Motion

Time Evolution of $\tilde{M}_i^j$ and $\tilde{E}_i^a$

$$\tilde{E}_i^a = -iN(\tilde{M}_k^j \delta_i^j - \tilde{M}_i^j)\tilde{E}_j^a$$  \hspace{1cm} (23)

$$\tilde{M}_i^j = 0$$  \hspace{1cm} (24)

Commutativity of Strong Limit and Equations of Motion

<table>
<thead>
<tr>
<th>Full Constraint</th>
<th>Large G limit</th>
<th>Reduced Constraint</th>
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<tr>
<td>Equation of Motion</td>
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</tr>
<tr>
<td>Full Equation of Motion</td>
<td>Large G limit</td>
<td>Reduced Equation of Motion</td>
</tr>
</tbody>
</table>
Equations of Motion

Reality Conditions

\[ q \in \mathbb{R} \rightarrow \dot{q} \in \mathbb{R} \rightarrow -i\tilde{M} \in \mathbb{R} \]  

(25)

\(-iM\) is a symmetric (from Gauss Constraint) real matrix and is therefore diagonalizable in some basis of our internal space.

Solutions

We can solve our simple diagonal equations in the lapse \(N = 1\) case:

\[
ds^2 = -d\tau^2 + \tau^{-2f_1} (^o E^1_1)^2 dx^2 + \tau^{-2f_2} (^o E^2_2)^2 dy^2 + \tau^{-2f_3} (^o E^3_3)^2 dz^2 \]  

(26)

With scalar constraint implying

\[
\sum f_i = \sum f_i^2 = 1
\]  

(27)
A New Calculus

The BKL Limit

We will examine the following scenario:
- $\tilde{E}^a_i \partial_a X \rightarrow 0$ where $\sqrt{\text{Det}(q)} T$ has limit $\forall T \in X$
- $A^i_a, \tilde{E}^a_i \in X$
- $\text{Det}(q) \rightarrow 0$
- Unreasonable Assumption: $\Gamma^i_a = 0$

Why Use This Model?
- Similar to “almost FL” models of Uggla, Elst et al.
- Numerical evidence from Garfinkle
- Gowdy model: $\partial_a X$ suppressed
- Allows for $\frac{\partial_a f}{f}$ bounded.
- From Kasner model we see $\tilde{E}^a_i, M^i_j$ have this behaviour.
A non-singular example

Consider Minkowski space in coordinates adapted to a hyperboloid:

\[ ds^2 = -d\rho^2 + \rho^2 [d\chi^2 + \cosh^2(\chi)(d\theta^2 + \cos^2(\theta)d\phi^2)] \]  \hspace{1cm} (28)

Here we see \( \rho \to 0 \) as taking \( \text{Det}(q) \to 0 \)
Non-Reduced Constraints and Equations of Motion

**Constraints**

- Full constraints the same as in the toy model
- Multiple of the Gauss constraint added to Hamiltonian - keeps evolution real:
  \[
  H = \int d^3x \left[ \frac{1}{2} \sum S - i N^a V_a + (\tilde{E}^a D_a \tilde{N}) G_i \right]
  \tag{29}
  \]

**Equations of Motion**

\[
\begin{align*}
\dot{\tilde{E}}^b_i &= -i \tilde{N} \tilde{E}^a_j D_a \tilde{E}^b_k \epsilon^{jk} \\
\dot{\tilde{M}}^i_j &= 2i \tilde{N} \tilde{E}^b_i \tilde{E}^a_k \partial_{[a} A_{b]} \epsilon^{ijkl} - i \tilde{N} \tilde{E}^a_k \partial_a \tilde{E}^b_i \epsilon^{kl}_i A^i_b \\
&- i \tilde{E}^b_i \partial_b (D_a \tilde{N} \tilde{E}^a_j) - i \tilde{E}^b_i \epsilon^{kl}_i A^k_b D_a \tilde{N} \tilde{E}^a_l \\
&+ i \tilde{N} \tilde{M}^{kj} \tilde{M}_{[ik]} 
\end{align*}
\tag{30, 31, 32, 33}
Reduced Equations of Motion

Equations of Motion

\[ G_i = \text{Re}[i\epsilon_{ijk} M^{jk}] \]  \hspace{1cm} (34)

\[ \tilde{E}^a_i = -i\tilde{N}(\tilde{M}_k^k \delta_i^j - \tilde{M}_i^j) \tilde{E}^a_j \]  \hspace{1cm} (35)

\[ \tilde{M}_i^j = 0 \]  \hspace{1cm} (36)

Commutativity of BKL Reduction and Equations of Motion

Full Constraint \xrightarrow{\text{BKL Reduction}} Reduced Constraint

\[ \downarrow \text{Equation of Motion} \hspace{1cm} \downarrow \text{Equation of Motion} \]

Full Equation of Motion \xrightarrow{\text{BKL Reduction}} Reduced Equation of Motion

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Relaxing $\Gamma = 0$

- We can relax our $\Gamma = 0$ condition
- $i\tilde{M}$ is no longer a symmetric matrix
  \[ \tilde{M}_i^j = i\tilde{N}\tilde{M}^{kj}\tilde{M}_{[ik]} \]
- General analytic solutions not found
- 'Bounces' appear in numerical simulation
- Perturbations to symmetric matrix grow
Conclusions

Results

- BKL Conjecture can be applied in Ashtekar Variables
- Resulting spacetime has Kasner singularity in 'worst case'
- Corresponds to known numerical results

Open Issues

- Work done in self-dual variables - changes in real variables?
- Spatial curvature is non-zero in general for bounce behaviour
- Need to add matter
References

- Uggla et al. “Past Attractor in Inhomogeneous Cosmology” gr-qc/0304002
- Garfinkle “Numerical Simulations of Stiff Fluid Gravitational Singularities” gr-qc/0506107
- Garfinkle “Numerical Simulations of Generic Singularities” gr-qc/0312117
- Rendall “The Nature of Spacetime Singularities” gr-qc/0503112