

Spinfoam graviton propagator

II

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Physics from the spinfoam kernel

The spinfoam graviton:

L. Modesto and C. Rovelli, [gr-qc/0502036]

$$W_{\mu\nu\rho\sigma}(x, y) = \frac{1}{\mathcal{N}} \sum_s \langle s | h_{\mu\nu}(x) | s \rangle \langle s | h_{\rho\sigma}(y) | s \rangle \Psi_q[s] K[s].$$

Consider the normalised projections

$$W_{ab} := \frac{1}{n_a^2 n_b^2} n_a^\mu(x) n_a^\nu(x) n_b^\rho(y) n_b^\sigma(y) W_{\mu\nu\rho\sigma}(x, y).$$

In the linearised continuum theory,

$$W_{ab} = f_\xi(\varphi_{ab}) \frac{1}{|x - y|^2}.$$

ξ is a gauge-fixing parameter.


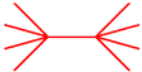

This is the result that should be reproduced by the spinfoam graviton in the semiclassical limit.

The Barrett–Crane kernel

$$K[s] = \sum_{\substack{\sigma \\ \partial\sigma=s}} \prod_f A_f \prod_e A_e \prod_v \{10j\}$$

Possible prescription: consider only trees in the spinfoam sum

L. Freidel, [hep-th/0505016]

Diagram	λ order	\hbar order	2-point function
	1	0	$\frac{1}{d^2} + \frac{\ell_P}{d^3} + \dots$ [arXiv:gr-qc/0604044]
	2	0	$\frac{1}{d^2} + \frac{\ell_P}{d^3} + \dots$ [arXiv:gr-qc/0604044]
	2	1	$\frac{\ell_P}{d^3} + \frac{\ell_P^2}{d^4} + \dots$ conjecture

In the following, we restrict to the single 4-simplex contribution

Asymptotics and the boundary state

On a single 4-simplex, $W_{ab}(x, y) = \frac{1}{\mathcal{N}} \sum_{\{j_l\}} F_{ab}(j, j_0, \alpha) \{10j\}$

The perturbative expansion can be addressed looking at the large spin limit. In this limit,

$$\{10j\} \sim e^{iS_R} + e^{-iS_R} + D(j)$$

J. C. Baez, J. D. Christensen and G. Egan, [gr-qc/0208010]

Introduce the background values j_0^l and $\theta_l \equiv \left. \frac{\partial S_R}{\partial j_l} \right|_{j_0}$ of the boundary geometry.

$$S_R(j) = \sum_l \theta_l j_l + \frac{1}{2} \sum_{l,k} G_{lk}(j_0) \delta j_l \delta j_k + \dots$$

If the boundary state contains a phase $\exp -i \sum_l \theta_l j_l$, only one term of the $\{10j\}$ asymptotics will survive.

The role of the boundary state

Consider the following Gaussian,

C. Rovelli, [gr-qc/0508124]

$$\Psi_q(j_l) = \exp \left\{ -\frac{\tilde{\alpha}(j_0)}{2} \sum_l \delta j_l^2 - i \sum_l \theta_l j_l \right\} \quad \tilde{\alpha} \sim \frac{1}{j_0}$$

Computing the leading order of the 2-point function amounts to evaluating the following Gaussian integral:

$$W_{ab} = \frac{1}{N} \int \prod_l dj_l \delta j_a \delta j_b e^{-\frac{1}{2} \sum_{lk} A_{lk} \delta j_l \delta j_k}$$

where

$$A_{lk} := \alpha(j_0) \delta_{lk} + iG_{lk}(j_0)$$

But $G_{lk} = \frac{\partial \theta_l}{\partial j_k}$ is not invertible! (Schlafli identity $\Rightarrow \det G_{kl} = 0$)

The quadratic term plays a gauge-fixing role: it allows to invert the kinetic term.

Numerical test

If these assumptions hold, then

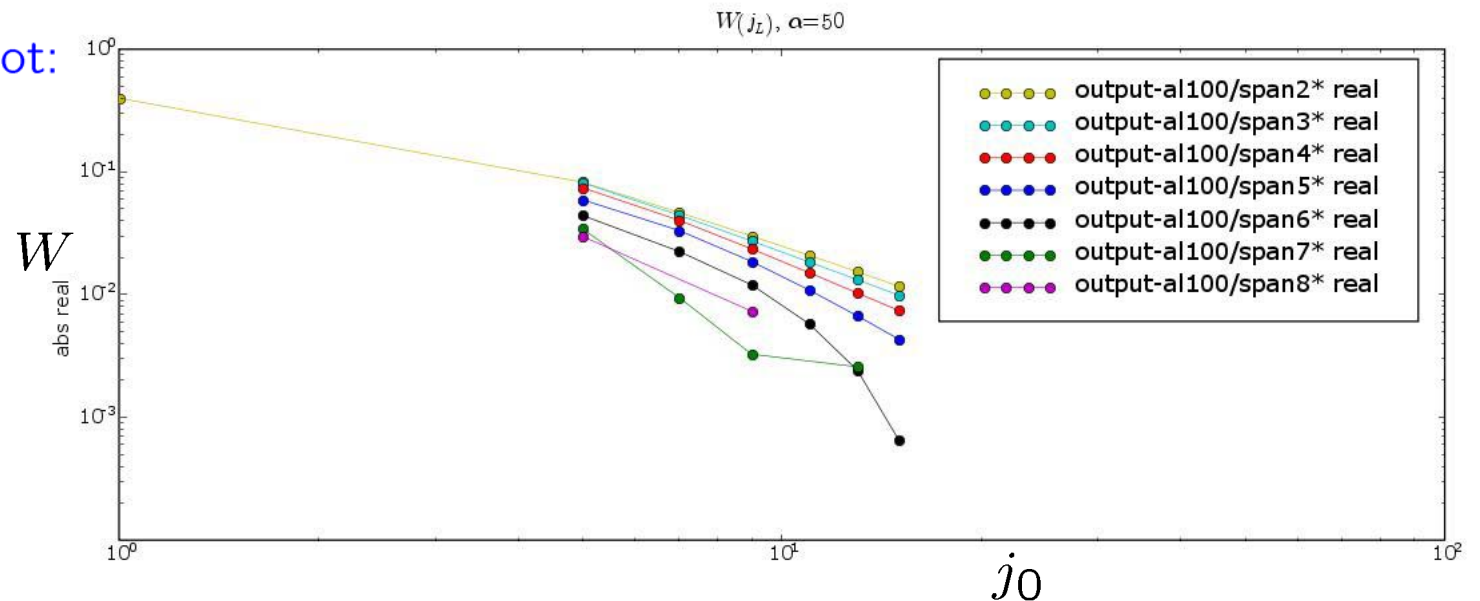
C. Rovelli, [gr-qc/0508124]

$$W_{ab} = (A^{-1})_{ab} = \frac{f_{ab}}{j_0}$$

Very hard to check numerically in the 4d case: work still in progress

J. D. Christensen, S. Speziale, to appear

On a log-log plot:

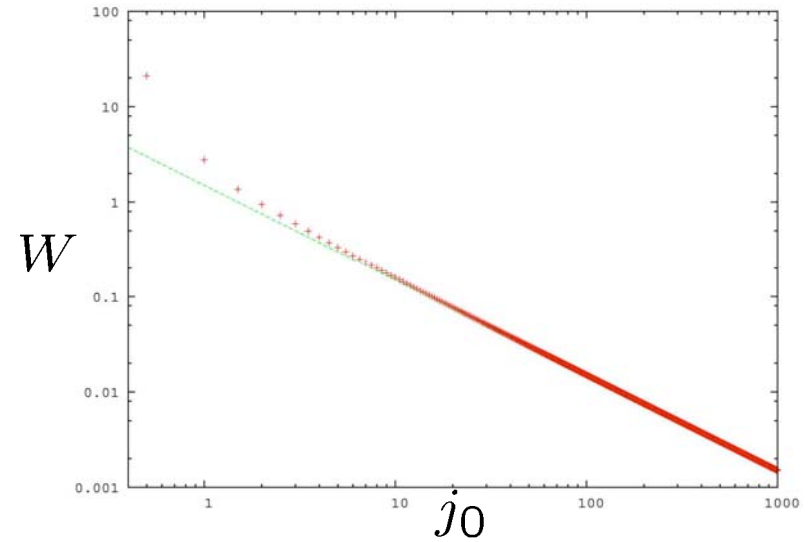


Numerical test in 3d

Confirmed numerically in the 3d case

S. Speziale, [arXiv:gr-qc/0512102]

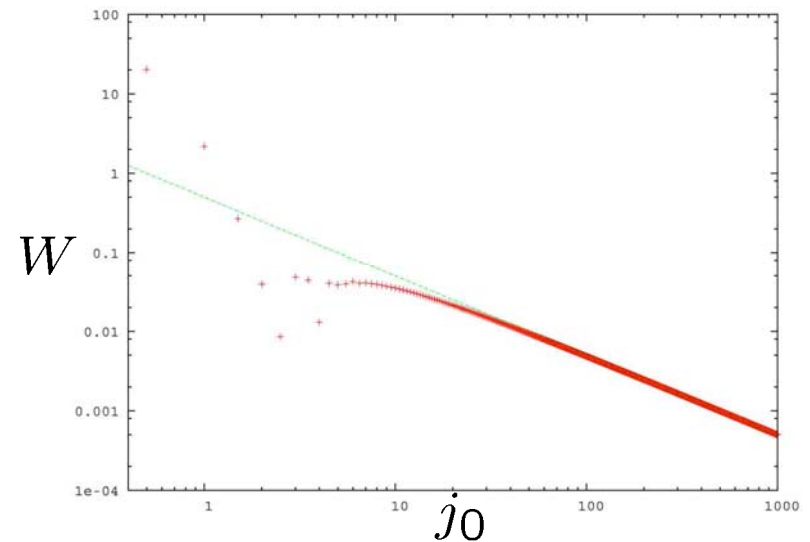
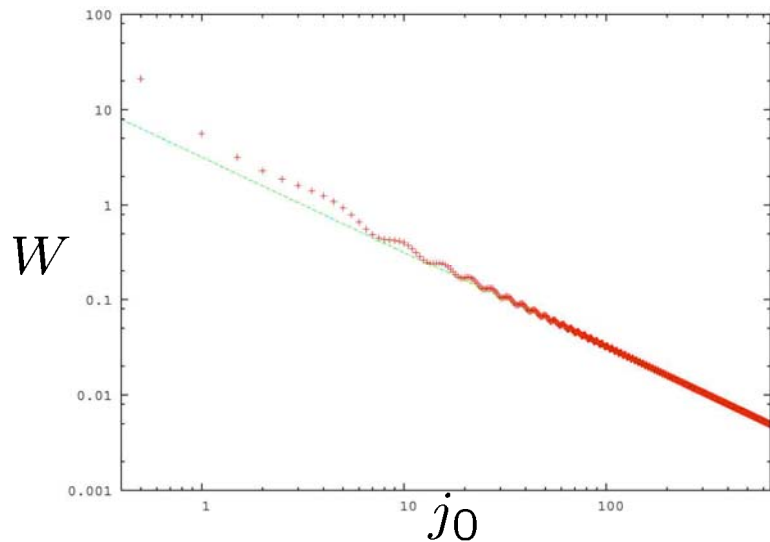
All log-log plots!



• also without quadratic term ($\alpha = 0$),

E. Livine, S. Speziale and J. Willis, [arXiv:gr-qc/0605123]

• also with $|\prod_l \cos \theta_l j_l|$



The new boundary state

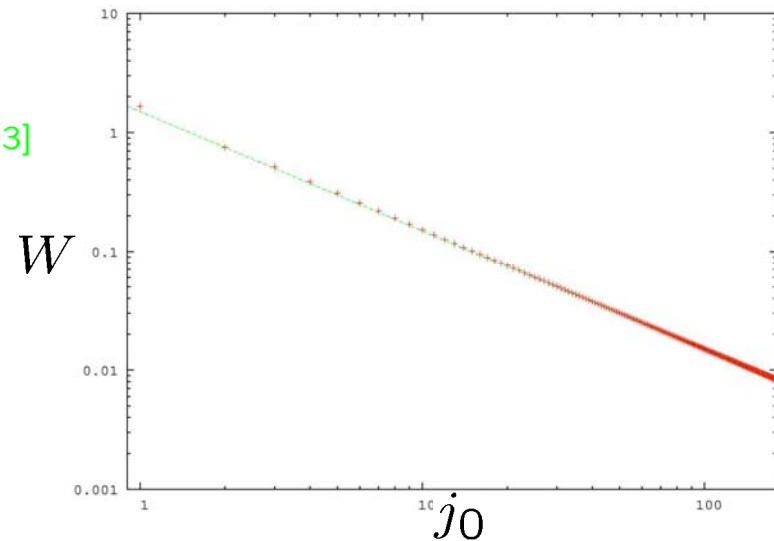
E. R. Livine and S. Speziale, [arXiv:gr-qc/0608131]

Factorised in link contributions, $\Psi_0(j_l) = \prod_l \psi_0(j_l)$.

$$\psi_0(j) = \frac{I_{|j-j_0|}(\frac{1}{\alpha}) - I_{j+j_0+1}(\frac{1}{\alpha})}{\sqrt{I_0(\frac{2}{\alpha}) - I_{2j_0+1}(\frac{2}{\alpha})}} \cos(d_j \theta) = \frac{1}{N} e^{-\frac{\tilde{\alpha}}{2} \delta j} \cos(d_j \theta) + \frac{1}{j_0} (\dots)$$

Tested in 3d

E. Livine, S. Speziale and J. Willis, [arXiv:gr-qc/0605123]



This state has a simple SU(2) Fourier transform,

$$\sum_j \psi_0(j) \chi_j(\phi) = \tilde{\psi}(\phi)$$

Integral representation of the propagator

The new boundary state allows to perform explicitly the sum

$$W_{ab}(x, y) = \frac{1}{\mathcal{N}} \sum_{\{j_l\}} F_{ab}(j, j_0, \alpha) \{10j\}$$

The key point: integral expression of the kernel $\{10j\} = \int \prod_l d\phi_l \mu(\phi_l) \prod_l \chi_{j_l}(\phi_l)$

J. W. Barrett, [math.qa/9803063]

L. Freidel and K. Krasnov, [hep-th/9903192]

$$\begin{aligned} W_{ab}(x, y) &= \frac{1}{\mathcal{N}} \int \prod_l d\phi_l \mu(\phi_l) \prod_l \left(\sum_j F_{ab}^{(l)}(j, j_0, \alpha) \chi_j(\phi_l) \right) \\ &= \frac{1}{\mathcal{N}} \int \prod_l d\phi_l \mu(\phi_l) \prod_l \mathcal{I}_{ab}^{(l)}(\phi_l, j_0, \alpha) \end{aligned}$$

Immediately generalises to higher-valent vertices

Saddle point approximation

E. R. Livine and S. Speziale, [arXiv:gr-qc/0608131]

Consider for simplicity an equilateral background: j_0 and $\theta = \arccos(-\frac{1}{4})$ for every link. Choose $\tilde{\alpha} = \frac{\alpha}{j_0}$. Then

$$W_{ab} = \frac{1}{\mathcal{N}} \int \prod_l d\phi_l \mu(\phi_l) \prod_l \mathcal{I}_{ab}^{(l)}(\phi_l, j_0, \alpha)$$

reads

$$W_{ab} = \sum_{\epsilon_l = \pm} \int d\kappa \int \prod_l d\phi_l F_{ab}(\phi_l) e^{-j_0 S[\phi_l, \epsilon_l, \kappa]},$$

where

$$S[\phi_l, \epsilon_l, \kappa] = \sum_l \left[\frac{2}{\alpha} \sin^2(\phi_l - \theta) - 2i\epsilon_l(\phi_l - \theta) \right] - i \frac{\kappa}{j_0} \det \mathbb{G}$$

is the standard action driving the asymptotics of the $\{10j\}$

J. W. Barrett and C. M. Steel, [gr-qc/0209023]

L. Freidel and D. Louapre, [hep-th/0209134]

The perturbative expansion is driven by the large j_0 limit.

The integral representation allows to recast the large j_0 limit into the saddle point expansion of the integral.

Saddle points

The saddle points are very similar to the ones found in the analysis of the $\{10j\}$ symbol.

Solutions of the equations of motion in the equilateral case:

$$\bar{\phi}_l = \theta + \frac{1 - \sigma_l}{2} \pi, \quad \bar{\epsilon}_l = \epsilon, \quad \bar{\kappa} = \epsilon \frac{j_0}{\sin \theta \cos \theta (1 - \cos \theta)^3}$$

Crucial remark: the solution $\phi_l = 0$, which is present in the asymptotics of the $\{10j\}$ symbol, and leads to the degenerate contribution $D(j_l)$, is now removed by the presence of the Gaussian term $\sin^2(\phi_l - \eta_l \theta)$.

This shows that the D term does not contribute.

The leading order is a Gaussian integral,

$$W_{ab} \simeq \sum_{\epsilon=\pm} \int d\delta\kappa \int \prod_l d\delta\phi_l F_{ab}(\delta\phi_l) e^{-j_0 \frac{\partial^2 S}{\partial \phi_l \partial \phi_k} \delta\phi_l \delta\phi_k - j_0 \frac{\partial^2 S}{\partial \phi_l \partial \kappa} \delta\phi_l \delta\kappa},$$

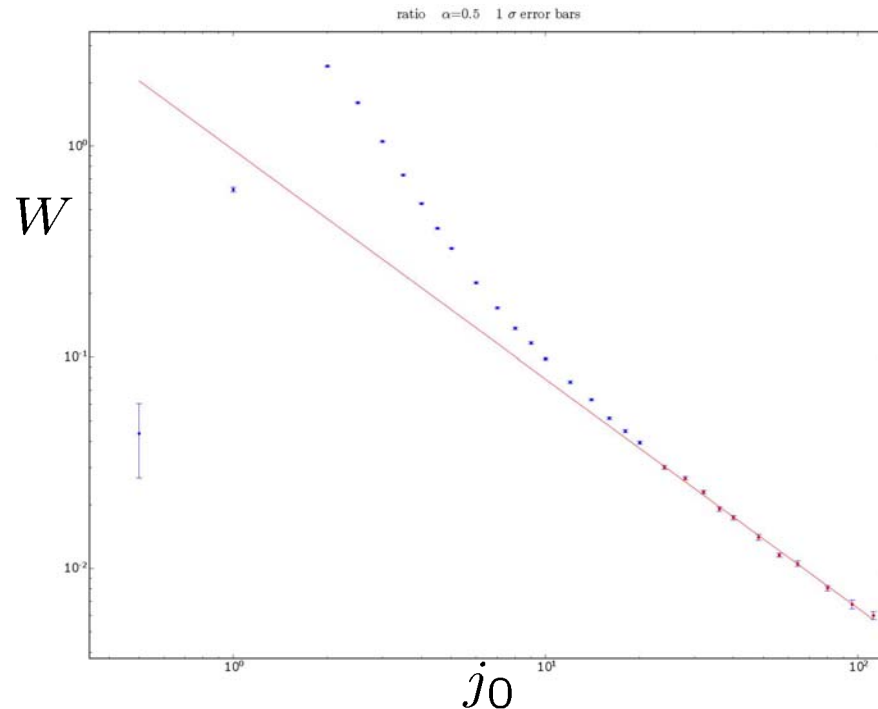
that gives $W_{ab} = \frac{f_{ab}(\alpha)}{j_0}$ where f_{ab} is a matrix with only three independent entries.

First numerical results

Leading order:

$$W_{ab} = \frac{f_{ab}(\alpha)}{j_0}$$

D. Christensen and S. Speziale, work in progress!



Tested for the various components, and for different values of α .

$\alpha = 0.5$ 1 cpu-year

In total (preliminary runs plus various α) about 15 cpu-years

- Direct confirmation of analytical calculations
- Indirect confirmation of the asymptotic structure

$$\{10j\} \sim \sum_{\tau} P(\tau) \cos\left(S_R[j_l] + \kappa_{\tau} \frac{\pi}{4}\right) + D(j_l)$$

More than dimensional analysis?

In the linearised continuum theory,

$$W_{ab} = f_{\xi}(\varphi_{ab}) \frac{1}{|x - y|^2}.$$

In the equilateral case, $|x - y|^2 \propto j_0$, thus

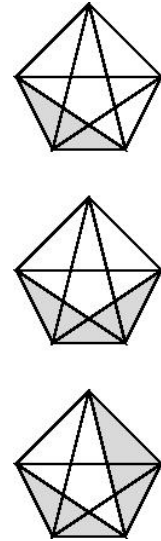
$$W_{ab} = f_{\xi}(\varphi_{ab}) \frac{N}{j_0}.$$

For a given gauge-fixing ξ , the matrix W_{ab} has only three independent entries.

This is precisely reproduced by the spinfoam calculations,

$$W_{ab} = \frac{f_{ab}(\alpha)}{j_0}.$$

A crucial test requires to consider arbitrary configurations.



Perturbative expansion

The perturbative expansion is driven by the large j_0 limit.

For the 4-simplex contribution, higher orders in the saddle point approximation

Leading order comes from the quadratic term in the Regge action

$$S_R(j) = \sum_l \theta_l j_l + \underbrace{\frac{1}{2} \sum_{l,k} G_{lk}(j_0) \delta j_l \delta j_k}_{\text{quadratic term}} + \underbrace{\frac{1}{6} \sum_{lkm} \frac{\partial^3 S_R}{\partial j_l \partial j_k \partial j_m} \Big|_{j_0} \delta j_l \delta j_k \delta j_m + \dots}_{\text{higher orders}}$$

Higher orders give corrections to the propagator.

So... are we simply doing Regge quantum gravity?

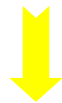
No! there is more: higher orders in the limit of the $\{10j\}$.

$$\{10j\} \sim \left[e^{iS_R} + e^{-iS_R} + D(j) \right] + \frac{1}{j} \left[\dots \right] + \dots$$

Perturbative expansion in the 3d toy model

$$W_{ab}(j_0) = \frac{1}{\mathcal{N}} \sum_j F_{ab}(j, j_0, \alpha) \{6j\} \quad \text{quantum gravity}$$

- $\{6j\}$ as integral over SU(2)



saddle point approximation

$$\frac{1}{\sqrt{V(\ell)}} \cos[S_R(\ell) + \frac{\pi}{4}] + \dots$$

G. Ponzano, T. Regge, (1968)
 J. D. Roberts, [math-ph/9812013]
 J. W. Barrett and C. M. Steel, *cited*
 L. Freidel and D. Louapre, *cited*

$$W_{ab}(j_0) = \frac{1}{\mathcal{N}} \sum_j \mu(j) F_{ab}(j, j_0, \alpha) e^{iS_R[j]} \quad \text{Regge path integral}$$

- Regge action



expanded around the background introduced by Ψ_q

$$S_R(j) = \sum_l \theta_l j_l + \frac{1}{2} \sum_{l,k} G_{lk}(j_0) \delta j_l \delta j_k + \frac{1}{6} \sum_{lkm} \left. \frac{\partial^3 S_R}{\partial j_l \partial j_k \partial j_m} \right|_{j_0} \delta j_l \delta j_k \delta j_m + \dots$$

$$W_{ab}(j_0) = \quad \text{perturbative expansion of the Regge path integral}$$

Leading order and higher order corrections

We computed LO, NLO and NNLO:

$$W_{12}(\ell) = a \frac{1}{\ell} + b \ell_{\text{P}} \frac{1}{\ell^2} + c \ell_{\text{P}}^2 \frac{1}{\ell^3} + \dots$$

S. Speziale, [gr-qc/0512102]

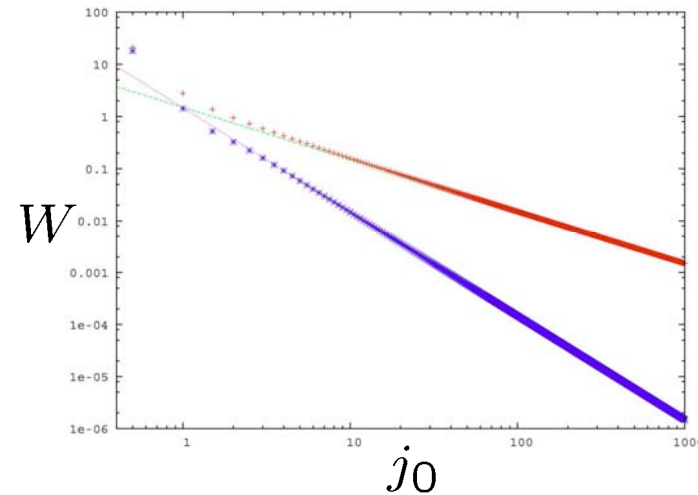
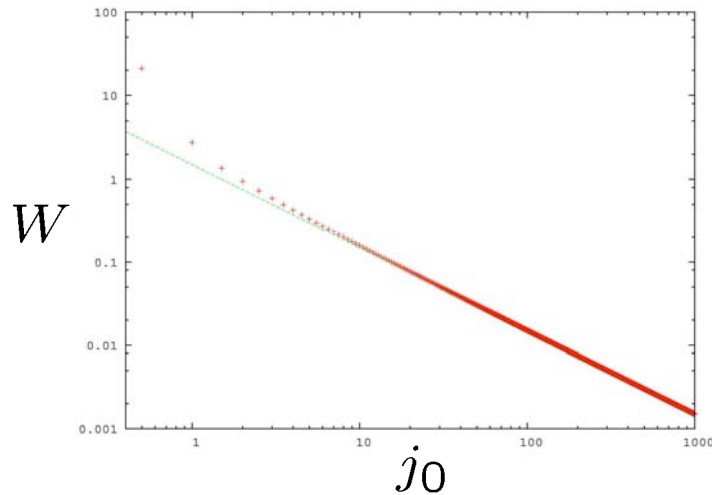
E. Livine, S. Speziale and J. Willis, [gr-qc/0605123]

Leading order

$$a \equiv a_{\text{R}} = -\frac{1}{2} + i\sqrt{2}$$

Next to leading order

$$b \equiv b_{\text{R}} = -\frac{7481 - i1048\sqrt{2}}{5184}$$



$$\{6j\} \sim \frac{1}{j^{3/2}} \cos(S_{\text{R}}) + \frac{1}{j^{5/2}} [\dots] + \dots$$

Next to next to leading order:

$$c = c_{\text{R}} + c_{\text{S}} = 0.2522 + i0.7263 + c_{\text{S}} = -0.0225 + i0.0730$$

Structure of the corrections

$$W_{12}(\ell) = a \frac{1}{\ell} + b \ell_{\text{P}} \frac{1}{\ell^2} + c \ell_{\text{P}}^2 \frac{1}{\ell^3} + \dots$$



Regge path integral

Regge path integral
and
quantum geometry corrections

In this toy model, quantum geometry corrections enter at NNLO, with the neat effect of reducing the magnitude of it.

Conclusions

Depending on the choice of boundary state, the spinfoam graviton can be written as a sum, or as an integral over $SU(2)$.

In both cases, the formula can be evaluated in a large spin expansion, and it shows the expected behaviour of the leading order.

For the integral representation, the analytical calculations are confirmed by numerical simulations.

As for higher order corrections, the 3d toy model suggests that interesting spinfoam effects could arise at two loops.

... and perspectives

- Extend results to many 4-simplicies
- Understanding the boundary state
- Understanding the gauge fixing: counting degrees of freedom
- Compute higher order corrections
- Add matter
- ...

Bibliography

4d results: leading order and propagator as group integrals

- C. Rovelli, “Graviton propagator from background-independent quantum gravity,” to appear on [Phys. Rev. Lett.](#) [[arXiv:gr-qc/0508124](#)]
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- E. R. Livine and S. Speziale, “Group integral techniques for the spinfoam graviton propagator,” [arXiv:gr-qc/0608131](#)

3d results: leading order and higher order corrections

- S. Speziale, “Towards the graviton from spinfoams: the 3d toy model,” [JHEP](#) **05** (2006) 039 [[arXiv:gr-qc/0512102](#)]
- E. Livine, S. Speziale and J. Willis, “Towards the graviton from spinfoams: higher order corrections in the 3d toy model,” [[arXiv:gr-qc/0605123](#)]