Twistorial structure of loop quantum gravity transition amplitudes

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ILQG 13-11-2012
mainly based on work with Wolfgang Wieland 1207.6348
Outline and goals

Classical Theory

- Description of the covariant phase space in terms of twistors algebra of the area matching and simplicity constraints
- Smearing the connection-tetrad algebra to the holonomy-flux algebra before or after solving the simplicity constraints is equivalent: $T^*SU(2)$ with AB holonomy
- Notion of simple twistors solving the simplicity constraints

Quantum theory

- Hilbert space represented via homogeneous functions on spinor space (instead of cylindrical functions)
- Dynamics as integrals in twistor space
- Embedding of the Regge data of the EPRL asymptotics in the initial phase space

Work and ideas shared with a number of collaborators

This talk: w Wolfgang Wieland 1207.6348 PRD12 w Miklos Langvik (to appear)
Smearing and simplicity constraints

Phase spaces:

\[ \text{SL}(2, \mathbb{C}) \text{ variables} \]

\[ \text{Ashtekar-Barbero variables} \]

\[ \text{Solve simplicity constraints} \quad (\text{primary and secondary}) \]

\[ \text{Does the diagram commute?} \]

LQG path: \[ \rightarrow \]

spin foam path: the other one

In the continuum:
- primary: simple bivectors, unique metric structure
- secondary: embedding of AB variables in covariant phase space: \( \Gamma = \Gamma(E) \)

In the discrete:
- primary: simple twistors, unique (twisted) geometry
- secondary: non-trivial embedding of \( T^*SU(2) \) in \( T^*SL(2, \mathbb{C}) \). Discrete \( \Gamma(E) \) ?
Smearing and simplicity constraints

Phase spaces:

\[
\begin{align*}
\text{Smearing} & \quad \Rightarrow \\
SL(2, \mathbb{C}) \text{ variables} & \quad \Rightarrow \\
T^*\text{SL}(2, \mathbb{C}) & \\
\downarrow & \\
Solve simplicity constraints & \\
(\text{primary and secondary}) & \\
\downarrow & \\
\text{Ashtekar-Barbero variables} & \quad \Rightarrow \\
T^*\text{SU}(2)&
\end{align*}
\]

**Strategy:**
do symplectic reduction of the primary constraints, and think of the secondary constraints as providing a non trivial gauge-fixing section.

⇒ allows to show commutativity of the procedure, while postponing knowing explicitly the form and solution of the secondary constraints

**Technique:**
twistorial tools: parametrize the phase spaces and the constraints in terms of twistors

⇒ allows to embed the non-linear HF phase space into a linear one, explicit calculations, plus direct link with variables used in the twisted geometry parametrization
Panorama

\[ \text{twistors} \quad \rightarrow \quad \text{C-area matching} \quad \text{covariant} \quad \text{twisted geometry} \quad \leftrightarrow \quad \text{covariant loop gravity} \quad \text{SL}(2, \mathbb{C}) \]

\[ \downarrow \quad \text{simplicity constraints} \quad \downarrow \quad \text{“open” twisted geometry} \quad \leftrightarrow \quad \text{loop gravity} \quad \text{SU}(2) \]

\[ \downarrow \quad \text{closure} \quad \downarrow \quad \text{Gauss law} \]

\[ \text{“closed” twisted geometries (polyhedra)} \quad \leftrightarrow \quad \text{gauge-inv. loop gravity} \]

\[ \downarrow \quad \text{shape matching} \]

Regge calculus
LQG and twisted geometries

On a fixed graph \( \mathcal{H}_\Gamma = L_2[SU(2)^L] \) \( \quad P_\Gamma = T^*SU(2)^L \)
represents a truncation of the theory to a finite number of degrees of freedom \( \text{w Rovelli, PRD 10} \)
These can be interpreted as discrete geometries, called twisted geometries \( \text{w Freidel, PRD 10, \ldots} \)

Each classical holonomy-flux configuration on a fixed graph can be visualized as a collection of adjacent polyhedra with extrinsic curvature between them

- Each polyhedron is locally flat: curvature emerges at the hinges, as in Regge calculus
- They induce a *discontinuous* discrete metric: two neighbouring polyhedra are attached by faces with same area but different shape
- \( \xi \): extrinsic geometry

- Realizes a link suggested by Immirzi, Smolin et al., with the subtlety of *discontinuity*: Regge calculus is too “rigid” to capture the kinematical degrees of freedom of a spin network
- The matching with Regge geometries becomes exact in 2+1 dimensions

The picture can be embedded in a \( \text{SL}(2, \mathbb{C}) \) formalism described by *simple* twistors
Twistors

- A pair of spinors, $\omega$ and $\pi$, chiral under Lorentz transformations,

$$ Z = \left( \begin{array}{c} \omega^A \\ \pi_{\bar{A}} \end{array} \right) \in \mathbb{T} := \mathbb{C}^2 \times \bar{\mathbb{C}}^2^* $$

Twistor “complex helicity”: $\pi_A \omega^A = [\pi|\omega\rangle$ SL(2, $\mathbb{C}$)-invariant

Twistor norm (helicity) $s := \text{Im}([\pi|\omega\rangle)$ SU(2, 2) invariant

- physical picture as a massless particle with a certain spin and momentum

- geometric picture via incidence relation $\omega^A = iX^{A\bar{A}} \pi_{\bar{A}}$
  - $X$ a light ray in Minkowski space for a null twistor $s = 0$
  - $X$ a congruence of light rays for a generic twistor

- can provide notion of non-linear graviton

We will see that twistors can be thought of as non-linear gravitons in a completely different way than Penrose’s original one, as classical counterparts of LQG’s quantum geometry
Representation of the Lorentz algebra

- $\mathbb{C}^2$ carries a representation of the Lorentz algebra (the celestial sphere)
- Similarly, there is a representation of the Lorentz algebra on twistor space $\mathbb{T}$

\[
\Pi^{AB} = \frac{1}{2} (L + iK)^{AB} = \frac{1}{2} \omega(A\pi B)
\]

- We are interested in a representation of a larger Poissonian algebra, the one of $T^*SL(2,\mathbb{C})$: the holonomy-flux algebra of (covariant) loop quantum gravity
- This can be achieved on $\mathbb{T}^2 \ni (Z, \tilde{Z})$,

\[
\Pi^{AB} = \frac{1}{2} \omega(A\pi B), \quad \tilde{\Pi}^{AB} = \frac{1}{2} \tilde{\omega}(A\tilde{\pi} B), \quad \h^A_B = \frac{\tilde{\omega}^A_B - \tilde{\pi}^A \omega_B}{\sqrt{\pi} \tilde{\omega} \sqrt{\pi \omega}}
\]

imposing a complex first-class constraint called complex area matching

\[
C = [\pi | \omega] - [\tilde{\pi} | \tilde{\omega}] = 0,
\]

$\mathbb{C}^8//C \cong T^*SL(2,\mathbb{C})$

That is, both Lorentz generators and holonomies can be expressed as simple functions on a space of two twistors, provided they have the same complex helicity
Twistor Poisson brackets

The symplectic structure simplifies in the larger space $\mathbb{T}^2$:

<table>
<thead>
<tr>
<th>$\mathbb{T}^2$</th>
<th>$C = 0$</th>
<th>$T^*\text{SL}(2, \mathbb{C})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${\omega^A, \omega^B} = 0$</td>
<td>${h, h} = 0$</td>
<td>${h, h} = 0$</td>
</tr>
<tr>
<td>${\pi_A, \omega^B} = \delta^B_A$</td>
<td>${\Pi^i, h} = \tau^i h$</td>
<td>${\Pi^i,\Pi^j} = \epsilon^{ijk} \Pi^k$</td>
</tr>
<tr>
<td>${\pi_A, \pi_B} = 0$</td>
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<td></td>
</tr>
</tbody>
</table>

In particular, momenta commute.

16 twistors $\mathbb{T}^2$ $\rightarrow$ $T^*\text{SL}(2, \mathbb{C})$ 12

$\mathbb{C}$-area matching $C$

simplicity constraints

8 simple twistors $\rightarrow$ $T^*\text{SU}(2)$ 6

SU(2) spinors

area matching $C$

Note: the complete system of constraints is reducible
Simplicity constraints

Gauge-fixed approach  Pick a time direction \( n_I \) (typically time gauge, \( n_I = (1, 0, 0, 0) \))
(cfr. with gauge-invariant approach e.g. Bianca and Jimmy)

- Allows to define SU(2) norm \( \|\omega\|^2 := \delta_{AA'}\omega^A\bar{\omega}^{A'} \) and \( \langle\omega|\pi\rangle := \delta_{AA'}\bar{\omega}^{A'}\pi^A \)

- Primary simplicity constraints: matching of left and right geometries

\[ K + \gamma L = 0 \quad \Leftrightarrow \quad \Pi = e^{i\theta}\bar{\Pi}, \quad \gamma = \cot \frac{\theta}{2} \]

and the same for tilded variables: \( \tilde{\Pi} = e^{i\theta}\tilde{\Pi} \)

- In terms of spinors: \( ([\pi|\omega\rangle = R + iI) \)

\[ F_1 = R - \gamma I = 0, \quad F_2 = \langle\omega|\pi\rangle = 0, \quad \bar{F}_2 = 0 \]

- On each link, \( (M, F_i, \bar{F}_i) \) form a second class system: \( \{F_2, \bar{F}_2\} \not\equiv 0 \)

which is further reducible: \( M \) and \( F_1 \) imply \( \bar{F}_1 \).

<table>
<thead>
<tr>
<th>3 indep. 1st class constraints</th>
<th>((C, D := F_1 + \bar{F}<em>1)) aut ((C</em>{\text{red}} \in \mathbb{R}, F_1, \bar{F}_1))</th>
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<td>4 2st class constraints</td>
<td>( F_2, \bar{F}_2 )</td>
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Counting

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<tr>
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16 twistors \(T^2\) \(\rightarrow\) \(T^*SL(2, \mathbb{C})\) 12
iclass matching \(C\)

\[\downarrow\]

\(\text{simplicity constraints}\) \(\downarrow\)

8 simple twistors \(\text{SU(2) spinors}\) \(\rightarrow\) \(T^*SU(2)\) 6
area matching \(C\)

Counting:

**Right-Down:**

<table>
<thead>
<tr>
<th>(T^2)</th>
<th>(C)</th>
<th>(T^*SL(2, \mathbb{C}))</th>
<th>(D, F_2, \tilde{F}_2)</th>
<th>(T^*SU(2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>-2x2</td>
<td>12</td>
<td>-1x2-4</td>
<td>6</td>
</tr>
</tbody>
</table>

**Down-Right:**

<table>
<thead>
<tr>
<th>(T^2)</th>
<th>(F)</th>
<th>(\mathbb{C}^4)</th>
<th>(C_{red})</th>
<th>(T^*SU(2))</th>
</tr>
</thead>
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</table>
Solution of the simplicity constraints

Focus first on half-link: \((\omega, \pi), \quad (F_1 = R - \gamma I, \ F_2 = \langle \omega | \pi \rangle)\)

- \(F_1 = 0 \quad \Rightarrow \quad [\pi | \omega \rangle = (\gamma + i)j, \ j \in \mathbb{R}\)
- orbits: \(\{F_1, \|\omega\|\} = \frac{1}{1 + \gamma^2} \|\omega\|\)
- convenient to introduce a \(F_1\)-gauge-invariant spinor

\[
z^A = \sqrt{2J} \frac{\omega^A}{\|\omega\| i \gamma + 1}, \quad \|z\| := \sqrt{2J}, \quad \{F_1, z^A\} = 0
\]

<table>
<thead>
<tr>
<th>constraint surface</th>
<th>orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\omega^A, \pi_A))</td>
<td>((\omega^A, j))</td>
</tr>
<tr>
<td>(F_i = 0)</td>
<td>(|\omega|)</td>
</tr>
</tbody>
</table>

Symplectic reduction: \(\mathbb{C}^4 // F \cong \mathbb{C}^2\)

- Solutions parametrized by a SU(2) spinor \(z^A\)
  (transforms linearly under rotations but not under boosts)
- **Simple twistors** (on the 5d constraint surface): \(Z = (\omega^A, j) = (z^A, \|\omega\|)\)
Area matching and $T^*\text{SU}(2)$

On the link:

\[16 \ (\omega^A, \pi^A, \tilde{\omega}^A, \tilde{\pi}^A) \in \mathbb{T}^2 \quad \rightarrow \quad (\Pi, h) \quad 12 \]

\[C\text{-area matching } C\]

\[\downarrow \quad \text{simplicity constraints} \quad \downarrow \]

\[8 \ (z^A, \tilde{z}^A) \in \mathbb{C}^4 \quad \rightarrow \quad (X, g) \quad 6 \]

\[\text{area matching } C_{\text{red}}\]

\[C_{\text{red}} = \|\tilde{z}\| - \|\tilde{z}\| \quad \mathbb{C}^4//C_{\text{red}} = T^*\text{SU}(2) \ni (X, g) \quad \text{Freidel PRD10}\]

Symplectic reduction by the linear primary simplicity constraints selects an $\text{SU}(2)$ holonomy-flux algebra from the covariant $\text{SL}(2, \mathbb{C})$ one

- **Flux**: $X = L = -K/\gamma$
- **Holonomy** $X \leftrightarrow h \sim A_{AB}^\gamma \leftrightarrow A$

Recall that $\tilde{\Pi} = -h\Pi h^{-1}$. From this plus $F_i = \tilde{F}_i = 0$, it follows that

\[
(h^\dagger h)^A_B \omega^B = e^{-\Xi} \omega^A, \quad (h^\dagger h)^A_B \pi^B = e^{\Xi} \pi^A
\]

Using this one proves that the reduced $g$ is the holonomy of the Ashtekar-Barbero connection, a non-trivial and $\gamma$-dependent mixing of real and imaginary parts of $\ln h = \Gamma + iK$:

\[
g \sim e^{\text{Re} \ln h} e^{\gamma \text{Im} \ln h} + o(\Xi)
\]
Geometric interpretation

Properties of the reduction:

- On $C = 0$ surface, 1st class diagonal simplicity constraint $D = F_1 + \tilde{F}_1$
- Each orbit of $D$ spanned by
  \[ \Xi := 2 \ln \left( \frac{\|\omega\|}{\|\tilde{\omega}\|} \right) \]
- In the time gauge,
  \[ \tilde{n}_I \Lambda(h)^I J n_J = - \cosh \Xi \]
  The dihedral angle parametrizes the orbits of the diagonal simplicity constraint
- Symplectic reduction eliminates the dependence on $\Xi$, which is the coordinate of the orbits of the diagonal simplicity constraint $D$
- Secondary constraints: provide a non-trivial gauge-fixing of the orbits and thus non-trivial embedding of $T^*SU(2)$ in $T^*SL(2, \mathbb{C})$, precisely as in the continuum
Geometric interpretation: covariant twisted geometries

- **gauge-fixed approach** pick a time direction $n_I$ (typically $n_I = (1, 0, 0, 0)$)

  \[ \tilde{n}_I \Lambda(h)^I_J n_J = - \cosh \Xi, \quad \Xi := 2 \ln \left( \frac{\|\omega\|}{\|\tilde{\omega}\|} \right) \]

  \[
  (\Pi, h) \mapsto \left( [\pi|\omega>, \zeta, \tilde{\zeta}, \alpha, \tilde{\alpha}, \xi, \Xi] \right)
  \]

  given by

  \[ \zeta = - \frac{\bar{\omega}^1}{\bar{\omega}^0}, \quad \alpha = \frac{e^{2i \arg(\omega^0)}}{[\pi|\omega]} F_2, \quad \xi := 2 \arg(\tilde{\omega}^0) - 2 \arg(\omega^0) + \gamma \Xi. \]

  reduction to twisted geometries:

  \[
  \Pi = - \frac{i}{2} [\pi|\omega] n(\zeta) T_\alpha \tau_3 T^{-1}_\alpha n^{-1}(\zeta) \quad R - \gamma I = \alpha = \tilde{\alpha} = 0 \quad X = jn(\zeta) \tau_3 n^{-1}(\zeta) \quad \Xi = 0 \\
  h = n(\tilde{\zeta}) T_\tilde{\alpha} e^{(-\xi + (\gamma - i) \Xi) \tau_3} T^{-1}_\tilde{\alpha} n^{-1}(\zeta) \quad \rightarrow \quad g = n(\tilde{\zeta}) e^{-\xi \tau_3} n^{-1}(\zeta) \\
  A = \Gamma + iK = A_{AB}^\gamma - (\gamma - i)K
  \]
More on the brackets and the role of $\gamma$

Symplectic potential of $T^*SL(2, \mathbb{C})$: 

$$(\Pi, h) \mapsto ([\pi | \omega \rangle = R + iI, \zeta, \bar{\zeta}, \alpha, \bar{\alpha}, \xi, \bar{\xi})$$

$$\Theta = -2 \text{Re} \left( \text{Tr}[\Pi h^{-1} dh] \right) =$$

$$= (R - \gamma I) d\Xi + I d\xi - \frac{iI}{1 + |\zeta|^2} (\zeta d\bar{\zeta} - \bar{\zeta} d\zeta) - \frac{1}{1 + |\zeta|^2} (\alpha [\pi | \omega \rangle d\bar{\zeta} + \bar{\alpha} \langle \omega | \pi \rangle d\zeta)$$

$$- \frac{iI}{1 + |\bar{\zeta}|^2} (\bar{\zeta} d\bar{\zeta} - \bar{\zeta} d\bar{\zeta}) - \frac{1}{1 + |\bar{\zeta}|^2} ([\pi | \omega \rangle \bar{\alpha} d\bar{\zeta} + \langle \omega | \pi \rangle \bar{\alpha} d\bar{\zeta}).$$

"Abelian" sector, described by the $(R, I, \xi, \Xi)$, with Poisson brackets

$$\{R, \Xi\} = 1, \quad \{R, \xi\} = \gamma, \quad \{I, \Xi\} = 0, \quad \{I, \xi\} = 1.$$

On the constraint surface, we recover the twisted geometry brackets, in particular

$$\{j, \xi\} = 1$$

where $\xi := 2 \arg(\bar{\omega}^0) - 2 \arg(\omega^0) + \gamma \Xi$ encodes the extrinsic curvature

(But to be able to extract it I need to know explicitly the solution of the secondary constraints, i.e. the embedding in the covariant phase space)
Geometry of a simple twistor

- Incidence relation, \( \omega^A = iX^{A{\hat A}}\pi_{\hat A} \), \( X \in M \) iff null twistor, \( s = 0 \)
- Simple twistors \( (F_i = 0) \):
  \[
  \omega^A = \frac{1}{r} n^{A{\hat A}} e^{i\vartheta/2} \pi_{\hat A}, \quad r := \frac{j\sqrt{1 + \gamma^2}}{\|\omega\|^2}
  \]

1. Not null, but **isomorphic** to null twistors:
   \[
   Z \mapsto Z_\gamma = (\omega^A, e^{i\vartheta/2} \pi_{\hat A}), \quad s(Z_\gamma) = 0
   \]
   Isomorphism depends on \( \gamma \), reduces to the identity for \( \gamma = \infty \)

2. \( F_2 = \langle \omega | \pi \rangle = 0 \) aligns the spinors’ null poles to the time normal
   \[
   \ell^I(\omega) + \frac{1}{r^2} k^I(\pi) = \sqrt{2}\|\omega\|^2 n^I
   \]

A simple twistor is a \( \gamma \)-null twistor with a time-like direction picked up

The projective twistor space \( \mathbb{P}N \) does not depend on the “scale” \( j \) whereas it depends on \( |\langle \omega | \pi \rangle| \)
Twistor networks

Graph $\Gamma$ decorated with a twistor on each half-link,

$$(\mathbb{T}^2)^L // C^L \cong T^* SL(2, \mathbb{C})^L$$

Quantization gives covariant spin networks $\Psi_{\Gamma,(p_l,k_l),Z_\infty[G_l]}$, with the covariant holonomy-flux algebra

- The standard Penrose interpretation is a collection of spinning massless particles
- A new interpretation in terms of geometries can be achieved thanks to the gauge-invariance at the nodes $(\Psi[h_{s(l)}^{-1} G_l h_{t(l)}] = \Psi[G_l])$:

$$\sum_{l \in n} J^L_l = \sum_{l \in n} J^R_l = 0$$

Established in two steps:

1. each left- and right-handed sector defines a twisted geometry
2. simplicity constraints impose the matching between the two geometries
Further comments and questions

- Twistor space carries a representation of the larger group SU(2,2) of conformal transformations of Minkowski: **Non-trivial effect on a twisted geometry!**
  See upcoming paper with Miklos Langvik

- Explicit form of the secondary constraints? **Open problem**
  Assume they correspond to a discretization of the continuum secondary constraints:
  \[ \text{torsionlessness, } \Gamma = \Gamma(E) \]
  Such a discrete Levi-Civita connection is well known in the context of Regge calculus
  Can it be constructed also without shape matching conditions? **Yes!**
  See brand new paper by Haggard-Rovelli-Vidotto-Wieland 1211.2166

- **Open question:** is there a consistent classical dynamics for twisted geometries, or only for the subsector of Regge geometries?
- Does shape mismatch play a role in the dynamics?
- Can torsion be consistently encoded in \( \xi \) alone?

  -- end of classical part --
Quantization

Dirac quantization: first quantize auxiliary space, then impose constraints: simplicity on half-links, then area matching on full link

- Quantize initial twistorial phase space, à la Schrödinger:

\[
\begin{align*}
[\hat{\pi}_A, \hat{\omega}^B] &= -i\hbar\delta_A^B, & f(\omega) &\in L^2(\mathbb{C}^2, d^4\omega) \\
[\hat{\pi}_A, \hat{\omega}^B] &= i\hbar\delta_A^B, & f(\hat{\pi})
\end{align*}
\]

- Convenient basis homogeneous functions

\[
f^{(\rho,k)}(\lambda\omega) = \lambda^{-k-1+i\rho}\bar{\lambda}^{k-1+i\rho}f^{(\rho,k)}(\omega)
\]

(carry a unitary, infinite dimensional representation of the Lorentz group)

- Simplicity constraints

\[
\hat{F}_1 f^{(\rho,k)}(\omega) = 0 \quad \Rightarrow \quad \rho = \gamma k
\]

\[
\mathcal{M} = F_2 \bar{F}_2 \quad \hat{M} f^{(\rho,k)}(\omega) = 0 \quad \Rightarrow \quad k = j
\]

- Solution space

\[
f_{jm}^{(\gamma,j,j)}(\omega) = \|\omega\|^{2(i\gamma j-j-1)}\langle j,m|j,\omega\rangle_{\text{Perelomov}}
\]

(Not a function of the $F_1$-reduced phase space, $\mathbb{C}^2 \ni z^A$. Half-density $\sqrt{d^2\omega}f_{jm}^{(\gamma,j,j)}$ is.)

- Area matching constraint

\[
\left(\hat{M} f^{\rho,k} \otimes f^{\tilde{\rho},\tilde{k}}\right)(\omega, \hat{\pi}) = 0 \quad \Rightarrow \quad (\tilde{\rho} = \rho, \tilde{k} = k)
\]

- Solution space

\[
G_{jm\bar{m}}^{(j)}(\omega, \hat{\pi}) := f_{jm}^{(\gamma,j,j)}(\omega) f_{j\bar{m}}^{(\gamma,j,j)}(\hat{\pi})
\]
From homogeneous functions to cylindrical functions

- Solution space $G^{(j)}_{\tilde{m},m}(\omega, \tilde{\pi}) := f_{\gamma j,j}^{(\gamma j,j)}(\omega)f_{\tilde{m},m}^{(\gamma j,j)}(\tilde{\pi})$
  (Morally Perelomov coherent states up to non-Lorentz-inv. norms $\|\omega\|$ and $\|\tilde{\pi}\|$)

- carries a representation of the holonomy-flux algebra though harmonic oscillators
  - Fluxes: Schwinger representation of $SL(2, \mathbb{C})$
  - Holonomy: ordering ambiguities, more natural operators twisted geometries $\hat{\zeta}, \hat{\xi},$ etc.

- And the usual cylidrical functions? The relation is a kernel in twistor space:

$$\langle G^{(j)}_{\tilde{m},m} \rangle(g) := \int_{\mathbb{T}^2} d\mu(Z, \tilde{Z}) e^{iS_{BF}(Z, \tilde{Z}, g)} G^{(j)}_{\tilde{m},m}(\omega, \tilde{\pi}) = \mu(j) D^{(\gamma j,j)}_{j\tilde{m},jm}(g),$$

- It allows us to recover the EPRL model

$$A(g) = \sum_{\gamma j} \mu_f(j) \text{Tr}_j \left( \prod_{w \in f} D^{(\gamma j,j)}(\tilde{g}_w g_w^{-1}) \right), \quad Z_C = \int \prod_{v, \tau} dg_{v, \tau}$$
Dynamics as integrals in twistor space

\[ \langle G^{(j)}_{\tilde{m},m}(g) \rangle = \int_{T^2} d\mu(Z, \tilde{Z}) e^{iS_{BF}(Z, \tilde{Z}, g)} G^{(j)}_{\tilde{m},m}(\omega, \tilde{\pi}) = \mu(j) D_{j\tilde{m},jm}(g), \]

Two ingredients:

1. Measure

\[ \int_{T^2} d\omega_A \wedge d\omega^A \wedge \ldots \delta_C(M) f(Z, \tilde{Z}) = \int_{T^*SL(2,\mathbb{C})} d^3\Pi \wedge d_{\text{Haar}} h \wedge c.c. f(\Pi, h) \]

FP fixing of the complex area matching condition

- direct geometric proof of gauge-invariance
- Morally the same as the Gaussian version by Etera, Maite and Johannes

2. BF action in terms of spinors

\[ S_w[B, A] = \Pi^A_B [t_w] h^B_A [\partial w] + c.c. = (g\tilde{g}^{-1})^A_B (\tilde{\omega}^B \pi_A + \tilde{\pi}^B \omega_A) + c.c. \]

- bilinear in the spinors
- mixes bulk and boundary elements

See paper for details of proof and explicit value of \( \mu(j) \)
What do we learn from this?

\[ \langle G_{m,m}^{(j)}(g) \rangle := \int_{T^2} d\mu(Z, \tilde{Z}) e^{iS_{BF}(Z, \tilde{Z}, g)} G_{m,m}^{(j)}(\omega, \tilde{\pi}) = \mu(j) D_{j\tilde{m}j\tilde{m}}^{(j\beta\tilde{j}\beta\tilde{j})}(g), \]

1. As a path integral in twistors, action gives trivial dynamics in phase space

\[ \langle G_{m,m}^{(j)}(g) \rangle = \int_{T^2} d\mu(Z, \tilde{Z}) e^{iS_{tot}(Z, \tilde{Z}, g)} \]

2. Non-trivial dynamics emerges by an interplay between boundary phase space variables (the twistors) and bulk degrees of freedom (the bulk holonomies \( g \))
key “gluing” role played by saddle point conditions

3. 4-simplex asymptotics and Regge data: 3d dihedral angles fit in the phase space (the \( \zeta \)'s), whereas areas \( j \) and extrinsic geometry \( \xi, \Xi \) are thrown away, but recovered, in two different ways:
   - areas identified with quantum spins, and summed over in the definition of the model
   - dihedral angles \( \Xi \) encoded in the bulk holonomies \( g \) which become the Regge-Levi-Civita holonomies at the saddle point

4. Relaxing wedge flatness, we can define a curvature tensor and decompose it into irreps: Weyl, Ricci and torsional parts
   - Still of Petrov type D, like in Regge calculus
   - Additional torsional components carried by certain pieces of the holonomy identified

All of this points in the same direction: can we rewrite the model in a fully coherent way, where the phase space variables are treated on equal footing?
Similar ideas have been expressed at length, Daniele, Etera, Maite, Valentin, Laurent, . . .
Conclusions and outlook

Can we express the dynamics as integrals over phase space? i.e. \[ \sum_j \rightarrow \int dj d\xi \]
(see Etera and Maite for an euclidean model of this) in such a way that exponential of a suitable discretization of GR immediately emerges, not only at holonomy saddle point. This should be a discretization of GR:

1. in the first order formalism
2. without shape matching

Again, the key open question is the existence of a dynamics for twisted geometries

Remarks:

- A priori this is not necessary: correct semiclassical limit may also emerge from coarse graining graphs, and not graph by graph But we need to find this out!
- Regardless, there should be many different discrete versions of GR, like there are of gauge theories on a lattice. Why do we know only one version?

Longer term questions

- Can the formulation as integrals in twistor space improve explicit calculations?
- Any connections with twistors of the complete, infinitely-many-dofs LQG theory?

– finis terrae –