Twistorial structure of loop quantum gravity transition amplitudes

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mainly based on work with Wolfgang Wieland 1207.6348

Outline and goals

Classical Theory

- Description of the covariant phase space in terms of twistors
  algebra of the area matching and simplicity constraints
- Smearing the connection-tetrad algebra to the holonomy-flux algebra before or after solving the simplicity constraints is equivalent: $T^*\mathrm{SU}(2)$ with AB holonomy
- Notion of *simple twistors* solving the simplicity constraints

Quantum theory

- Hilbert space represented via homogeneous functions on spinor space
  (instead of cylindrical functions)
- Dynamics as integrals in twistor space
- Embedding of the Regge data of the EPRL asymptotics in the initial phase space

Work and ideas shared with a number of collaborators
L. Freidel, E. Livine, W. Wieland, B. Dittrich, C. Rovelli, M. Dupuis, J. Tambornino, ...

<u>This talk:</u> w Wolfgang Wieland 1207.6348 PRD12 w Miklos Langvik (to appear)

Smearing and simplicity constraints

Phase spaces:

$$
\begin{array}{ccc}
\mathrm{SL}(2,\mathbb{C}) \text{ variables} & \overset{\text{Smearing}}{\Longrightarrow} & T^{*}\mathrm{SL}(2,\mathbb{C}) \\
\Downarrow & \begin{array}{c}\textit{Solve simplicity constraints} \\ \textit{(primary and secondary)}\end{array} & \Downarrow \\
\text{Ashtekar-Barbero variables} & \Longrightarrow & T^{*}SU(2)
\end{array}
$$

Does the diagram commute?

LQG path: $\hookrightarrow$ spin foam path: the other one

In the continuum:

- primary: simple bivectors, unique metric structure
- secondary: embedding of AB variables in covariant phase space: $\Gamma = \Gamma(E)$

In the discrete:

- primary: simple twistors, unique (twisted) geometry
- secondary: non-trivial embedding of $T^{*}\mathrm{SU}(2)$ in $T^{*}\mathrm{SL}(2,\mathbb{C})$. Discrete $\Gamma(E)$?

Smearing and simplicity constraints

Phase spaces:

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{C}) \text{ variables} & \overset{\text{Smearing}}{\Longrightarrow} & T^*\mathrm{SL}(2,\mathbb{C}) \\ \Downarrow & \begin{array}{c}\textit{Solve simplicity constraints}\\ \textit{(primary and secondary)}\end{array} & \Downarrow \\ \text{Ashtekar-Barbero variables} & \Longrightarrow & T^*\mathrm{SU}(2) \end{array}$$

Strategy: do symplectic reduction of the primary constraints, and think of the secondary constraints as providing a non trivial gauge-fixing section.

$\Rightarrow$ allows to show commutativity of the procedure, while postponing knowning explicitly the form and solution of the secondary constraints

Technique: twistorial tools: parametrize the phase spaces and the constraints in terms of twistors

$\Rightarrow$ allows to embed the non-linear HF phase space into a linear one, explicit calculations, plus direct link with variables used in the twisted geometry parametrization

Panorama

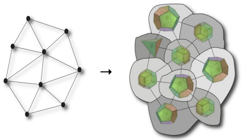
twistors	$\rightarrow$ *$\mathbb{C}$-area matching*	covariant twisted geometry	$\Longleftrightarrow$	covariant loop gravity $\mathrm{SL}(2,\mathbb{C})$
$\downarrow$	*simplicity constraints*	$\downarrow$		$\downarrow$
SU(2) spinors	$\rightarrow$ *area matching*	"open" twisted geometry	$\Longleftrightarrow$	loop gravity $\mathrm{SU}(2)$
		$\downarrow$ *closure*		$\downarrow$ *Gauss law*
		"closed" twisted geometries (polyhedra)	$\Longleftrightarrow$	gauge-inv. loop gravity
		$\downarrow$ *shape matching*		
		Regge calculus		

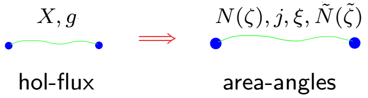
LQG and twisted geometries

On a fixed graph $\mathcal{H}_\Gamma = L_2[\mathrm{SU}(2)^L]$ $P_\Gamma = T^*\mathrm{SU}(2)^L$

represents a truncation of the theory to a finite number of degrees of freedom w Rovelli, PRD 10

These can be interpreted as discrete geometries, called twisted geometries w Freidel, PRD 10, . . .





Each classical holonomy-flux configuration on a fixed graph can be visualized as a collection of adjacent polyhedra with extrinsic curvature between them

- Each polyhedron is locally flat: curvature emerges at the hinges, as in Regge calculus
- They induce a *discontinuous* discrete metric: two neighbouring polyhedra are attached by faces with same area but different shape
- ξ: extrinsic geometry

- Realizes a link suggested by Immirzi, Smolin et al., with the subtlelty of *discontinuity*: Regge calculus is too "rigid" to capture the kinematical degrees of freedom of a spin network
- The matching with Regge geometries becomes exact in 2+1 dimensions

The picture can be embedded in a $\mathrm{SL}(2,\mathbb{C})$ formalism described by *simple* twistors

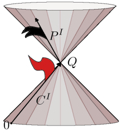
Twistors

- A pair of spinors, ω and π, chiral under Lorentz transformations,

$$Z = \begin{pmatrix} \omega^A \\ \bar{\pi}_{\dot{A}} \end{pmatrix} \in \mathbb{T} := \mathbb{C}^2 \times \bar{\mathbb{C}}^{2*}$$

Twistor "complex helicity":	$\pi_A \omega^A = [\pi\vert\omega\rangle$	$\mathrm{SL}(2,\mathbb{C})$-invariant
Twistor norm (helicity)	$s := \mathrm{Im}\big([\pi\vert\omega\rangle\big)$	$\mathrm{SU}(2,2)$ invariant

- physical picture as a massless particle with a certain spin and momentum
- geometric picture via incidence relation $\omega^A = i X^{A\dot{A}} \bar{\pi}_{\dot{A}}$
  - X a light ray in Minkowski space for a *null* twistor $s = 0$
  - X a congruence of light rays for a generic twistor



- can provide notion of non-linear graviton

We will see that twistors can be thought of as non-linear gravitons in a completely different way than Penrose's original one, as classical counterparts of LQG's quantum geometry

Representation of the Lorentz algebra

- $\mathbb{C}^2$ carries a representation of the Lorentz algebra (the celestial sphere)
- Similarly, there is a representation of the Lorentz algebra on twistor space $\mathbb{T}$

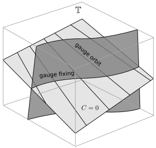
$$\Pi^{AB} = \frac{1}{2}(L + iK)^{AB} = \frac{1}{2}\omega^{(A}\pi^{B)}$$

- We are interested in a representation of a larger Poissonian algebra, the one of $T^*\mathrm{SL}(2,\mathbb{C})$: the holonomy-flux algebra of (covariant) loop quantum gravity
- This can be achieved on $\mathbb{T}^2 \ni (Z, \tilde{Z})$,

$$\Pi^{AB} = \frac{1}{2}\omega^{(A}\pi^{B)}, \quad \tilde{\Pi}^{AB} = \frac{1}{2}\tilde{\omega}^{(A}\tilde{\pi}^{B)}, \quad h^{A}{}_{B} = \frac{\tilde{\omega}^{A}\pi_B - \tilde{\pi}^{A}\omega_B}{\sqrt{\tilde{\pi}\tilde{\omega}}\,\sqrt{\pi\omega}}$$

imposing a complex first-class constraint called *complex area matching*

$$C = [\pi|\omega\rangle - [\tilde{\pi}|\tilde{\omega}\rangle = 0,$$
$$\mathbb{C}^8 /\!/ C \cong T^*SL(2,\mathbb{C})$$



That is, both Lorentz generators and holonomies can be expressed as simple functions on a space of two twistors, provided they have the same complex helicity

Twistor Poisson brackets

The symplectic structure simplifies in the larger space $\mathbb{T}^2$:

$\mathbb{T}^2$	$C=0$	$T^*\mathrm{SL}(2,\mathbb{C})$
$\{\omega^A,\omega^B\}=0$ $\{\pi_A,\omega^B\}=\delta_A^B$ $\{\pi_A,\pi_B\}=0$	$\Longrightarrow$	$\{h,h\}=0$ $\{\Pi^i,h\}=\tau^i h$ $\{\Pi^i,\Pi^j\}=\epsilon^{ijk}\Pi^k$

In particular, momenta commute.

$$\begin{array}{cccccc} 16 & \text{twistors } \mathbb{T}^2 & \xrightarrow[\mathbb{C}\text{-area matching } C]{} & T^*\mathrm{SL}(2,\mathbb{C}) & 12 \\ & \downarrow & \textit{simplicity constraints} & \downarrow & \\ 8 & \begin{array}{c}\textit{simple twistors}\\ \mathrm{SU}(2)\text{ spinors}\end{array} & \xrightarrow[\text{area matching } C]{} & T^*\mathrm{SU}(2) & 6 \end{array}$$

Note: the complete system of constraints is reducible

Simplicity constraints

Gauge-fixed approach Pick a time direction n_I (typically time gauge, $n_I = (1,0,0,0)$)
(cfr. with gauge-invariant approach e.g. Bianca and Jimmy)

- Allows to define SU(2) norm $\|\omega\|^2 := \delta_{A\dot{A}}\omega^A\bar{\omega}^{\dot{A}}$ and $\langle\omega|\pi\rangle := \delta_{A\dot{A}}\bar{\omega}^{\dot{A}}\pi^A$
- Primary simplicity constraints: matching of left and right geometries

$$K + \gamma L = 0 \qquad \Leftrightarrow \qquad \Pi = e^{i\theta}\bar{\Pi}, \qquad \gamma = \cot\frac{\theta}{2}$$

  and the same for tilded variables: $\tilde{\Pi} = e^{i\theta}\bar{\tilde{\Pi}}$

- In terms of spinors: $\big([\pi|\omega\rangle = R + iI\big)$

$$F_1 = R - \gamma I = 0, \qquad F_2 = \langle\omega|\pi\rangle = 0, \qquad \bar{F}_2 = 0$$

- On each link, $(M, F_i, \tilde{F}_i)$ form a **second class** system: $\{F_2, \bar{F}_2\} \not\approx 0$
  which is further **reducible**: M and F_1 imply $\tilde{F}_1$.

3 indep. 1st class constraints	$\big(C, D := F_1 + \tilde{F}_1\big)$ aut $\big(C_{\rm red} \in \mathbb{R}, F_1, \tilde{F}_1\big)$
4 2st class constraints	$F_2, \tilde{F}_2$

Counting

3 indep. 1st class constraints	$\left(C, D := F_1 + \tilde{F}_1\right)$ aut $\left(C_{\text{red}} \in \mathbb{R}, F_1, \tilde{F}_1\right)$
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$$\begin{array}{llccr}
16 & \text{twistors } \mathbb{T}^2 & \xrightarrow[\mathbb{C}\text{-}\textit{area matching } C]{} & T^*\mathrm{SL}(2,\mathbb{C}) & 12 \\
 & \quad\downarrow & \textit{simplicity constraints} & \downarrow & \\
8 & \begin{array}{l}\textit{simple twistors} \\ \mathrm{SU}(2) \text{ spinors}\end{array} & \xrightarrow[\textit{area matching } C]{} & T^*\mathrm{SU}(2) & 6
\end{array}$$

Counting:

Right-Down:	$\mathbb{T}^2$	C	$T^*\mathrm{SL}(2,\mathbb{C})$	$D, F_2, \tilde{F}_2$	$T^*\mathrm{SU}(2)$
	16	-2x2	12	-1x2-4	6
Down-Right:	$\mathbb{T}^2$	F	$\mathbb{C}^4$	C_{red}	$T^*\mathrm{SU}(2)$
	16	-2x2-4	8	-1x2	6

Solution of the simplicity constraints

Focus first on half-link: $\quad (\omega,\pi), \quad \big(F_1 = R - \gamma I, \;\; F_2 = \langle\omega|\pi\rangle\big)$

- $F_1 = 0 \quad \Rightarrow \quad [\pi|\omega\rangle = (\gamma + i)j, \;\; j \in \mathbb{R}$
- orbits: $\quad \{F_1, \|\omega\|\} = \frac{1}{1+\gamma^2}\|\omega\|$
- convenient to introduce a F_1-gauge-invariant spinor

$$z^A = \sqrt{2J}\frac{\omega^A}{\|\omega\|^{i\gamma+1}}, \qquad \|z\| := \sqrt{2J}, \qquad \{F_1, z^A\} = 0$$

$$\boxed{\begin{array}{ccccc} & \text{constraint surface} & & \text{orbits} & \\ (\omega^A, \pi^A) & \longrightarrow & (\omega^A, j) & \longrightarrow & z^A \\ & F_i = 0 & & \|\omega\| & \end{array}}$$

Symplectic reduction: $\quad \mathbb{C}^4 /\!/ F \cong \mathbb{C}^2$

- Solutions parametrized by a SU(2) spinor z^A
  (transforms linearly under rotations but not under boosts)
- *Simple* twistors (on the 5d constraint surface): $Z = (\omega^A, j) = (z^A, \|\omega\|)$

Area matching and $T^*\mathrm{SU}(2)$

On the link:

$$
\begin{array}{cccccc}
16 & (\omega^A, \pi^A, \tilde\omega^A, \tilde\pi^A) \in \mathbb{T}^2 & \underset{\mathbb{C}\text{-area matching } C}{\longrightarrow} & (\Pi, h) & 12 \\
& \downarrow & \textit{simplicity constraints} & \downarrow & \\
8 & (z^A, \tilde z^A) \in \mathbb{C}^4 & \underset{\text{area matching } C_{\mathrm{red}}}{\longrightarrow} & (X, g) & 6
\end{array}
$$

$$\boxed{C_{\mathrm{red}} = \|z\| - \|\tilde z\| \qquad \mathbb{C}^4//C_{\mathrm{red}} = T^*\mathrm{SU}(2) \ni (X, g) \qquad \text{Freidel PRD10}}$$

Symplectic reduction by the linear primary simplicity constraints selects an $\mathrm{SU}(2)$ holonomy-flux algebra from the covariant $\mathrm{SL}(2,\mathbb{C})$ one

- Flux: $X = L = -K/\gamma$
- Holonomy $X \leftrightarrow h \ \sim \ A^\gamma_{AB} \leftrightarrow A$

  Recall that $\tilde\Pi = -h\Pi h^{-1}$. From this plus $F_i = \tilde F_i = 0$, it follows that

  $$(h^\dagger h)^A{}_B \omega^B = e^{-\Xi}\omega^A, \qquad (h^\dagger h)^A{}_B \pi^B = e^{\Xi}\pi^A$$

  Using this one proves that the reduced g is the holonomy of the Ashtekar-Barbero connection, a non-trivial and γ-dependent mixing of real and imaginary parts of $\ln h = \Gamma + iK$:

  $$g \sim e^{\mathrm{Re}\ln h} e^{\gamma \mathrm{Im}\ln h} + o(\Xi)$$

Geometric interpretation

Properties of the reduction:

- On $C = 0$ surface, 1st class diagonal simplicity constraint $D = F_1 + \tilde{F}_1$
- Each orbit of D spanned by
$$\Xi := 2 \ln \left(\frac{\|\omega\|}{\|\tilde{\omega}\|} \right)$$
- In the time gauge,
$$\tilde{n}_I \Lambda(h)^I{}_J n_J = -\cosh \Xi$$

  The dihedral angle parametrizes the orbits of the diagonal simplicity constraint
- Symplectic reduction eliminates the dependence on Ξ, which is the coordinate of the orbits of the diagonal simplicity constraint D
- Secondary constraints: provide a non-trivial gauge-fixing of the orbits and thus non-trivial embedding of $T^*\mathrm{SU}(2)$ in $T^*\mathrm{SL}(2,\mathbb{C})$, precisely as in the continuum

Geometric interpretation: covariant twisted geometries

- gauge-fixed approach pick a time direction n_I (typically $n_I = (1,0,0,0)$)

$$\tilde{n}_I \Lambda(h)^I{}_J n_J = -\cosh \Xi, \qquad \Xi := 2\ln\left(\frac{\|\omega\|}{\|\tilde{\omega}\|}\right)$$

$$\boxed{(\Pi, h) \mapsto \Big([\pi|\omega\rangle, \zeta, \tilde{\zeta}, \alpha, \tilde{\alpha}, \xi, \Xi\Big)}$$

given by

$$\zeta = -\frac{\bar{\omega}^1}{\bar{\omega}^0}, \qquad \alpha = \frac{e^{2i\arg(\omega^0)}}{[\pi|\omega\rangle} F_2, \qquad \xi := 2\arg(\tilde{\omega}^0) - 2\arg(\omega^0) + \gamma\Xi.$$

reduction to twisted geometries:

$\Pi = -\frac{i}{2}[\pi\|\omega\rangle n(\zeta) T_\alpha \tau_3 T_\alpha^{-1} n^{-1}(\zeta)$	$R - \gamma I = \alpha = \tilde{\alpha} = 0$	$X = j n(\zeta)\tau_3 n^{-1}(\zeta)$
	$\longrightarrow$	
$h = n(\tilde{\zeta}) T_{\tilde{\alpha}} e^{(-\xi + (\gamma - i)\Xi)\tau_3} T_\alpha^{-1} n^{-1}(\zeta)$	$\Xi = 0$	$g = n(\tilde{\zeta}) e^{-\xi\tau_3} n^{-1}(\zeta)$

$$A = \Gamma + iK = A^\gamma_{AB} - (\gamma - i)K$$

More on the brackets and the role of γ

Symplectic potential of $T^*\mathrm{SL}(2,\mathbb{C})$: $\quad (\Pi, h) \mapsto \Big([\pi|\omega\rangle = R + iI, \zeta, \tilde\zeta, \alpha, \tilde\alpha, \xi, \Xi\Big)$

$$\begin{aligned}\Theta &= -2\mathrm{Re}\Big(\mathrm{Tr}[\Pi h^{-1} dh]\Big) = \\ &= (R-\gamma I)d\Xi + I d\xi - \frac{iI}{1+|\zeta|^2}(\zeta d\bar\zeta - \bar\zeta d\zeta) - \frac{1}{1+|\zeta|^2}(\alpha[\pi|\omega\rangle d\bar\zeta + \bar\alpha\langle\omega|\pi]d\zeta) \\ &\qquad - \frac{iI}{1+|\tilde\zeta|^2}(\tilde\zeta d\bar{\tilde\zeta} - \bar{\tilde\zeta} d\tilde\zeta) - \frac{1}{1+|\tilde\zeta|^2}([\pi|\omega\rangle\tilde\alpha d\bar{\tilde\zeta} + \langle\omega|\pi]\bar{\tilde\alpha} d\tilde\zeta).\end{aligned}$$

"Abelian" sector, described by the (R, I, ξ, Ξ), with Poisson brackets

$$\{R, \Xi\} = 1, \qquad \{R, \xi\} = \gamma, \qquad \{I, \Xi\} = 0, \qquad \{I, \xi\} = 1.$$

On the constraint surface, we recover the twisted geometry brackets, in particular

$$\{j, \xi\} = 1$$

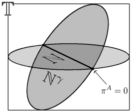
where $\xi := 2\arg(\tilde\omega^0) - 2\arg(\omega^0) + \gamma\Xi$ encodes the extrinsic curvature
(But to be able to extract it I need to know explicitly the solution of the secondary constraints, i.e. the embedding in the covariant phase space)

Geometry of a simple twistor

- Incidence relation, $\omega^A = iX^{A\dot{A}}\bar{\pi}_{\dot{A}}$, $\quad X \in M$ iff null twistor, $s = 0$
- *Simple twistors* $(F_i = 0)$:

$$\omega^A = \frac{1}{r} n^{A\dot{A}} e^{i\frac{\theta}{2}} \bar{\pi}_{\dot{A}}, \qquad r := \frac{j\sqrt{1+\gamma^2}}{\|\omega\|^2}$$

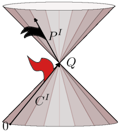
1. Not null, but **isomorphic** to null twistors:



$$Z \;\mapsto\; Z_\gamma = (\omega^A, e^{i\theta/2}\bar{\pi}_{\dot{A}}), \quad s(Z_\gamma) = 0$$

   Isomorphism depends on γ, reduces to the identity for $\gamma = \infty$

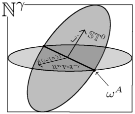
2. $F_2 = \langle\omega|\pi\rangle = 0$ aligns the spinors' null poles to the time normal



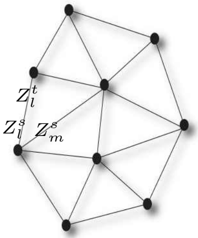
$$\ell^I(\omega) + \frac{1}{r^2} k^I(\pi) = \sqrt{2}\|\omega\|^2 n^I$$

A simple twistor is a γ-null twistor with a time-like direction picked up

The projective twistor space $\mathbb{PN}$ does not depend on the "scale" j whereas it depends on $|\langle\omega|\pi\rangle|$



Twistor networks



Graph Γ decorated with a twistor on each half-link,

$$(\mathbb{T}^2)^L /\!/ C^L \cong T^* SL(2,\mathbb{C})^L$$

Quantization gives covariant spin networks $\Psi_{\Gamma,(\rho_l,k_l),\mathcal{I}_n}[G_l]$, with the covariant holonomy-flux algebra

- The standard Penrose interpretation is a collection of spinning massless particles
- A new interpretation in terms of geometries can be achieved thanks to the gauge-invariance at the nodes ($\Psi[h_{s(l)}^{-1}\, G_l\, h_{t(l)}] = \Psi[G_l]$):

$$\sum_{l\in n} \vec{J}_l^L = \sum_{l\in n} \vec{J}_l^R = 0$$

Established in two steps:

1. each left- and right-handed sector defines a twisted geometry
2. simplicity constraints impose the matching between the two geometries

Further comments and questions

- Twistor space carries a representation of the larger group SU(2,2) of conformal transformations of Minkowski: Non-trivial effect on a twisted geometry!
  See upcoming paper with Miklos Langvik
- Explicit form of the secondary constraints? Open problem
  Assume they correspond to a discretization of the continuum secondary constraints:
  $$\text{torsionlessness}, \qquad \Gamma = \Gamma(E)$$
  Such a discrete Levi-Civita connection is well known in the context of Regge calculus
  Can it be constructed also without shape matching conditions? Yes!
  See brand new paper by Haggard-Rovelli-Vidotto-Wieland 1211.2166
- Open question: is there a consistent classical dynamics for twisted geometries, or only for the subsector of Regge geometries?
- Does shape mismatch play a role in the dynamics?
- Can torsion be consistently encoded in ξ alone?

– end of classical part –

Quantization

Dirac quantization: first quantize auxiliary space, then impose constraints: simplicity on half-links, then area matching on full link

- Quantize initial twistorial phase space, **à la Schrödinger**:

$$[\hat{\pi}_A, \hat{\omega}^B] = -i\hbar\delta_A^B, \qquad f(\omega) \in L^2(\mathbb{C}^2, d^4\omega) \qquad \left(\hat{\omega} = \omega,\ \hat{\pi} = -i\hbar\frac{\partial}{\partial\omega}\right)$$
$$[\hat{\tilde{\pi}}_A, \hat{\tilde{\omega}}^B] = i\hbar\delta_A^B, \qquad f(\tilde{\pi})$$

- Convenient basis homogeneous functions $f^{(\rho,k)}(\lambda\omega) = \lambda^{-k-1+i\rho}\bar{\lambda}^{k-1+i\rho}f^{(\rho,k)}(\omega)$ (carry a unitary, infinite dimensional representation of the Lorentz group)
- Simplicity constraints

$$\begin{aligned} & \hat{F}_1 f^{(\rho,k)}(\omega) = 0 \quad \Rightarrow \quad \rho = \gamma k \\ \mathcal{M} = F_2\bar{F}_2 \qquad & \hat{\mathcal{M}} f^{(\rho,k)}(\omega) = 0 \quad \Rightarrow \quad k = j \end{aligned}$$

- Solution space $f_{jm}^{(\gamma j,j)}(\omega) = \|\omega\|^{2(i\gamma j-j-1)}\langle j,m|j,\omega\rangle_{\text{Perelomov}}$
(Not a function of the F_1-reduced phase space, $\mathbb{C}^2 \ni z^A$. Half-density $\sqrt{d^2\omega}f_{jm}^{(\gamma j,j)}$ is.)
- Area matching constraint

$$\left(\hat{M}f^{\rho,k} \otimes f^{\tilde{\rho},\tilde{k}}\right)(\omega,\tilde{\pi}) = 0 \ \Rightarrow \ (\tilde{\rho} = \rho, \tilde{k} = k)$$

- Solution space $G_{m\tilde{m}}^{(j)}(\omega,\tilde{\pi}) := f_{jm}^{(\gamma j,j)}(\omega)f_{j\tilde{m}}^{(\gamma j,j)}(\tilde{\pi})$

From homogeneous functions to cylindrical functions

- Solution space $G^{(j)}_{m\tilde{m}}(\omega,\tilde{\pi}) := f^{(\gamma j,j)}_{jm}(\omega) f^{(\gamma j,j)}_{j\tilde{m}}(\tilde{\pi})$
  (Morally Perelomov coherent states up to non-Lorentz-inv. norms $\|\omega\|$ and $\|\tilde{\pi}\|$)

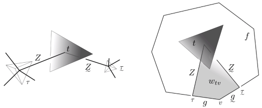
- carries a representation of the holonomy-flux algebra though harmonic oscillators
  - Fluxes: Schwinger representation of $\mathrm{SL}(2,\mathbb{C})$
  - Holonomy: ordering ambiguities, more natural operators twisted geometries $\hat{\zeta},\hat{\xi}$, etc.

- And the usual cylidrical functions? The relation is a kernel in twistor space:

$$\boxed{\left\langle G^{(j)}_{\tilde{m},m}\right\rangle(g) := \int_{\mathbb{T}^2} d\mu(Z,\tilde{Z})\, e^{iS_{BF}(Z,\tilde{Z},g)}\, G^{(j)}_{\tilde{m},m}(\omega,\tilde{\pi}) = \mu(j)\, D^{(\gamma j,j)}_{j\tilde{m}\, jm}(g),}$$

- It allows us to recover the EPRL model

$$A(g) = \sum_j \mu_f(j)\, \mathrm{Tr}_j\Big(\prod_{w\in f} D^{(\gamma j,j)}(\tilde{g}_w g_w^{-1})\Big), \qquad Z_{\mathcal{C}} = \int \prod_{v,\tau} dg_{v\tau}$$



Dynamics as integrals in twistor space

$$\langle G^{(j)}_{\tilde m,m}\rangle(g) := \int_{\mathbb{T}^2} d\mu(Z,\tilde Z)\, e^{iS_{BF}(Z,\tilde Z,g)}\, G^{(j)}_{\tilde m,m}(\omega,\tilde\pi) = \mu(j)\, D^{(\beta j,j)}_{j\tilde m\, jm}(g),$$

Two ingredients:

1. Measure

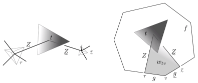
$$\int_{\mathbb{T}^2_{\rm gf}} d\omega_A \wedge d\omega^A \wedge \ldots\ \delta_{\mathbb{C}}(M)\, f(Z,\tilde Z) = \int_{T^*\mathrm{SL}(2,\mathbb{C})} d^3\Pi \wedge d_{\rm Haar}h \wedge c.c.\ f(\Pi,h)$$

   FP fixing of the complex area matching condition

   - direct geometric proof of gauge-invariance
   - Morally the same as the Gaussian version by Etera, Maite and Johannes

2. BF action in terms of spinors

$$S_w[B,A] = \Pi^A{}_B[t_w] h^B{}_A[\partial w] + c.c. = \left(g\tilde g^{-1}\right)^A{}_B(\tilde\omega^B\pi_A + \tilde\pi^B\omega_A) + c.c.$$



   - bilinear in the spinors
   - mixes bulk and boundary elements

See paper for details of proof and explicit value of $\mu(j)$

What do we learn from this?

$$\langle G^{(j)}_{\tilde{m},m}\rangle(g) := \int_{\mathbb{T}^2} d\mu(Z,\tilde{Z})\, e^{iS_{BF}(Z,\tilde{Z},g)}\, G^{(j)}_{\tilde{m},m}(\omega,\tilde{\pi}) = \mu(j)\, D^{(\beta j,j)}_{j\tilde{m}\, jm}(g),$$

1. As a path integral in twistors, action gives trivial dynamics in phase space

$$\langle G^{(j)}_{\tilde{m},m}\rangle(g) = \int_{\mathbb{T}^2} d\mu(Z,\tilde{Z})\, e^{iS_{tot}(Z,\tilde{Z},g)}$$

2. non-trivial dynamics emerges by an interplay between boundary phase space variables (the twistors) and bulk degrees of freedom (the bulk holonomies g)
   key "gluing" role played by saddle point conditions
3. 4-simplex asymptotics and Regge data: 3d dihedral angles fit in the phase space (the ζ's), whereas areas j and extrinsic geometry ξ, Ξ are thrown away, **but recovered**, in two different ways:
   - areas identified with quantum spins, and summed over in the definition of the model
   - dihedral angles Ξ encoded in the bulk holonomies g which become the Regge-Levi-Civita holonomies at the saddle point
4. Relaxing wedge flatness, we can define a curvature tensor and decompose it into irreps: Weyl, Ricci and torsional parts
   - Still of **Petrov type D**, like in Regge calculus
   - Additional torsional components carried by certain pieces of the holonomy identified

All of this points in the same direction: can we rewrite the model in a fully coherent way, where the phase space variables are treated on equal footing?

Similar ideas have been expressed at length, Daniele, Etera, Maite, Valentin, Laurent, ...

Conclusions and outlook

Can we express the dynamics as integrals over phase space? i.e. $\sum_j \rightarrow \int dj d\xi$
(see Etera and Maite for an euclidean model of this)
in such a way that exponential of a suitable discretization of GR immediately emerges, not only at holonomy saddle point. This should be a discretization of GR:

1. in the first order formalism
2. without shape matching

Again, the key open question is the existence of a dynamics for twisted geometries

Remarks:

- A priori this is *not* necessary: correct semiclassical limit may also emerge from coarse graining graphs, and not graph by graph But we need to find this out!
- Regardless, there should be many different discrete versions of GR, like there are of gauge theories on a lattice. Why do we know only one version?

Longer term questions

- Can the formulation as integrals in twistor space improve explicit calculations?
- Any connections with twistors of the complete, infinitely-many-dofs LQG theory?

– finis terrae –