Quantum group spin nets: refinement limit and relation to spin foams

Bianca Dittrich, Mercedes Martin-Benito and S.St.: arXiv:1312.0905
Bianca Dittrich and S.St.: arXiv:1311.7565

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What is the refinement limit of spin foam models?

Spin foams: path integral approach related to Loop Quantum Gravity

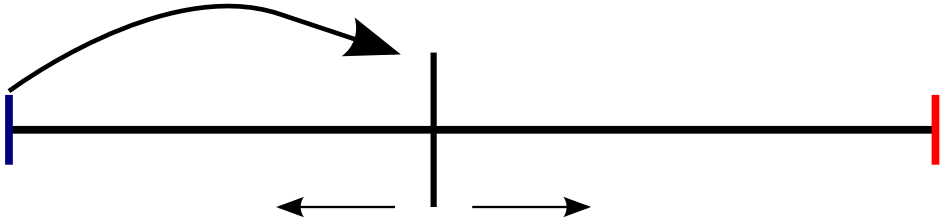
[Barrett, Crane, Rovelli, Reisenberger, Engle, Livine, Pereira, Freidel, Krasnov, ...]

- Lattice as a regulator
  - Semi-classical limit of large building blocks $\rightarrow$ Regge action (discrete gravity) [Barrett, Baez, Freidel, and many others]
  - Generically breaks diffeomorphism symmetry
- Do spin foams possess a smooth phase with dynamics consistent with General Relativity?
- Problems for renormalization / coarse-graining schemes:
  - Identification of relevant d.o.f.
  - No background scale [Bahr, Dittrich '09, Bahr, Dittrich, S.St. '11, Rovelli '11, Dittrich '12]
  - Phases beyond (ordered) BF and degenerate phase?

Goal: Coarse-grain analogue spin foam models on Quantum groups.

What about the simplicity constraints?

simplicity constraints



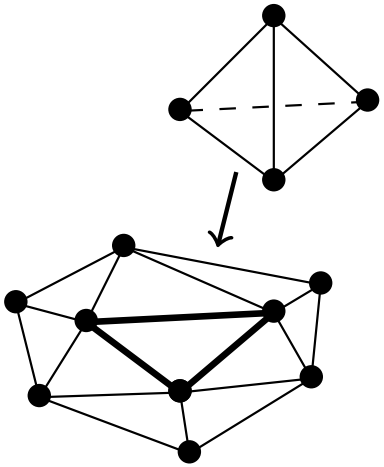
ordered (BF) ? degenerate (HT)

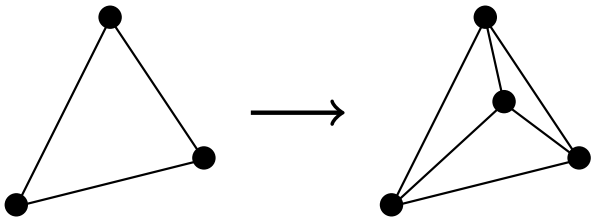
Outline

1. Constructing the physical vacuum
2. From spin foams to spin nets
3. Spin nets as 'melonic' spin foams
4. Quantum group basics
5. Coarse-Graining via Tensor Network Renormalization
6. Results: Fixed points and phase diagrams
7. Conclusion / Outlook

Time evolution via Pachner moves [Dittrich, Höhn, '11, '13]

Glue a d–simplex onto a triangulated $(d-1)$ hypersurface (local time evolution move):





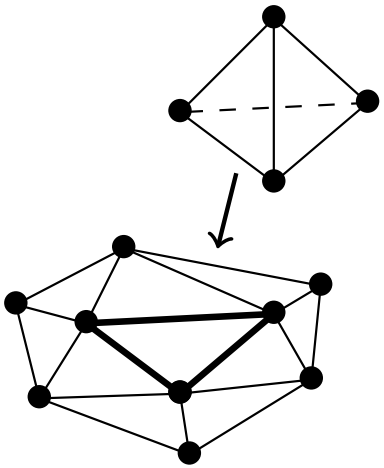
Constraints in the canonical formulation:

- Pre–constraints (conditions for evolution to take place)
- Post–constraints (satisfied after evolution has taken place)

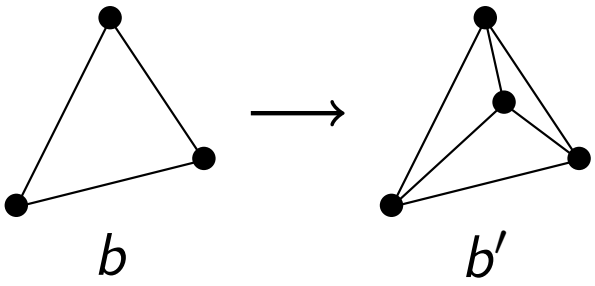
These constraints include diffeo– and Hamiltonian constraints in gravitational theories.

Time evolution via Pachner moves – Quantum theory [Dittrich, S.St., '13, Dittrich, Geiller, '14, Höhn '14]

Hilbert space $\mathcal{H}_b$ associated to boundary b.



$$\iota_{bb'} : \mathcal{H}_b \to \mathcal{H}_{b'}$$



Interpret time evolution as dynamical embedding map $\iota_{bb'}$.

- Under refinement: new degrees of freedom in vacuum state (or gauge).
- Need non–trivial refining maps: **Entangling maps** required (like 2–2 move in 3D or 3–2, 2–3 in 4D).

Cylindrical consistency for embedding maps [Dittrich, S.St. '13]

These embedding maps have to satisfy (cylindrical) consistency conditions (for $b \prec b' \prec b''$):

$$\iota_{b'b''} \circ \iota_{bb'} = \iota_{bb''} \quad .$$

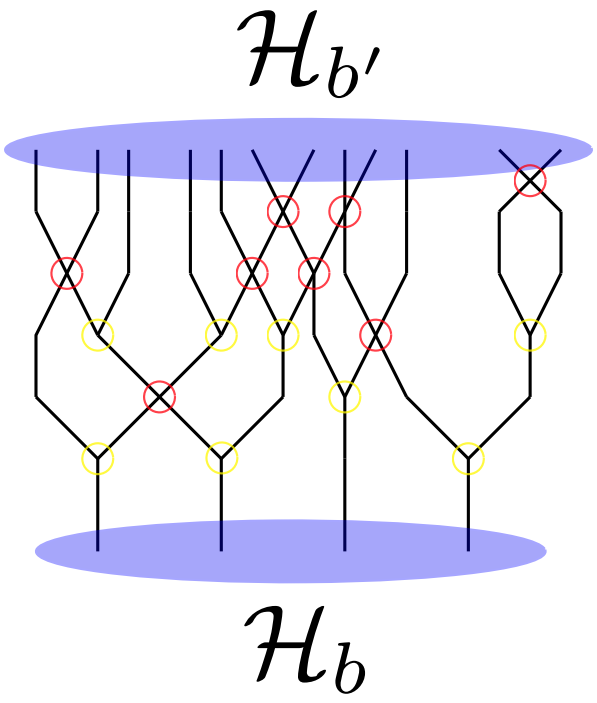
Define a **projective limit** $\mathcal{H} := \cup_b \mathcal{H}_b / \sim$, with equivalence relation ($b \prec b''$ and $b' \prec b''$):

$$\psi_b \sim \psi'_{b'} \quad \Longleftrightarrow \quad \iota_{bb''}(\psi_b) = \iota_{b'b''}(\psi'_{b'}) \quad .$$

- The equivalence class of states $[\psi_b]$ depends only implicitly on the boundary b.
- In gravity, time evolution acts as a **projector** on the physical Hilbert space:
- Evolved / refined state is the **same physical state** represented on a finer boundary.

Consistent time evolution [Dittrich, S.St., '13]

Many ways to evolve from b to b':



- **Dynamical cylindrical consistency** of embedding maps $\iota_{bb'}$.
- **Path independence** of time evolution [Kuchar, '76].
- Equivalent to discrete **Dirac algebra** of constraints [Bonzom, Dittrich, '13]
- **Diffeosymmetry** in the **discrete** [Bahr, Dittrich '09, Bahr, Dittrich, S.St. '11]

How to do in practice?

These conditions are highly non–trivial to satisfy and generically violated by simplicial discretizations.

- Realized perfectly in topological theories.
  - Fixed point presented later.
  - New vacuum for LQG [Dittrich, Geiller, '14] and also Dittrich's ILQGS talk two weeks ago
- Improve the discretization via coarse graining:
  - At fixed points: **perfect discretisation** [Bahr, Dittrich '09, Bahr, Dittrich, S.St. '11]
- Two options:
  - Keep basic building blocks, get **non–local couplings**
  - **Effective building blocks** (represented as amplitude maps [Oeckl '03]) that interact locally [Dittrich '12].

Tensor network renormalization [Levin, Nave '06, Gu, Wen '09] fits into the latter idea.

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Lattice gauge theories

Spin foams can be understood as generalized lattice gauge theories: [Bahr, Dittrich, Hellmann, Kamiński, '12]

Let's consider the classical actions of gravity and Yang–Mills theory:

Plebankski action (with simplicity constraints)

$$S_{Pleb} = \int B \wedge F + \phi B \wedge B \quad ; \quad B \sim *(e \wedge e) \quad .$$

- Impose simplicity constraints $B \sim *(e \wedge e)$ via Lagrange multiplier ϕ

Yang–Mills action with coupling constant g

$$S_{YM} = \int B \wedge F + g^2 B \wedge (\star B) \quad .$$

- Definition requires background metric.
- Weak coupling limit $g \to 0$ is BF theory.

Lattice gauge theories vs. spin foams

Partition function:

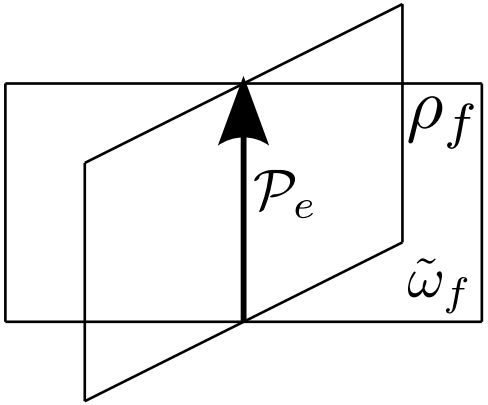
$$Z = \sum_{\rho_f} \left(\prod_f \tilde{\omega}_f(\rho_f) \right) \prod_e \mathcal{P}_e(\{\rho_f\}_{f \supset e})$$

	Lattice gauge theory	Spin foams
Parameter space:	**face weights**: $\tilde{\omega}_f(\rho_f)$	replace **Haar projector**: $\mathcal{P}_e(\{\rho_f\}_{f \supset e})$ smaller subspace [Bahr,Dittrich,Hellmann,Kamiński '12]
Phases / fixed points:	• BF / weak coupling • degenerate / strong coupling •BF on normal subgroups	Same phases as for lattice gauge theories **More phases?**

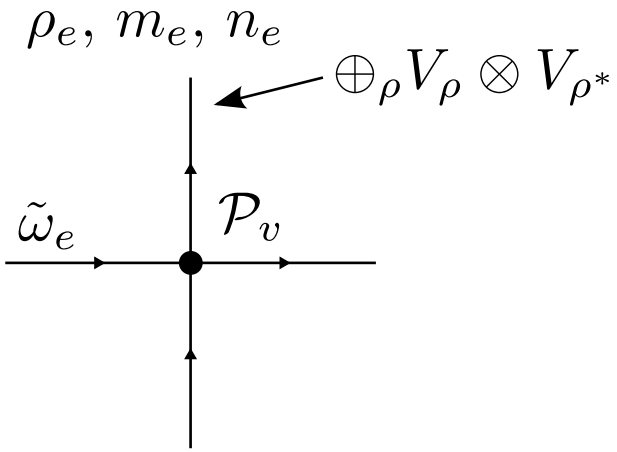
To apply numerical simulations, we have to make two simplifications:

- Dimensional reduction
- 'Regularization' - finite amount of rep. labels

Dimensional reduction



lattice gauge theory
in dual variables



vertex model
in dual variables

- Gauge symmetry (Spin foam model) $\rightarrow$ global symmetry (spin net model)
- Statistical properties of 4D gauge theories and (associated) 2D spin systems are very similar [Kogut '79]. For example:
  - 2D Ising model $\leftrightarrow$ 4D $\mathbb{Z}_2$ lattice gauge theory
  - QCD: 2D Heisenberg model $\leftrightarrow$ 4D $SU(2)$ lattice gauge theory

Attention!

Spin nets are not spin networks!

Regularization

We require finite summations in partition function

- Finite groups [Bahr,Dittrich,Ryan '11]
  - Abelian models [Dittrich,Eckert,Martin-Benito '11]
  - Non-Abelian: S_3 group of permutations [Dittrich,Martin-Benito,Schnetter '13]
- Quantum groups (cosmological constant) with q root of unity: full models [Christensen, Khavkine '07, Fairbain, Meusburger '10, Han '10]
  - Finite amount of representations
  - Same representation theory for all quantum groups

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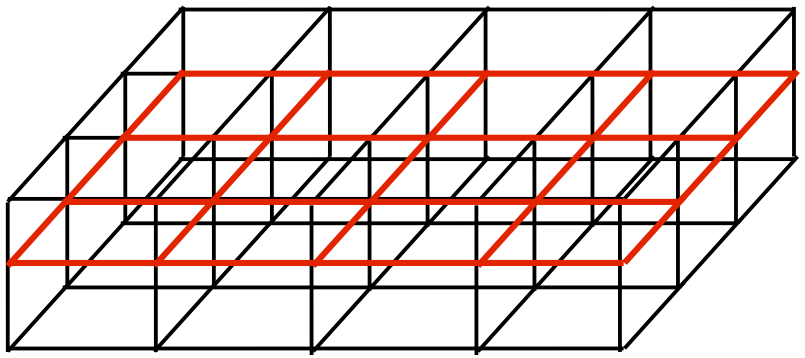
Spin nets as dimensional reduction of spin foams

Consider a $3D$ hypercubic lattice, actually just 'one slice of cubes'.

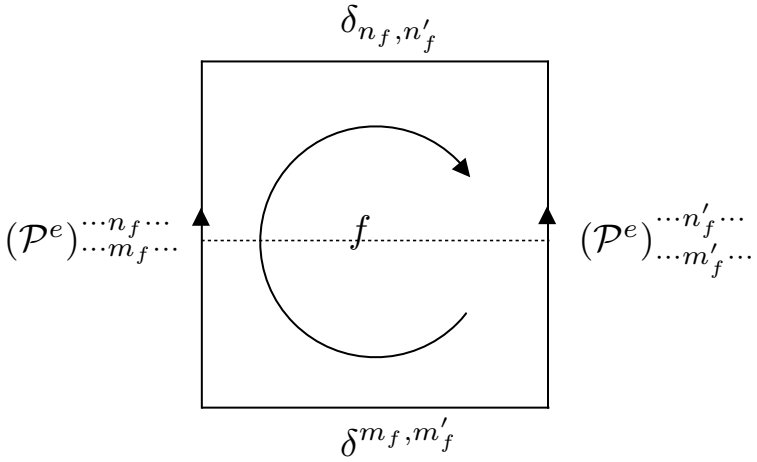


$$Z_{SF} = \sum_{j_f} \prod_f \tilde{\omega}_f(j_f) \prod (\mathcal{P}^e)^{n_{f_1}\, n_{f_2}\, n_{f_3}\, n_{f_4}}_{m_{f_1}\, m_{f_2}\, m_{f_3}\, m_{f_4}} \quad .$$

What is the connection to spin nets?

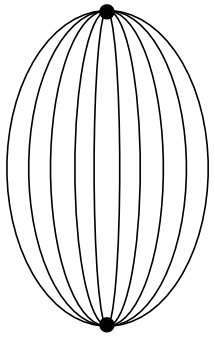
Supermelons

On the horizontal edges, we replace the intertwiners by $\delta_{n_f,n_{f'}}\delta^{m_f,m_{f'}}$.



This gives the spin net partition function:

$$Z_{net} = \sum_{j_e} \prod_e \tilde{\omega}_e(j_e) \prod_e (\mathcal{P}^v)^{n_{e_1}\, n_{e_2}\, n_{e_3}\, n_{e_4}}_{m_{e_1}\, m_{e_2}\, m_{e_3}\, m_{e_4}} \quad .$$



Geometrically, this resembles a 'super-melon'[Riello '13].

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The quantum group $\mathrm{SU}(2)_k$ [1]

$\mathrm{SU}(2)_k$ understood as $\mathcal{U}_q(\mathfrak{su}(2))$, the q-deformation of $\mathcal{U}(\mathfrak{su}(2))$.
We pick q as a root of unity, $q = \exp\left(\frac{i\pi}{k+2}\right)$. $k \in \mathbb{N}$ is the level.

- Three generators $J_\pm$, J_z:

$$[J_z, J_\pm] = \pm J_\pm$$

$$[J_+, J_-] = \frac{q^{J_z} - q^{-J_z}}{q^{1/2} - q^{-1/2}} = [2J_z] \quad .$$

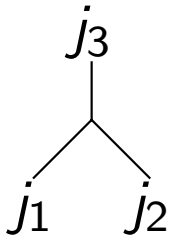
- $[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{\sin\left(\frac{i\pi n}{2k+4}\right)}{\sin\left(\frac{i\pi}{2k+4}\right)}$ is the **(periodic) quantum number**.
- finite dim. representations labelled by $j \in \frac{\mathbb{N}}{2}$, V_j is $(2j+1)$ dimensional.
- $d_j = [2j+1]$ is the **quantum dimension**.
- Reps. $j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$ are called **admissible**, $j > \frac{k}{2}$ are called quantum trace zero and will be modded out.

[1] [L.C. Biedenharn, M.A. Lohe, "Quantum Group Symmetries and q-Tensor Algebras,"]

Diagrammatic Calculus

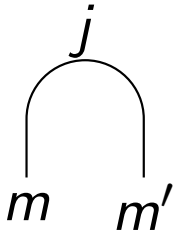
To deal with the subtleties of quantum groups, e.g. definition of the dual, we introduce a diagrammatic calculus.

- Clebsch-Gordans:

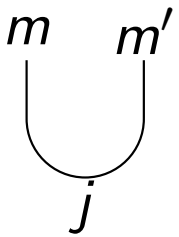


$$:= {}_q\mathcal{C}^{j_1\, j_2\, j_3}_{m_1\, m_2\, m_3} = {}_q C^{j_1\, j_2\, j_3}_{m_1\, m_2\, m_3} \sqrt{(d_{j_3})^{-1}} \quad .$$

- 'Caps' and 'cups':



$$:= (-1)^{j-m} q^{\frac{m}{2}} \delta_{m,-m'} \quad ,$$



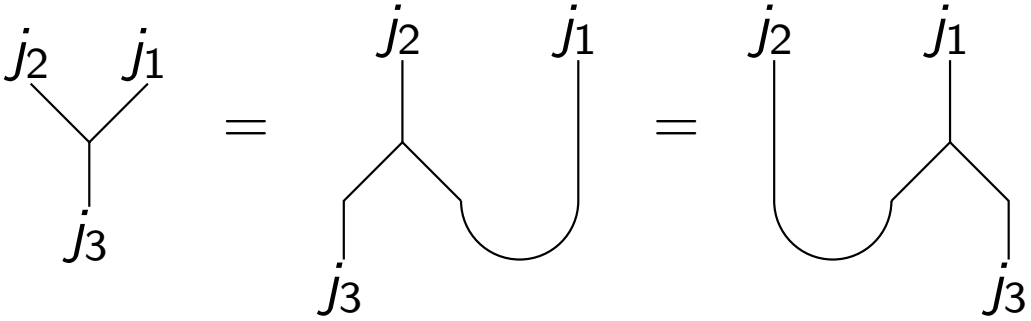
$$= (-1)^{j+m} q^{\frac{m}{2}} \delta_{m,-m'} \quad .$$

Diagrammatic Calculus II

Let's look at two basic diagrammatic calculations:

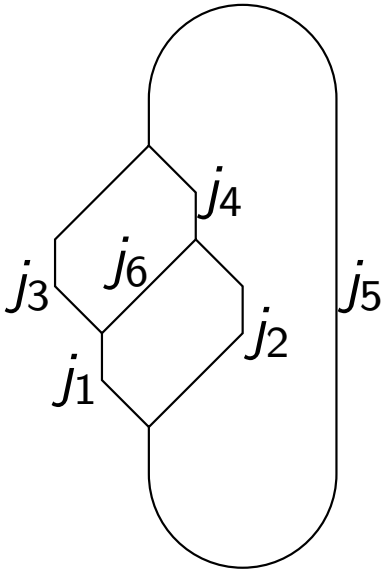
- The Clebsch-Gordons for $\bar{q}$:

$${}_{\bar{q}}\mathcal{C}^{j_1\, j_2\, j_3}_{m_1\, m_2\, m_3} =$$



$$.$$

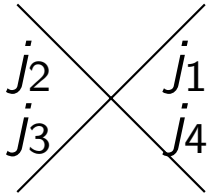
- The $6j$ symbol:



$$= \begin{Bmatrix} j_1 & j_2 & j_5 \\ j_4 & j_3 & j_6 \end{Bmatrix} = \frac{(-1)^{j_1+j_2+j_3+j_4}}{\sqrt{d_{j_5}\, d_{j_6}}} \begin{bmatrix} j_1 & j_2 & j_5 \\ j_4 & j_3 & j_6 \end{bmatrix}$$

Haar projecter $\mathcal{P}$

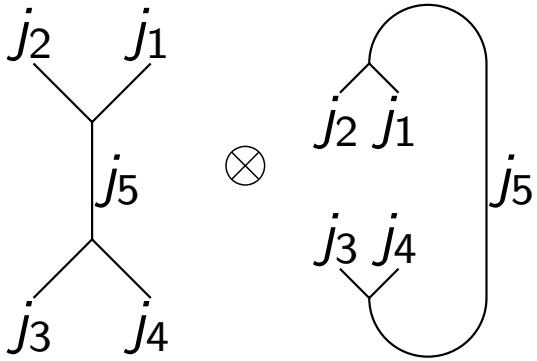
To enforce the quantum group symmetry onto a tensor t, act on both pairs of magnetic indices with the Haar projector $\mathcal{P}$ (recall: $\mathcal{H}_e = \oplus_j V_j \otimes V_{j*}$!)

$\mathcal{P}\ t\ \mathcal{P} = \mathcal{P}$ 

$\mathcal{P}$.

$\mathcal{P}$ is defined as:

$$\mathcal{P}_{(\{m\},\{m'\})}(j_1, j_2, j_3, j_4) = \sum_{j_5} (-1)^{j_1+j_2+j_3+j_4} d_{j_5}$$



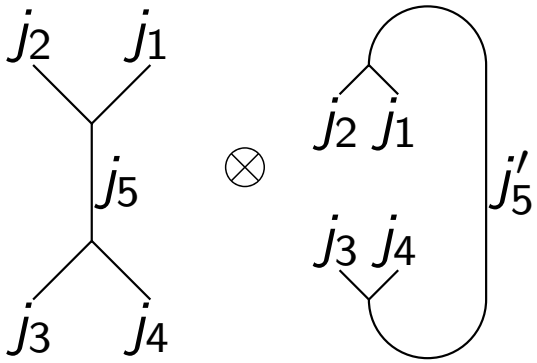
.

Proof that $\mathcal{P} \cdot \mathcal{P} = \mathcal{P}$ is straightforward with diagrammatic calculus.

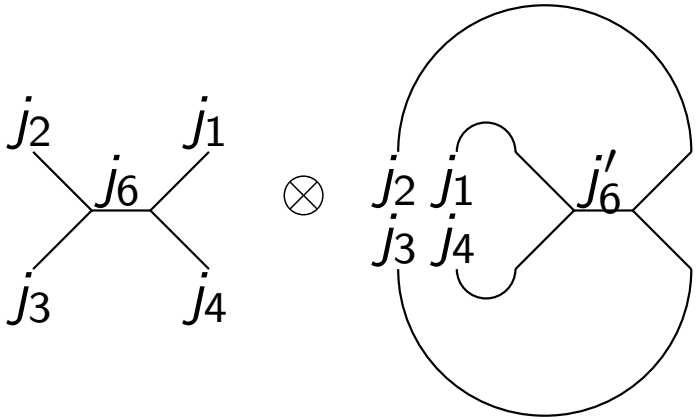
Initial tensor and recoupling basis

The quantum group symmetries give the tensors a particular form:

$$t(\{j\},\{m\},\{n\}) = \sum_{j_5, j_5'} \hat{t}_1^{(j_5, j_5')}(j_1, j_2; j_3, j_4)\; d_{j_5} d_{j_5'}$$



$$= \sum_{j_6, j_6'} \hat{t}_2^{(j_6, j_6')}(j_1, j_4; j_2, j_3)\; d_{j_6} d_{j_6'}$$



$\hat{t}_1$ and $\hat{t}_2$ are the tensors written in the recoupling basis. This plays a crucial role for the **symmetry preserving algorithm**.

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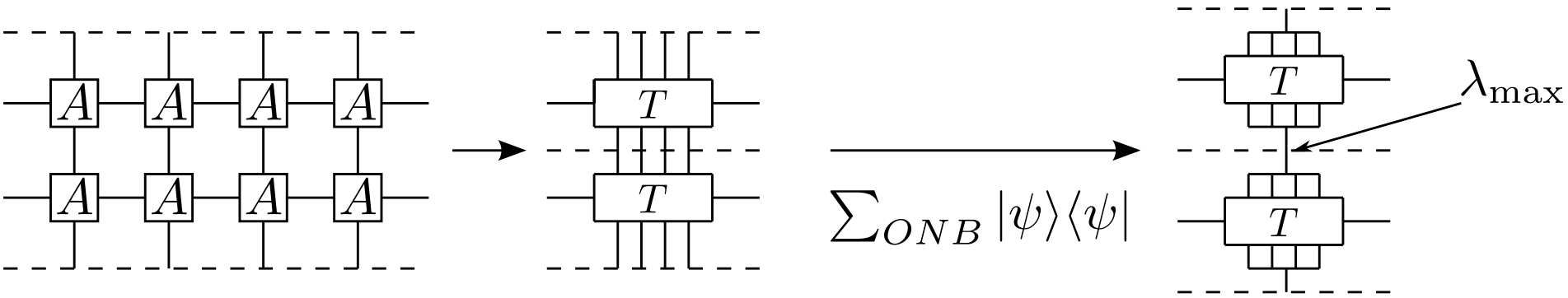
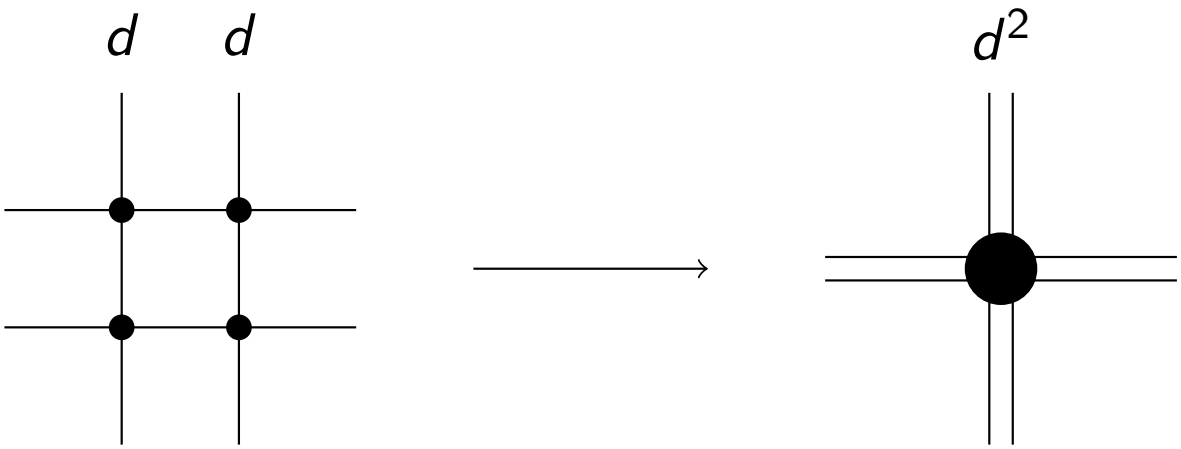
Coarse graining to the continuum

- Effective dynamics at different scales / refinement steps
- Infinite refinement (fixed point) $\rightarrow$ continuum limit
- Problem: Real space renormalization schemes limited [Migdal, Kadanoff '70s]
  - Non-local couplings
  - Truncation / approximations not under control (error?)
- New developments in Condensed Matter / Quantum Information
  - Tensor Network Renormalization [Levin, Nave '06, Gu, Wen '09]



The idea of tensor network renormalization

- We consider a 2D, quadratic lattice. Each lattice site carries a tensor containing **all** ingredients of the model, in particular the intertwiners.
- Coarse graining means combining tensors into a new effective tensor:



- Boundary data grows exponentially – introduce a suitable truncation via **dynamical embedding maps**.
- The new tensors can be regarded as new effective building blocks (that interact locally).

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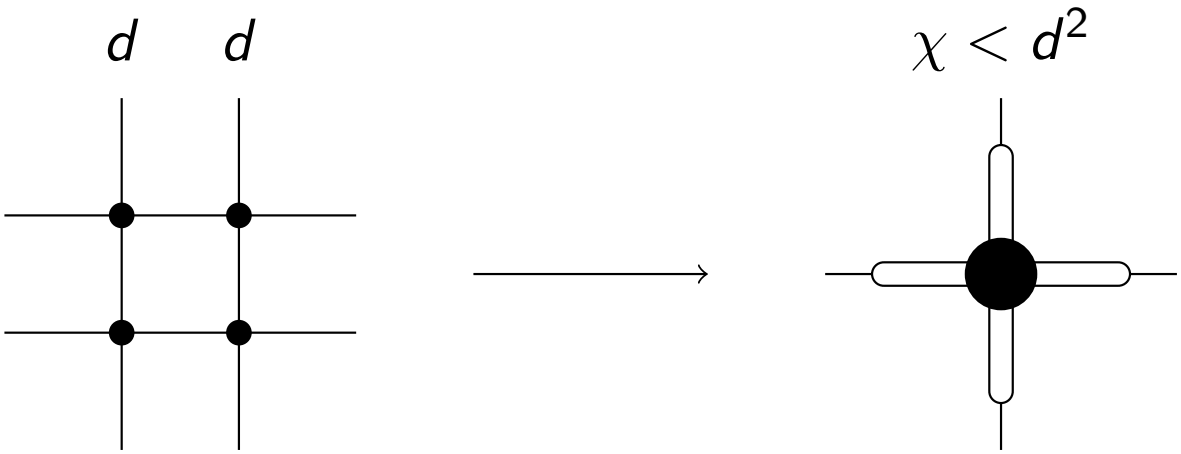

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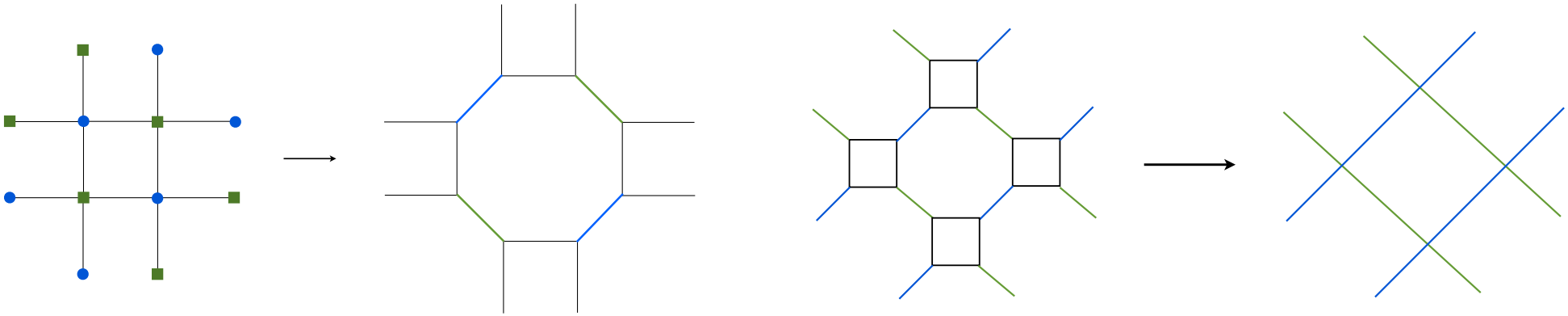
The algorithm - General description [Levin,Nave '06, Gu, Wen '09]

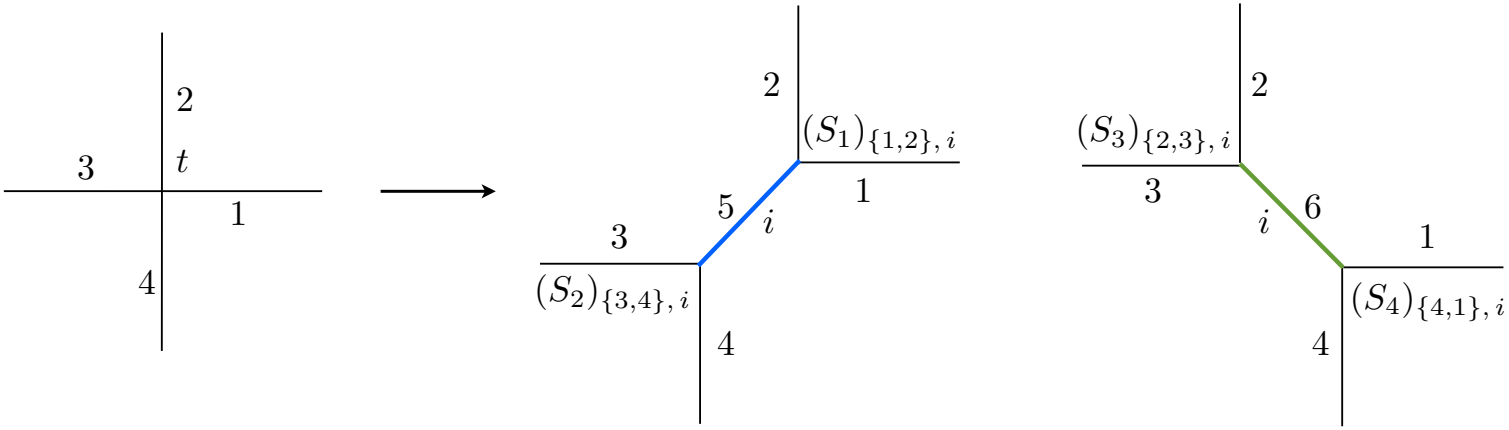


- The 4–valent vertices are split into two 3–valent vertices via **Singular Value Decomposition (SVD)**:

$$t_{(ab);(cd)} =: M_{AB} \approx \sum_{i=1}^{\chi<N} U_{Ai}\lambda_i (V)^{\dagger}_{iB} \quad .$$

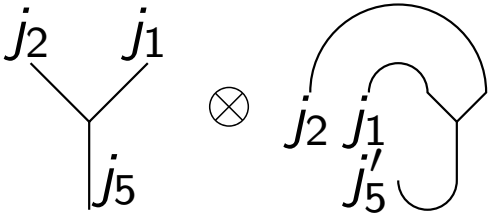
- Truncation in the number of singular values $\chi < N$, **bond dimension**.
- Contract 3-valent tensors (along black edges) to obtain a new **effective tensor** on a new 4–valent lattice.

The algorithm - preserving the symmetries [2]



- We perform the SVD in the **recoupling basis**, e.g. $\hat{t}_1^{(j_5,j_5')}$ for each pair (j_5, j_5'), e.g.:

$$(S_1)^{j_5,j_5'}_{m_5,n_5}(\{I\}_{\{1,2\}}, i) = \sqrt{d_{j_5} d_{j_5'} (\lambda_1)^{(j_5,j_5')}_{ii}} (U_1)^{(j_5,j_5')}_{\{j_1,j_2\},i}$$

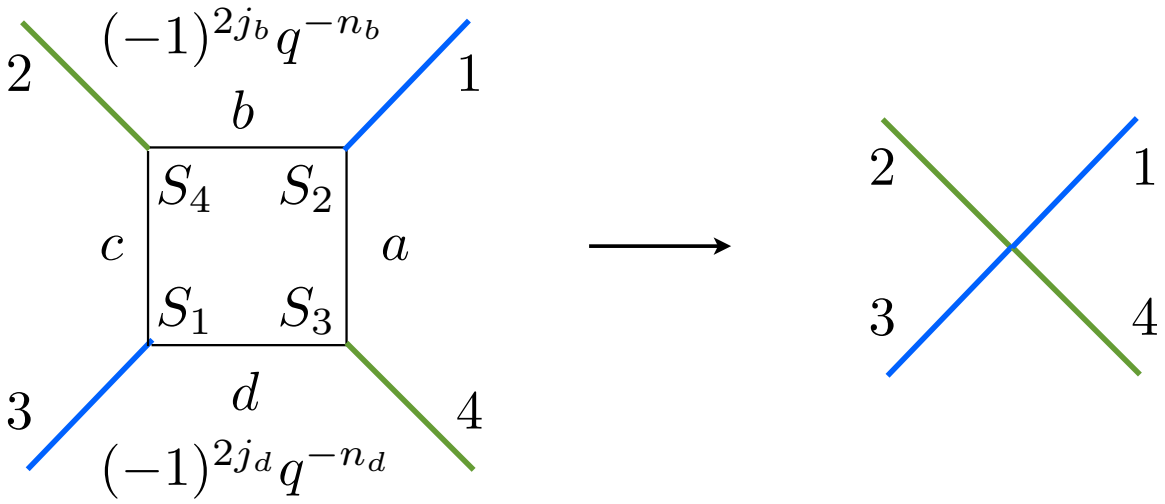


- **Intertwiner channel** (j_5, j_5') appears with multiplicity $\mu_{j_5,j_5'}$.
- New edge **Hilbert space**: $\oplus_{\mu_{(j,j')}} \mathcal{H}_j \otimes \mathcal{H}_{j'}$.

[2] [Dittrich, Martín-Benito, Schnetter '13]

Contraction and flow equation

For the contraction we have to combine all contributions according to the following diagram:



$\rightarrow$ Flow equation for the tensor.

Take home message:

- Relevant d.o.f. chosen dynamically (via SVD).
- Symmetry preserving algorithm – intertwiner channel (j, j').
- System leaves initial space of models.

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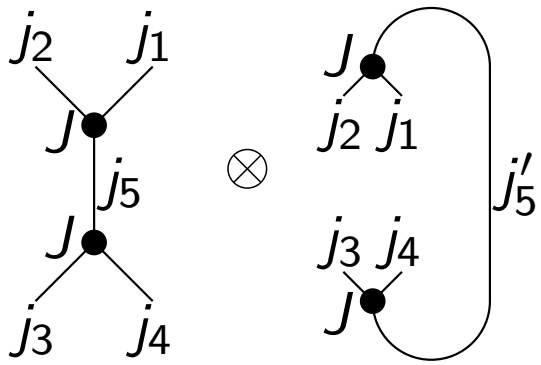
Initial tensor

- Linear combination of **fixed point intertwiners** [Dittrich,Kamiński '13]:

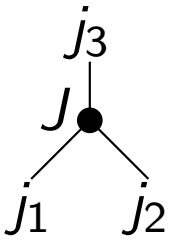
$$t(\{j\},\{m\},\{n\}) = \sum_J \alpha_J \left(\sum_{j_5, j_5'} \right.$$



$$\left. \right) \quad ,$$

with $\sum_J \alpha_J = 1$, $J \in \mathbb{N}$.

- Fat vertex denotes modified Clebsch–Gordan coefficients, labelled by a maximum spin J:



$$:= a^J(j_1, j_2, j_3)\ {}_q\mathcal{C}^{j_1\, j_2\, j_3}_{m_1\, m_2\, m_3} \quad .$$

- Factors $a^J(j_1, j_2, j_3) \sim \begin{bmatrix} j_1 & j_2 & j_3 \\ \frac{J}{2} & \frac{J}{2} & \frac{J}{2} \end{bmatrix}$.

Example: $k = 8$ – Fixed point pattern

For level $k = 8$, the maximal spin $j_{\max} = 4$. We have five parameters α_J, $J \in \{0, 1, \ldots, 4\}$. Consider first that only **one** intertwiner is excited:

J, α_J	fixed point
$J = 0$, $\alpha_0 = 1$	degenerate
$J = 1$, $\alpha_1 = 1$	factorizing $J = 1$
$J = 2$, $\alpha_2 = 1$	factorizing $J = 2$
$J = 3$, $\alpha_3 = 1$	analogue BF
$J = 4$, $\alpha_4 = 1$	"'mixed"

- Three **non–trivial fixed points** beyond BF and degenerate!
- **No fine–tuning** required to flow there.
- Fixed points describe triangulation invariant / topological models.

Pattern generalizes to other quantum groups.

How to identify the different phases?

Look at the **excited intertwiner channels** – their **singular values**:
Matrix of singular values (j, j'):
Degenerate:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Only the trivial representation $j = 0$ is allowed.

How to identify the different phases?

Look at the **excited intertwiner channels** – their **singular values**:
Matrix of singular values (j, j'):
Analogue BF:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Only **diagonal** intertwiner channels (j, j^*) allowed.

How to identify the different phases?

Look at the **excited intertwiner channels** – their **singular values**:
Matrix of singular values (j, j'):
Factorising $J = 1, 2$:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

All channels (j, j') excited, with $j, j' \leq J$. Left and right copy decouple.

How to identify the different phases?

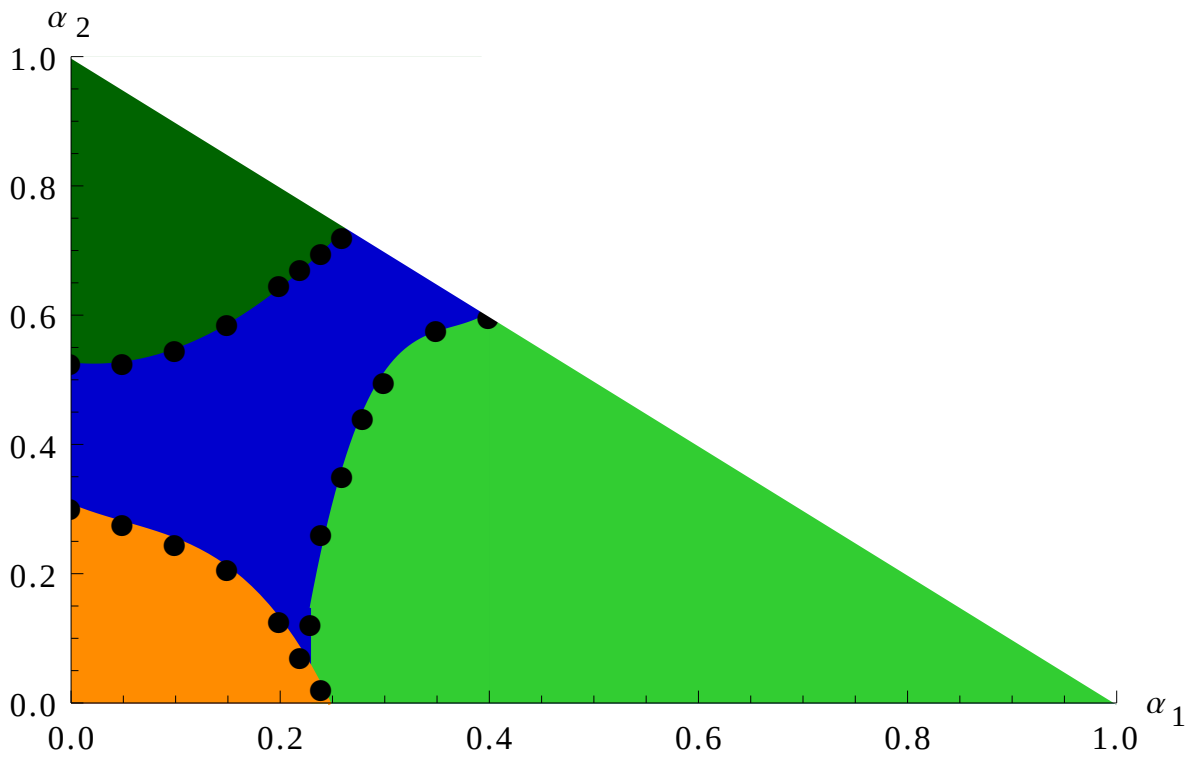
Look at the **excited intertwiner channels** – their **singular values**:
Matrix of singular values (j, j'):
"Mixed":

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

All channels (j, j') with $(j + j')$ even are excited.
Let's look at a phase diagram.

Phase diagram $k = 8^3$

We look at the slice, where $\alpha_0 = \alpha_3 = 0$.

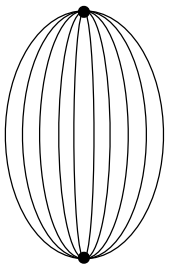


- We find four extended phases, the the analogue BF phase, the factorizing phase with $J = 1$, the factorizing phase with $J = 2$ and the "mixed" phase.

[3] $\alpha_4 = 1 - \alpha_1 - \alpha_2$.

What do spin net fixed points mean in spin foams?

Recall the 'super–melon':



Highly anisotropic spin foam: What do the fixed points imply in this situation?

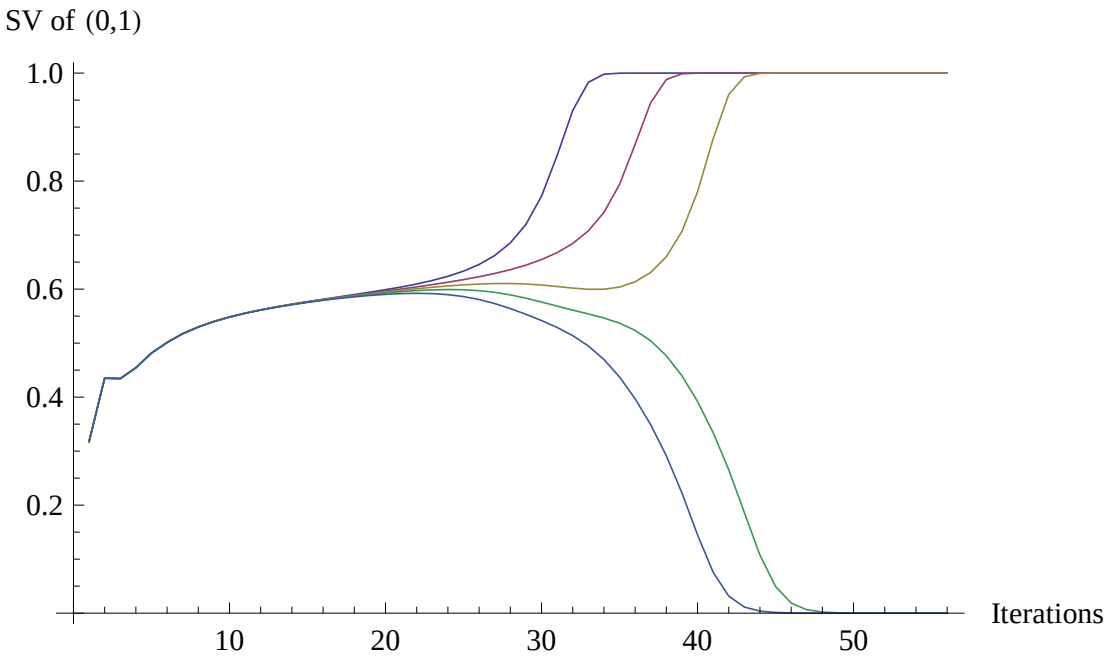
- *Factorizing*: Both vertices **completely decouple**, it decomposes into upper and lower part. Decoupling effect too strong.
- *Analogue BF*: The **'maximal' glueing** of the two vertices, which is in general weakened by simplicity constraints.
- *'Mixed':* We expect this to be **in between** BF and factorizing, so it **couples** the two vertices, but **weaker** than BF.

Two counteracting effects

'Glueing' (analogue BF) $\leftrightarrow$ 'Decoupling' via simplicity constraints

What happens at the phase transition?

Singular values for different initial data during the coarse graining process close to the phase transition (here between BF and factorising $J = 1$):



- 'Plateau' close to phase transition: more iterations, more relevant degrees of freedom... very similar to Ising model!

2nd order phase transition?

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Conclusions

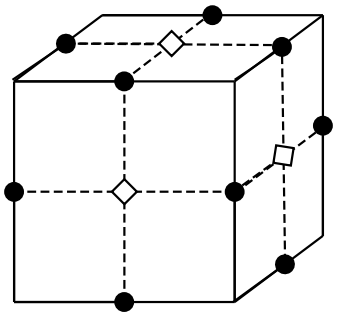
- Construction principle for the physical vacuum: **dynamical cylindrical consistency**
- Definition of quantum group spin net models and the **symmetry preserving** coarse graining algorithm.
- **Conjecture: Intertwiner d.o.f. are relevant (also for spin foams)**.
- Rich fixed point structure with extended phases and phase transitions.
- **No fine tuning** needed to flow to interesting fixed points.
- Although starting from a fixed lattice: Fixed points (under coarse-graining) describe **fully triangulation invariant models**
- Interpretation of spin nets as 'melonic' spin foams.
- Glueing (BF) vs. Decoupling (simplicity constraints) $\rightarrow$ **'mixed' fixed point**
- 2nd order phase transition?

Outlook

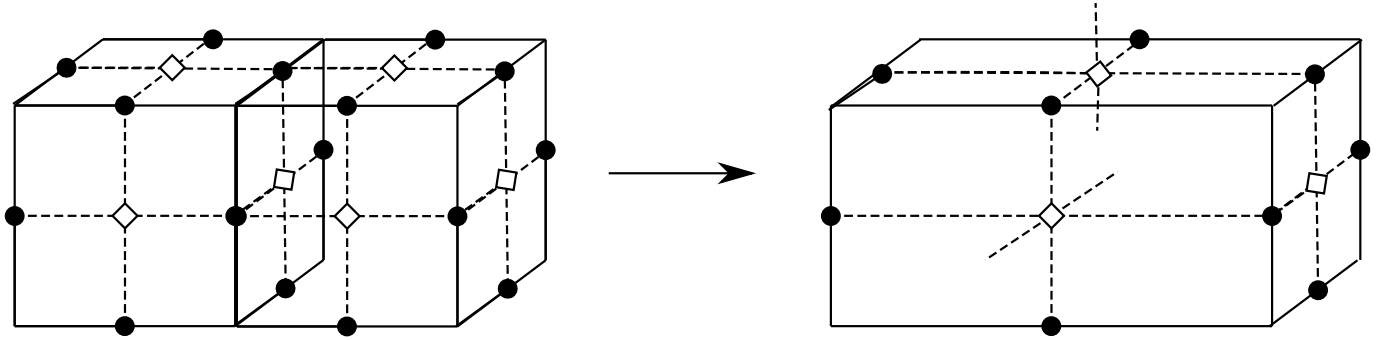
- Explore larger symmetry groups, e.g. $\mathrm{SU}(2)_k \times \mathrm{SU}(2)_k$, analogous to the Barrett-Crane model. Also explore analogue EPRL- and FK-models.
- Further investigate the interplay of 'glueing / decoupling' and the imposition of simplicity constraints.
- Explore behaviour at phase transition further.
- Tune to phase transition to get propagating degrees of freedom.

Outlook: A coarse graining algorithm for spin foams [w.i.p.]

Also spin foams fit into the tensor network language: Define auxiliary tensors [Dittrich, Eckert, Martin-Benito '11]



Glue two cubes together and define new effective tensors [Liu, Meurice, Qin, Unmuth–Yockey, Xiang, Xie, Yu, Zou '13, arXiv:1307.6543]:



Phase diagrams of spin foams in reach!

Thank you for your attention!