Path integral measure and triangulation independence in discrete gravity

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Introduction

Regularisation / Discretisation

Space of geometries
||
$\mathcal{G}/Diff(\mathcal{M})$

$$\int \mathcal{D}g \; e^{i\, S_{EH}[g]}$$

Sum over triangulations
vs.
One fixed triangulation

Diffeomorphisms

Necessary to better understand relation between discrete and continuous geometries.

Measure in Euclidean Regge Calculus

Discretization of the (formal) continuum path integral [Hamber, Williams '99]:

$$I_{disc} = \int \prod_e dl_e^2 \prod_\Delta V_\Delta^\alpha \exp\{-S_R\} \quad , \tag{1}$$

where α is *a priori* unknown.
Some ansätze are non-local [Lund, Regge '74 (unpublished)]:

$$I_{RL} = \int \prod_e dl_e^2 \prod_\Delta \sqrt{\det(G_{ee'})} \exp\{-S_R\} \quad , \tag{2}$$

where $G_{ee'}$ is the discretised De-Witt super metric.

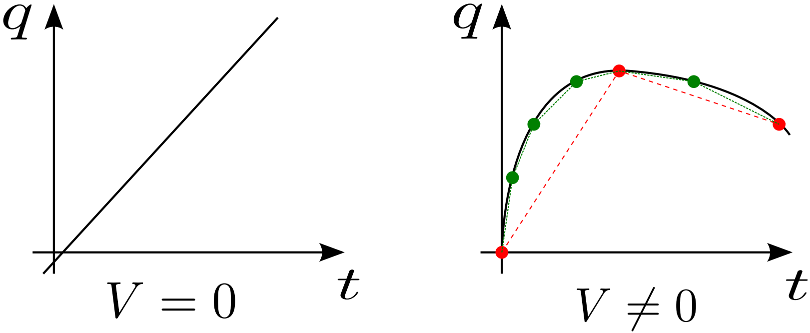
Teaser

By requiring triangulation independence of the path integral, we will obtain a measure similar to the asymptotics of spin foam models!

Outline

In general, discretisations break the symmetries (Diffeomorphisms in GR) of the continuous theory.

Example: 1D parametrised particle

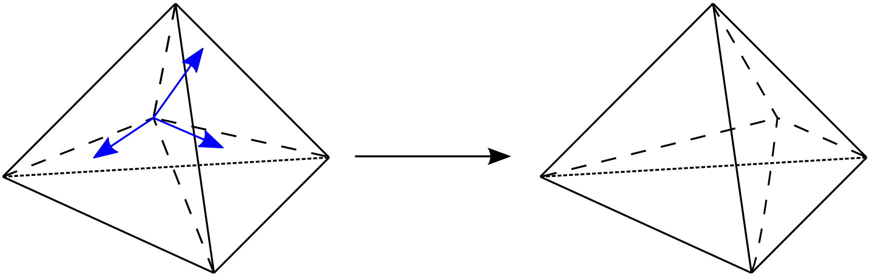


Perfect Discretisation

One can construct discretisations that respect the continuous symmetries:

- Parametrised Particle [Marsden, West '01]
- Regge Calculus without Cosmological constant [Hamber, Williams '81]
- Regge Calculus with Cosmological constant [Bahr, Dittrich '09]

Discrete (diffeomorphism) symmetries are realized by vertex translations.



Why triangulation independence? [Bahr, Dittrich, St. '11]

Discretised 1D parametrised (quantum) (an)harmonic oscillator:

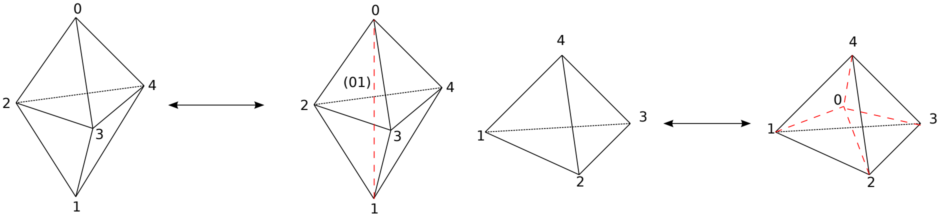
- Wilsonian Renormalisation Group Procedure
  $\rightarrow$ flows towards fixed point
- Fixed Point: Reparametrisation invariant!
- Starting from a discretisation invariant under vertex translations:
  $\Longrightarrow$ discretisation independence

Reparametrisation invariance $\Longleftrightarrow$ Triangulation independence

Triangulation independence in discrete gravity [Pachner '91]

Question: Assign same (path integral) amplitude to different triangulations of the same manifold?

Triangulations of the same manifold are related by a series of Pachner moves:



Goal

Achieve triangulation independence by requiring invariance under Pachner moves.

Linearised Regge Calculus

Euclidean Path integral

Euclidean path integral for a Regge discretisation of 3D or 4D triangulation:

$$\int_{l_e|_{e\subset\partial M}} \prod_{e\subset\text{bulk}} dl_e\ \mu(l_e)\ \exp\{-S\} \tag{3}$$

where S is given by:

$$S := -\sum_{h\subset\text{bulk}} V_h \underbrace{\left(2\pi - \sum_{\sigma^D\supset h} \theta_h^{(\sigma^D)}\right)}_{=:\omega_h^{(\text{bulk})}} - \sum_{h\subset\text{bdry}} V_h \underbrace{\left(\pi - \sum_{\sigma^D\supset h} \theta_h^{(\sigma^D)}\right)}_{=:\omega_h^{(\text{bdry})}} \tag{4}$$

Linearising Regge

To circumvent generalised triangle inequalities, we linearise around a classical background solution $l_e^{(0)}$ and integrate over perturbations λ_e:

$$l_e = l_e^{(0)} + \lambda_e \tag{5}$$

The action expanded up to second order in λ_e:

$$S = S^{(0)}\Big|_{l_e = l_e^{(0)}} + \underbrace{\frac{\partial S}{\partial l_e}\Big|_{l_e = l_e^{(0)}}}_{=0} \lambda_e + \frac{1}{2}\frac{\partial^2 S}{\partial l_e \partial l_{e'}}\Big|_{l_e = l_e^{(0)}} \lambda_e \lambda_{e'} \tag{6}$$

where

$$\frac{\partial S}{\partial l_e} = -\sum_{h \supset e} \frac{\partial V_h}{\partial l_e}\omega_h = 0 \tag{7}$$

is the Regge equation of motion for the edge l_e.

Hessian matrix

The components of the Hessian are given by:

3D:

$$\frac{\partial^2 S}{\partial l_e \partial l_{e'}} = -\frac{\partial \omega_e}{\partial l_{e'}} \tag{8}$$

4D:

$$\frac{\partial^2 S}{\partial l_e \partial l_{e'}} = -\sum_h \frac{\partial A_h}{\partial l_{e'}} \frac{\partial \omega_h}{\partial l_e} - \sum_{h \subset \text{bulk}} \frac{\partial^2 A_h}{\partial l_e \partial l_{e'}} \omega_h^{(\text{bulk})} - \sum_{h \subset \text{bdry}} \frac{\partial^2 A_h}{\partial l_e \partial l_{e'}} \omega_h^{(\text{bdry})} \tag{9}$$

So in both cases one has to compute $\frac{\partial \omega_h}{\partial l_e}$.

Computing $\frac{\partial \omega_h}{\partial l_e}$

Exact formula for $\frac{\partial \theta_h}{\partial l_e}$ (for one D-simplex): [Dittrich, Freidel, Speziale '07]

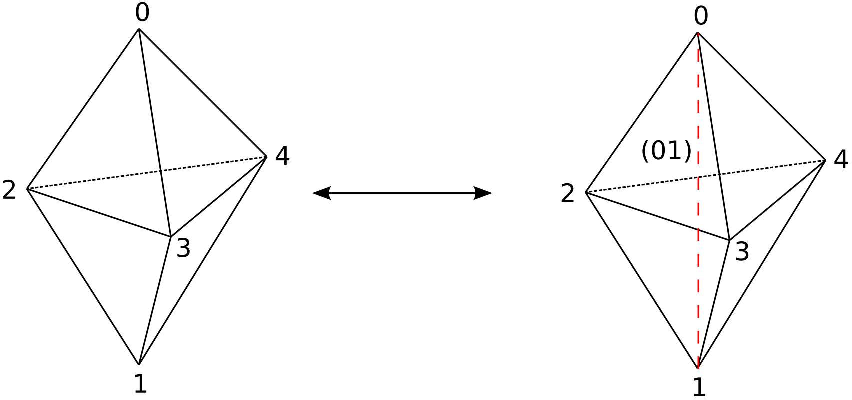
$$\frac{\partial \tilde{\theta}_{kl}}{\partial l_{hm}} = \frac{1}{D^2} \frac{l_{hm}}{\sin(\tilde{\theta}_{kl})} \frac{V_h V_m}{V^2} \Big(\cos(\tilde{\theta}_{kh}) \cos(\tilde{\theta}_{ml}) + \cos(\tilde{\theta}_{km}) \cos(\tilde{\theta}_{hl}) + + \cos(\tilde{\theta}_{kl}) \Big(\cos(\tilde{\theta}_{kh}) \cos(\tilde{\theta}_{km}) + \cos(\tilde{\theta}_{lh}) \cos(\tilde{\theta}_{lm}) \Big) \Big) \tag{10}$$

Shortcut:

- Eq. (10) for $(kl) = (hm)$: $\frac{\partial \tilde{\theta}_{kl}}{\partial l_{kl}} = \frac{1}{D(D-1)} \frac{l_{kl} V_{kl}}{V}$
- $\frac{\partial l}{\partial l'}$ under the condition that the $D+1$ volume of the Pachner complex vanishes. [Korepanov '00, '01]
- Exterior curvature angles equal before and after Pachner move.

3D Regge

3-2 Pachner move



- One dynamical edge (01).
- $l_{01}^{(0)}$ is uniquely determined by the boundary data, such that $\omega_{01} = 0$.

Hessian for the $3-2$ move

With the ingredients mentioned above, one obtains:

Hessian matrix for $3-2$ move:

$$H^{(3)}_{(ij),(km)} = H^{(2)}_{(ij),(km)} + (\pm 1)\,\frac{l_{ij}\,l_{km}}{6}\,\frac{V_{\bar{i}}V_{\bar{j}}V_{\bar{k}}V_{\bar{m}}}{\prod_n V_{\bar{n}}} \tag{11}$$

- Additional term in (11) only contains volumes and edge lenghts.
- Solving e.o.m. for λ_{01} addtional term in (11) vanishes.
  $\Longrightarrow$ Action is invariant under $3-2$ move.

Path integral for the 3-2 Pachner move

$$P_{3-2} = \int d\lambda_{01}\, \mu(l) \exp\left[- \sum_{(ij),(km)} \frac{1}{2} H^{(3)}_{(ij),(km)} \lambda_{ij}\lambda_{km}\right] \qquad (12)$$

Invariance under 3-2 Pachner move

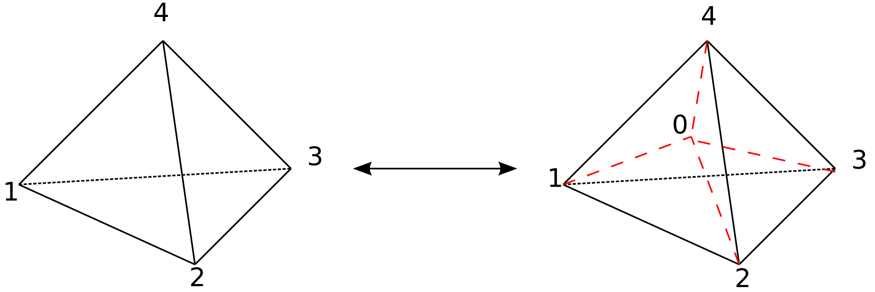
- The action is invariant under the $3-2$ Pachner move:

$$P_{3-2} \propto \exp\left[- \sum_{(ij)\neq(01),(km)\neq(01)} \frac{1}{2} H^{(2)}_{(ij),(km)} \lambda_{ij}\lambda_{km}\right] \qquad (13)$$

- The measure $\mu(l)$ is invariant if

$$\mu(l) = \frac{\prod_e \frac{l_e}{\sqrt{12\pi}}}{\prod_\tau \sqrt{V_\tau}} \qquad (14)$$

The 4-1 move



$$P_{4-1} = \int \prod_i d\lambda_{0i}\, \mu(l) \exp\left[- \sum_{(ij),(km)} \frac{1}{2} H^{(4)}_{(ij),(km)} \lambda_{ij} \lambda_{km} \right] \qquad (15)$$

Obstacles

- 3 gauge degrees of freedom
- $H^{(4)}_{(0i),(0i)}$ has the wrong sign - conformal mode problem [Gibbons, Hawking, Perry '78]

Path integral for the 4-1 Pachner move

Using the same $\mu(l)$ from the $3-2$ move, we obtain

$$P_{4-1} = \frac{\prod_b \frac{l_b}{\sqrt{12\pi}}}{\sqrt{V_{\bar{0}}}} \exp\left[-\sum_{b,b'} H^{(1)}_{b,b'} \lambda_b \lambda_{b'}\right] \underbrace{\int \frac{\prod_{i\neq 1} \frac{l_{0i}}{\sqrt{12\pi}} d\lambda_{0i}}{V_{\bar{1}}}}_{\text{Infinitesimal 3D volume element}} \qquad (16)$$

Summary 4-1 move

- Action is invariant.
- 3 gauge degrees of freedom (corresponding to a 3D volume).
- Measure $\mu(l)$ also invariant.

Summary 3D Regge

- The path integral

$$P = \int \prod_{e \subset \text{bulk}} d\lambda_e \, \mu(l) \exp\left[- \sum_{e,e'} H_{e,e'} \lambda_e \lambda_{e'} \right] \tag{17}$$

  is triangulation independent.
- The Hessian is of the simple form (for the $s - t$ Pachner move, $s > t$):

$$H^{(s)}_{(ij),(km)} = H^{(t)}_{(ij),(km)} + (\pm 1) \frac{l_{ij} l_{km}}{6} \frac{V_{\bar{i}} V_{\bar{j}} V_{\bar{k}} V_{\bar{m}}}{\prod_v V_{\bar{v}}} \tag{18}$$

- The invariant measure is given by

$$\mu(l) = \frac{\prod_e \frac{l_e}{\sqrt{12\pi}}}{\prod_\tau \sqrt{V_\tau}} \quad . \tag{19}$$

Agrees with Ponzano-Regge Asymptotics!

4D Regge

4D Regge

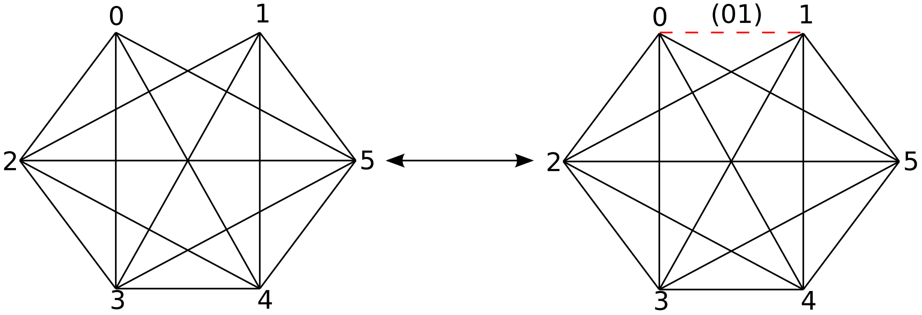
The situation in 4D is more complicated:

$$\frac{\partial^2 S}{\partial l_e \partial l_{e'}} = -\sum_h \frac{\partial A_h}{\partial l_{e'}} \frac{\partial \omega_h}{\partial l_e} - \sum_{h \subset \text{bulk}} \frac{\partial^2 A_h}{\partial l_e \partial l_{e'}} \omega_h^{(\text{bulk})} - \sum_{h \subset \text{bdry}} \frac{\partial^2 A_h}{\partial l_e \partial l_{e'}} \omega_h^{(\text{bdry})} \tag{20}$$

The calculation for $\frac{\partial \omega_h}{\partial l_e}$ (restricted to the case $\omega_h^{(\text{bulk})} = 0$) is completely analogous to the 3D case.

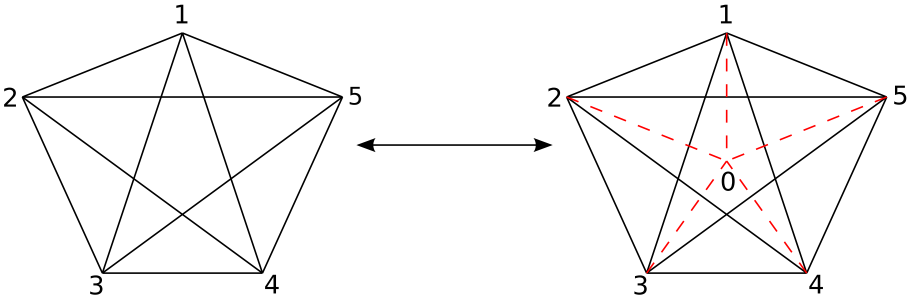
Amazingly, the result for the Hessian is almost identical to the 3D case as well!

4-2 Pachner move



- One dynamical edge (01).
- 4 bulk triangles (that contain (01)). $\omega_h^{(\text{bulk})} = 0$ is a solution to the Regge equation, but then $l_{01}^{(0)}$ uniquely determined.

5-1 Pachner move



- 5 dynamical edges, $(0i)$, for $i = 1, \ldots, 5$ and 4 gauge degrees of freedom.
- As in 3D, $H^{(5)}_{(0i),(0i)}$ has wrong sign, conformal mode problem.

Results for $s - t$ Pachner move, $s > t$

- The Hessian is given by:

$$H^{(s)}_{(op),(mn)} = H^{(t)}_{(op),(mn)} + (\pm 1)\, D\, \frac{l_{op} l_{mn}}{96} \frac{V_{\bar{o}} V_{\bar{p}} V_{\bar{n}} V_{\bar{n}}}{\prod_l V_{\bar{l}}} \quad , \tag{21}$$

where D is a nonlocal factor:

$$D^{(s)}_{op} = \sum_{k \neq o,p} (\pm 1)\, V_{\bar{k}} \left(l^2_{ok} + l^2_{pk} - l^2_{op} \right) \quad . \tag{22}$$

- Action is invariant under Pachner moves.
- To make the path integral as invariant as possible:

$$\mu(l) = \frac{\prod_e \frac{l_e}{\sqrt{192\pi}}}{\prod_\Delta \sqrt{V_\Delta}} \tag{23}$$

- It is not clear whether to assign D to a vertex or an edge!

Massless Scalar Field in 2D [wip with Dittrich]

The action for a massless scalar field on a triangulation is given by:

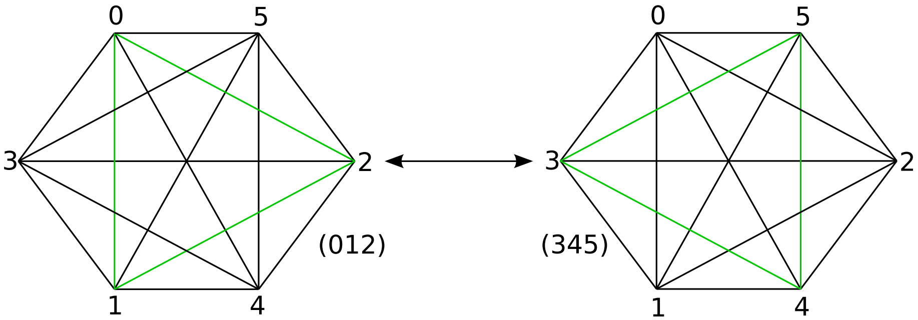
$$S = \sum_e \left(\phi_{s(e)} - \phi_{t(e)}\right)^2 \left(\cot(\alpha_1^{(\mathrm{opp})}) + \cot(\alpha_2^{(\mathrm{opp})})\right) \tag{24}$$

- Action is invariant under 3-1 move, not 2-2 move.
- Gaussian integration gives a factor:

$$\left(\frac{l_{12}^2}{A_{\bar{3}}} + \frac{l_{13}^2}{A_{\bar{2}}} + \frac{l_{23}^2}{A_{\bar{1}}}\right)^{\frac{1}{2}} \tag{25}$$

How to assign nonlocal factor to geometrical quantities?

3-3 move



- Only boundary edges, no dynamical edge.
- The configuration is only determined by the triangle shared by all 4-simplices (green).
- Boundary configuration might describe curvature.

Results for the 3-3 move

- For the Hessian matrix one obtains:

$$H^{(012)}_{(op),(mn)} - H^{(345)}_{(op),(mn)} = (\pm 1)\, D^{(012)} \frac{l_{op} l_{mn}}{96} \frac{V_{\bar{o}} V_{\bar{p}} V_{\bar{m}} V_{\bar{n}}}{\prod_l V_{\bar{l}}} \quad , \tag{26}$$

where $D^{(012)} = -D^{(345)} \neq 0$ in general.

Regge action is not invariant under the $3-3$ move.

- The measure

$$\mu(l) = \frac{\prod_e \frac{l_e}{\sqrt{192\pi}}}{\prod_\Delta \sqrt{V_\Delta}} \tag{27}$$

is only invariant if

$$V_{\bar{0}} V_{\bar{1}} V_{\bar{2}} = V_{\bar{3}} V_{\bar{4}} V_{\bar{5}} \quad . \tag{28}$$

Summary 4D Regge

- The Hessian is given by the following form:

$$H^{(s)}_{(op),(mn)} = H^{(t)}_{(op),(mn)} + (\pm 1)\, D\, \frac{l_{op} l_{mn}}{96} \frac{V_{\bar{o}} V_{\bar{p}} V_{\bar{m}} V_{\bar{n}}}{\prod_l V_{\bar{l}}} \quad , \qquad (29)$$

  where D is a nonlocal factor.
- Action is invariant for 4-2 and 5-1 move, not for 3-3 move.
- The measure (up to factors of D) is given by:

$$\mu(l) = \frac{\prod_e \frac{l_e}{\sqrt{192\pi}}}{\prod_\Delta \sqrt{V_\Delta}} \quad . \qquad (30)$$

- Up to the nonlocal factor D, action and measure are invariant for the $4-2$ and $5-1$ move. Neither is invariant (in general) for the $3-3$ move.

Conclusion and Outlook

Summary

- Goal: Triangulation independent path integral for Regge, linearised around flat background solution.
- Using the condition $\omega^{(\text{bulk})} = 0$, we derived the Hessian in terms of geometrical quantities:

$$H^{(s)}_{(op),(mn)} = H^{(t)}_{(op),(mn)} + (\pm 1)\; D\; \frac{l_{op} l_{mn}}{\text{const.}} \frac{V_{\bar{o}} V_{\bar{p}} V_{\bar{m}} V_{\bar{n}}}{\prod_l V_{\bar{l}}} \quad , \tag{31}$$

  where $D = 1$ in 3D. In 4D, D is a nonlocal factor.
- The linearised Regge action is invariant for all Pachner moves with exception of the $3-3$ Pachner move in 4D.
- The (as) invariant (as possible) measure is given by:

$$\mu(l) = \frac{\prod_e \frac{l_e}{\sqrt{2 \cdot \text{const.} \pi}}}{\prod_{\sigma^{(D)}} \sqrt{V_{\sigma^{(D)}}}} \tag{32}$$

Conclusion

- 3D (linearised) Regge gravity:
  - Triangulation independent (topological theory)
  - Measure similar to spin foam asymptotics although just linearised theory!
- 4D (linearised) Regge gravity:
  - Nonlocal, but local part of path integral similar to 3D gravity.
    $\rightarrow$ provide interpretation for spin foams in 4D.
  - Not invariant under all Pachner moves
  - Regge action not invariant under $3-3$ move.
- Triangulation independence fixes ambiguities.
- Vertex translation symmetry $\leftrightarrow$ triangulation independence

Outlook

- Spin foam model invariant under $5-1$ move?
  - Renormalization group procedure in the local couplings.
  - If achieved, no further subdivision necessary / reasonable:
    $\rightarrow$ smallest local "atom of spacetime"
- First order Regge calculus
  - Area and angle variables are "closer" to spin foams
- Scalar Field on a triangulation
  - $n-1$ move in higher dimensions

Thank you for your attention!