Relational observables and cosmological perturbation theory

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References:
A perturbative approach to Dirac observables and their space-time algebra, Bianca Dittrich, jt, CQG 24 No. 4, 757-783, [gr-qc/0610060]

Gauge invariant perturbations around symmetry reduced sectors of general relativity: applications to cosmology, Bianca Dittrich, jt, CQG 24 No. 18, 4543-4585, [gr-qc/0702093]
Plan of the talk

- Motivation
- Relational observables
  - General idea and framework
  - A perturbative approach
- Cosmological perturbation theory
  - Standard approach
  - Relational approach
- An example: Bianchi-I as perturbation around FRW
- Conclusions and outlook
Motivation

Relational observables in cosmology, why?

- GR is a background–independent theory, gauge invariant dynamics can be formulated using relational observables.
- Cosmology is the subarea of GR that deals with solutions that are close to homogeneous and isotropic on large scales.
- Using relational observables in cosmological perturbation theory has several advantages:
  - Gauge invariance
  - Background independence
  - Interpretation of the results
- Can reproduce standard results of linearized theory.
- First cosmological perturbation scheme that keeps validity in arbitrary orders (e.g. backreaction effects).
Relational observables: General idea

**heuristic idea**

- classical GR is a *gauge theory* (4d-diffeomorphism group)
- classical GR: theory on a differentiable manifold $M$, not on a metric one
- $\phi(x), x \in M$ is NOT an observable
- Observables: *relations between dynamical fields!*
  
e.g. $\phi(x) \leftrightarrow g_{\mu\nu}(x)$

**non-relational**

fixed space and time

- matter fields

**relational**

- geometry influences matter
- matter influences geometry
- auxilliary structure: differentiable manifold

- matter fields
- gravitational fields

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Relational observables in cosmology
Relational observables: General idea

Hamiltonian constrained theories

- Phase space: \( \{q_i, p_j\} = \delta_{ij}, \quad i, j \in I \)

- First class constraint algebra: \( \{C_i, C_j\} = f_{ijk}C_k \)

- Constraints generate gauge orbits in phase space:
  
  \[ \alpha_i \beta_l C_l(x) := \sum_{k=0}^{\infty} \frac{1}{k!} \{x, \beta_l C_l\}_{(k)} \quad x: \text{arbitrary phase space point} \]

  \[ \{., \beta_l C_l\}_{(k)} := \{., \beta_l C_l\}_{(k-1)}, \beta_m C_m \] (iterated Poisson brackets)

- Observables have to be constants along the gauge orbit

  \[ \Rightarrow O \text{ is a \textit{DIRAC OBSERVABLE}} \]

  \[ \Leftrightarrow \{O, C_j\} \simeq 0 \quad \forall j \]
Relational observables: General idea

Gauge invariant observables for systems with one constraint [Rovelli '90s]

- consider “reparametrized free particle” in 1 spatial dimension
  \[ 1 = \{x, p_x\} = \{T, p_T\} \]
  \[ C = p_T + \frac{1}{2m} p_x^2 \]
- \( T \) or \( x \) are NOT observables
- But: we can construct a Dirac observable in the following way

**Complete observables** [Rovelli]

\[ F_{[x,T]}(\tau) = \alpha_C^T(x) \big|_{\alpha_C^T(T) = \tau} = x + \frac{1}{m} p_x (\tau - T) \]

- \( F_{[x,T]}(\tau) \) is a Dirac observable
- \( F_{[x,T]}(\tau) \) has a relational interpretation: “\( F \) gives the value of \( x \) at that point on the gauge orbit where \( T \) takes the value \( \tau \)”
Relational observables: General idea

Generalization for infinite dimensional systems [Dittrich '04]

- infinite dimensional phase space: \( \{q_i(\sigma), p_j(\sigma')\} = \delta_{ij}\delta(\sigma, \sigma') \)
  \( i, j \in \text{finite index set}, \sigma, \sigma' \in \text{infinite index set} \) (e.g. coordinates on the spatial hypersurfaces)

- first class constraint algebra \( \{C_j, C_k\} \simeq f_{ijk}C_k \)

- choose one clock function \( T \) per constraint

- construct new set of constraints:
  \( \tilde{C}_k := (A^{-1})^j_k C_j \quad A^K_j := \{T^K, C_j\} \) (generate the same gauge orbit)

- choose phase space function \( f \)

Complete observables for field theories [Dittrich '04]

\[
F[f, T^K](\tau^K) = \sum_0^\infty \frac{1}{n!} \{f, \tilde{C}_K \} \times (\tau^K_1 - T^K_1) \times \cdots \times (\tau^K_n - T^K_n)
\]
Relational observables: General idea

Example: Free Maxwell theory

- phase space given by $U(1)$-connection $A_a(\sigma)$ and electric fields $E^a(\sigma)$, $a = 1, 2, 3$ and $\sigma \in \Sigma_3$

$$\{A_a(\sigma), E^b(\sigma')\} = \delta_a^b \delta(\sigma, \sigma')$$

- $U(1)$-Gauss-constraint: $G = \partial_a E^a(\sigma)$

- choose clock function $T(\sigma) := \partial^a A_a(\sigma)$

- compute new set of constraints:
  $$\tilde{C} := [\{T, G\}]^{-1} G = -\Delta^{-1} \partial_a E^a(\sigma)$$

- complete observables:

$$F[E^a(\sigma), T=\partial^a A_a(\sigma)](\tau) = E^a(\sigma)$$

$$F[A_a(\sigma), T=\partial^a A_a(\sigma)](\tau) = A_a(\sigma) - \Delta^{-1} \partial_a \partial^b A_b(\sigma)$$

- complete observables are the transversal vector modes!
Problems

- interesting result on a formal level
- but there are some problems:
  - infinite sum
  - convergence properties will strongly depend on choice of clock-variables
  - have to invert infinite dimensional matrix
    \[ A^K_j (\sigma, \sigma') := \{ T^K(\sigma), C_j(\sigma') \} \]
  - nevertheless, complete observables are an ideal starting point for perturbation theory
A perturbative approach

Perturbation theory around symmetry reduced sectors of the theory

- split phase space into
  - symmetry reduced sector (0th order variables)
  - anything else (higher order variables)
- can be implemented by a projection operator $\mathcal{P}$ acting on a canonical basis
- insert split into $F[f, T^K](\tau^K)$ and sort by powers of "fluctuations"
- "perturbative complete observable": restrict series to the first $(n+1)$ terms in the series:
  \[ \{ [n]F[f, T^K](\tau^K), C_j \} \simeq O(n) \]
A perturbative approach

Application to homogeneous, isotropic cosmologies

- phase space of GR given by connection $A_a^i(\sigma)$ and “electric” field $E^a(\sigma)$, Gauss–, Diffeo– and Hamiltonian constraint
- projection operator $\mathcal{P}$ averages over dynamical degrees of freedom:

$$\mathcal{P} := \frac{1}{3} \delta^i_a \int_{\Sigma_3} \delta^a_i (\ldots)$$

$$A_a^i(\sigma) = \mathcal{P} \cdot A_a^i + (id - \mathcal{P}) \cdot A_a^i$$

$$=: A \beta \delta^i_a + a_a^b(\sigma) \beta \delta^i_b$$

0th order 1st order

- $A$ describes the usual FRW-connection (homogeneous and isotropic), $a_a^b(\sigma)$ inhomogeneous and anisotropic fluctuations
Standard approach to cosmological perturbation theory

Homogeneous and isotropic universes

- FRW-solution (for vanishing spatial curvature)  $ds^2 = a^2(\eta)(d\eta^2 - dx^2)$
- put FRW-solution in Einstein's field equations and add matter (e.g. $T^\mu_\nu = \text{diag}(\rho, -p, -p, -p)$)
  $\Rightarrow$ Friedman-equations:  $(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3} \rho \quad \dot{\rho} = -3H(\rho + p)$

How to deal with inhomogeneities? [Brandenberger, Feldman, Mukhanov '92]

- perturbation theory:  $g_{\mu\nu} = (^{\text{FRW}})g_{\mu\nu} + \delta g_{\mu\nu}$
- characterize 10 entries of $\delta g_{\mu\nu}$ according to their behaviour under $SO(3)$:
  - 4 scalar modes
  - 4 vector modes
  - 2 tensor modes

What are the physical, i.e. gauge invariant degrees of freedom?
Linear perturbation theory

- in linear order:
  - modes do not couple
  - can concentrate on scalar modes
  - gauge invariant modes are known: Bardeen Potentials [Bardeen ’80]
    invariant under small gauge trafo’s, commute with linearized constraints

- insert Bardeen Potentials and desired matter contribution (e.g. scalar fields) into Einstein’s eq. and keep only terms linear in \( \delta g_{\mu \nu} \) and \( \delta T^\mu_\nu \)
  \( \Rightarrow \) system of DE’s
  \( \Rightarrow \) dynamical evolution of matter- and gravitational perturbations on a FRW–background

- most important tool to study physics of the early universe
Standard approach to cosmological perturbation theory

Problems with higher order perturbation theory

- modes couple
  - problematic to extract gauge-inv. observables already to 2nd order
  - difficult to distinguish physical perturbations from mere coordinate artefacts
  - cannot trust dynamical evolution

- backreactions
  - split into background and perturbations gets ambiguous in higher order
  - background dynamics gets influenced by perturbations starting at 2nd order
    - i.e. FRW is only a 0th order approximation to the true dynamical behaviour of homogeneous and isotropic quantities

- messy calculations
Relational approach to cosmological perturbation theory

Outline

- based on **Relational Observables**
  - "coordinates" are not put in by hand but defined in a **purely dynamical** way
    - use physical fields to define what **space** and **time** are
  - "**background**" emerges through an averaging procedure over dynamical degrees of freedom on the spatial hypersurfaces
  - symmetry reduced sector of homogeneous and isotropic fields
    - (**background**) stays entirely dynamical
      - important for consistency in higher order perturbation theory, e.g. **backreaction effects**
  - "**Feynman-like interpretation**" in terms of **free propagation** and **interactions**
    - **free propagation**: linear evolution of perturbations on FRW background
    - **interactions**: couplings between different modes
Relational approach to cosmological perturbation theory

projection and symplectic structure

- phase space of GR: \( \{ A^i_a(\sigma), E^b_k(\sigma') \} = \kappa \delta^b_a \delta^i_j \delta(\sigma, \sigma'), \) Gauss–, Diffeo– and Hamiltonian constraint (use weight 2 version throughout this talk)

- projection operator onto homogeneous, isotropic subspace:
  \[ P := \frac{1}{3} \delta^i_a \int_{\Sigma_3} \delta^a_i (\ldots) \]

split canonical basis:

\[ A^i_a(\sigma) = P \cdot A^i_a + (id - P) \cdot A^i_a =: A^i_a \beta \delta^i_a + a^b_a(\sigma) \beta \delta^i_b \]

\[ 0\text{th order} \quad 1\text{st order} \]

\[ E^a_i(\sigma) = P \cdot E^a_i + (id - P) \cdot E^a_i =: E^a_i \beta^{-1} \delta^a_i + e^a_b(\sigma) \beta^{-1} \delta^b_i \]

\[ 0\text{th order} \quad 1\text{st order} \]

symplectic structure

\[ \{ A, E \} = \frac{\kappa}{3} \]

\[ \{ a^b_a(\sigma), e^c_d(\sigma') \} = \kappa \delta^c_a \delta^b_d \delta(\sigma, \sigma') - \frac{\kappa}{3} \delta^b_a \delta^c_d \]
Relational approach to cosmological perturbation theory

some details

- technical details:
  - assume compact spatial topology $\Sigma_3 = T^3$
  - work in Fourier–space
  - choose fixed value for Immirzi parameter: $\beta = \frac{i}{2}$

constraints for GR + scalar field

Gauss: $G_j = \partial_a E_j^a + \epsilon_{jkl} A_k^a E_l^a$

Diffeo: $V_a = \frac{1}{\kappa} F_{ab}^j E_j^b + \frac{1}{\gamma} \pi \partial_a \phi$

Hamiltonian: $C = \beta^2 \epsilon_{jkl} F_{ab}^j E_k^a E_l^b$

$\quad \quad + \frac{1}{\gamma} \left( \frac{1}{2} \pi^2 + \frac{1}{2} \beta^2 E_j^a E_j^b \partial_a \phi \partial_b \phi + \beta^3 \det(E) V(\phi) \right)$
Relational approach to cosmological perturbation theory

Some details

perturbative structure of the constraints

<table>
<thead>
<tr>
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<th>$k = 0$</th>
<th>$k \neq 0$</th>
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<tbody>
<tr>
<td>Gauss</td>
<td>$O(1) + O(2) + \ldots$</td>
<td>$O(1) + O(2) + \ldots$</td>
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<tr>
<td>Diffeo</td>
<td>$O(2) + O(3) + \ldots$</td>
<td>$O(1) + O(2) + \ldots$</td>
</tr>
<tr>
<td>Hamiltonian</td>
<td>$O(0) + O(2) + \ldots$</td>
<td>$O(1) + O(2) + \ldots$</td>
</tr>
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- $^{(0)}C(0)$ is the FRW Hamiltonian:

$$^{(0)}C(0) = \frac{1}{\kappa} 6 \beta^2 A^2 E^2 + \frac{1}{\gamma} \left( \frac{1}{2} \pi^2 + E^3 V(\Phi) \right)$$

- integrated Diffeo-constraint starts at 2nd order
  \[ \Rightarrow \text{linearization instabilities} \]  
  [see e.g. Fischer, Marsden '73 or Moncrief '75/'76]

2 possibilities:

- use massless scalar fields as clocks
- one can show that one can deal with the linearization instabilities AFTER solving for the remaining constraints
Clock variables

4 scalar fields as clocks

\[ \varphi^0(\sigma) = \Phi^0 + \phi^0(\sigma) \quad \pi_0 = \Pi_0 + \rho_0(\sigma) \]
\[ \varphi^A(\sigma) = \delta^A_a \sigma^a + \phi^A(\sigma) \quad \pi_A(\sigma) = 0 + \rho_A(\sigma) \quad A = 1, 2, 3 \]

- **clock functions:**
  - \( T^{D_a}(k) = \sum_A \varphi^A(k) \delta^a_A \) (associated to the Diffeomorphism constraint)
  - \( T^C(k) = \varphi^0(k) \) (associated to the Hamiltonian constraint)
  - \( T^{G_a} = \epsilon^{abc} e_{bc}(k) \) (associated to the Gauss constraint)

+ no linearization instabilities

- hard to compare with standard approach
Clock variables

clock variables out of gravitational modes

- non-local ($\Delta^{-1}$) functions containing certain tensorial modes of the inhomogeneous, anisotropic parts of the gravitational fields

- e.g. longitudinal clocks:
  - $T^G_a = \epsilon^{abc}(A T e_{bc} + L T e_{bc} + T L e_{bc})$
  - $T^D_a = \Delta^{-1}(-\partial^a L L e^d_d + \frac{1}{2}\partial^a T e^d_d - \partial_e T L e^{ae} - \partial_d L T e^{da})$
  - $T^C = -\Delta^{-1}(\frac{1}{2} T a^d_d - L L a^d_d)$

- plus homogeneous clock to measure “time”

- +: very close to the standard treatment

- -: non-local terms in higher orders
First order dynamics for perturbations on FRW

- calculate $[1] F_{[f, T^K]}(\tau^K)$ for a first order quantity $f$ (e.g. scalar, vector or tensor modes of the linearized gravitational field)

- using the longitudinal clocks we can reproduce the standard results of linear cosmological perturbation theory, but in a purely relational way

$\Rightarrow$ system of DE’s for $[1] F_{[f, T^K]}(\tau^K)$

$\Rightarrow$ generalized propagator functions on curved backgrounds, etc. . . .
Consider complete observables for a zeroth order quantity $f$.

$$[0] F_{[f, T^K]}(\tau^K) =: \alpha_{\text{free}}^{t-T^0(0)}(f)$$ coincides with usual FRW dynamics for homogeneous and isotropic quantities $f$ (e.g. scale factor, total spatial volume, ...).

But there are higher order corrections!

Calculate $[2] F_{[f, T^K]}(\tau^K)$ for homogeneous and isotropic $f$:

$$[2] F_{[f, T^K]}(\tau^K) = \alpha_{\text{free}}^{t-T^0(0)}(f)$$

$$+ \int_0^{t-T^0(0)} ds \alpha_{\text{free}}^{t-T^0(0)} \left\{ \alpha_{\text{free}}^s(f), (2) \tilde{C} \right\}$$

+ gauge inv. extension of these terms.
**The model**

- **homogeneous but anisotropic** cosmologies, exactly solvable
  \[ A^j_a = \beta \text{diag}(A_1, A_2, A_3) \quad E^a_j = \beta^{-1} \text{diag}(E_1, E_2, E_3) \]
  \[ \phi(x) = \phi, \quad \pi(x) = \pi \]

- **Hamiltonian constraint**
  \[ C = \frac{2\beta^2}{\kappa} (E_1 A_1 E_2 A_2 + E_1 A_1 E_3 A_3 + E_2 A_2 E_3 A_3) + \frac{1}{2\gamma} \pi^2 \]

- treat Bianchi-I as perturbation around FRW
  \[ A := \frac{1}{3} \sum_i A_i, \quad E := \sum_i E_i \]
  \[ a_i := A_i - A, \quad e_i := E_i - E \]

**Observables**

- consider complete observable associated to \( E \) and the clock \( T := \frac{\phi}{\pi} \)
Motivation
Relational observables
Cosmological perturbation theory
An example: Bianchi-I as perturbation around FRW
Outlook and Conclusions

An example: Bianchi-I as perturbation around FRW

Observables

- **exact solution**
  \[ F[E,T](\tau) = \frac{1}{3} \sum_i E_i \exp[2\beta^2 A_i E_i(\tau - \frac{\phi}{\pi})] \exp[-2\beta^2 \sum_j A_j E_j(\tau - \frac{\phi}{\pi})] \]

- **0th order approximation (FRW)**
  \[ [0] F[E,T](\tau) = E \exp[-\omega(\tau - \frac{\phi}{\pi})] \]

- **2nd order approximation (backreaction effects)**
  \[ [2] F[E,T](\tau) = \exp[-\omega(\tau - \frac{\phi}{\pi})] \]
  \[ \left( E + \frac{2}{3} \beta^2 A(\tau - \frac{\phi}{\pi}) \sum_i e_i e_i - \frac{2}{3} \beta^2 E(\tau - \frac{\phi}{\pi}) \sum_i a_i e_i \right. \]
  \[ \left. + \frac{1}{6} \beta^2 \omega A(\tau - \frac{\phi}{\pi})^2 \sum_i e_i e_i + \frac{1}{6} \beta^2 \omega E^2 A(\tau - \frac{\phi}{\pi})^2 \sum_i a_i e_i \right) \]

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Relational observables in cosmology
An example: Bianchi-I as perturbation around FRW

Observables

initial conditions: $A = 1$, $E = 1m^2$, $a_1 = a_2 = 0.1$, $a_3 = -0.2$, $e_1 = e_2 = 0.1m^2$, $e_3 = -0.2m^2$
An example: Bianchi-I as perturbation around FRW

What we learned from Bianchi-I:

- calculation of backreaction effects using relational observables is straightforward.
- FRW is not a good approximation to the true dynamical behaviour of homogeneous and isotropic quantities for all times.
- Backreaction effects are physical effects (gauge invariance), not mere coordinate artefacts.
- Model is too simple to draw any conclusion for the “real world”.
- Therefore, use relational observables to calculate backreactions in a physically more realistic model.
- For which range of initial conditions does perturbation theory make sense?
Relational observables in cosmology

Outlook and Conclusions

- Relational observables are well suited to describe dynamics in GR
- Applying the relational framework to cosmology has several advantages compared to the standard approach:
  - issue of gauge invariance is very clear
  - one scheme for arbitrary orders
  - nice interpretation: “background” emerges through averaging over dynamical degrees of freedom
- ideal framework to calculate backreaction effects in cosmology
- can backreaction effects possibly resolve the dark energy problem? [see e.g. Buchert gr-qc/0612166]