Relational observables and cosmological perturbation theory

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References:

A perturbative approach to Dirac observables and their space-time algebra, Bianca Dittrich, jt, CQG **24** No. 4, 757-783, [gr-qc/0610060]

Gauge invariant perturbations around symmetry reduced sectors of general relativity: applications to cosmology, Bianca Dittrich, jt, CQG **24** No. 18, 4543-4585, [gr-qc/0702093]

Plan of the talk

- Motivation
- Relational observables
  - General idea and framework
  - A perturbative approach
- Cosmological perturbation theory
  - Standard approach
  - Relational approach
- An example: Bianchi-I as perturbation around FRW
- Conclusions and outlook

Motivation

Relational observables in cosmology, why?

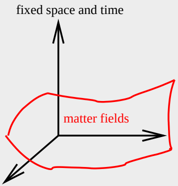
- GR is a background–independent theory, gauge invariant dynamics can be formulated using relational observables
- cosmology is the subarea of GR that deals with solutions that are close to homogeneous and isotropic on large scales
- using relational observables in cosmological perturbation theory has several advantages:
  - gauge invariance
  - background independence
  - interpretation of the results
- can reproduce standard results of linearized theory
- first cosmological perturbation scheme that keeps validity in arbitrary orders (e.g. backreaction effects)

Relational observables: General idea

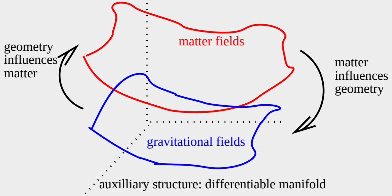
heuristic idea

- classical GR is a gauge theory (4d-diffeomorphism group)
- classical GR: theory on a differentiable manifold M, not on a metric one
- $\phi(x), x \in M$ is NOT an observable
- Observables: relations between dynamical fields!
  e.g. $\phi(x) \leftrightarrow g_{\mu\nu}(x)$

non-relational



relational



Relational observables: General idea

Hamiltonian constrained theories

- phase space $\{q_i, p_j\} = \delta_{ij}, \quad i, j \in \mathcal{I}$
- first class constraint algebra $\{C_i, C_j\} = f_{ijk} C_k$
- constraints generate gauge orbits in phase space:

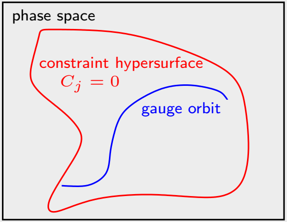
  gauge orbit: $\alpha_{\beta_l C_l}(x) := \sum_0^\infty \frac{1}{k!}\{x, \beta_l C_l\}_{(k)}$ $\quad x$: arbitrary phase space point

  $\{., \beta_l C_l\}_{(k)} := \{\{., \beta_l C_l\}_{(k-1)}, \beta_m C_m\}$ (iterated Poisson brackets)

- observables have to be constants along the gauge orbit

  $\Rightarrow$ O is a DIRAC OBSERVABLE

  $\Leftrightarrow \{O, C_j\} \simeq 0 \quad \forall j$



Relational observables: General idea

Gauge invariant observables for systems with one constraint [Rovelli '90s]

- consider "reparametrized free particle" in 1 spatial dimension
  $1 = \{x, p_x\} = \{T, p_T\}$
  $C = p_T + \frac{1}{2m} p_x^2$
- T or x are NOT observables
- But: we can construct a Dirac observable in the following way

Complete observables [Rovelli]

$$F_{[x,T]}(\tau) = \alpha_C^t(x)\Big|_{\alpha_C^t(T)=\tau} = x + \frac{1}{m} p_x(\tau - T)$$

- $F_{[x,T]}(\tau)$ is a Dirac observable
- $F_{[x,T]}(\tau)$ has a relational interpretation: "F gives the value of x at that point on the gauge orbit where T takes the value τ"

Relational observables: General idea

Generalization for infinite dimensional systems [Dittrich '04]

- infinite dimensional phase space: $\{q_i(\sigma), p_j(\sigma')\} = \delta_{ij}\delta(\sigma,\sigma')$
  $i, j \in$ finite index set, $\sigma, \sigma' \in$ infinite index set (e.g. coordinates on the spatial hypersurfaces)
- first class constraint algebra $\{C_j, C_k\} \simeq f_{ijk}C_k$
- choose one clock function T per constraint
- construct new set of constraints:
  $\tilde{C}_k := (A^{-1})^j_K C_j \quad A^K_j := \{T^K, C_j\}$ (generate the same gauge orbit)
- choose phase space function f

Complete observables for field theories [Dittrich '04]

$$F_{[f,T^K]}(\tau^K) = \sum_0^\infty \frac{1}{n!}\{\ldots\{f, \tilde{C}_{K_1}\}, \ldots, \tilde{C}_{K_n}\} \times (\tau^{K_1} - T^{K_1}) \times \cdots \times (\tau^{K_n} - T^{K_n})$$

Relational observables: General idea

Example: Free Maxwell theory

- phase space given by $U(1)$-connection $A_a(\sigma)$ and electric fields $E^a(\sigma), a = 1, 2, 3$ and $\sigma \in \Sigma_3$

  $\{A_a(\sigma), E^b(\sigma')\} = \delta_a^b \delta(\sigma, \sigma')$
- $U(1)$-Gauss-constraint: $G = \partial_a E^a(\sigma)$
- choose clock function $T(\sigma) := \partial^a A_a(\sigma)$
- compute new set of constraints:
  $\tilde{C} := [\{T, G\}]^{-1} G = -\Delta^{-1} \partial_a E^a(\sigma)$
- complete observables:

$$
\begin{aligned}
F_{[E^a(\sigma), T = \partial^a A_a(\sigma)]}(\tau) &= E^a(\sigma) \\
F_{[A_a(\sigma), T = \partial^a A_a(\sigma)]}(\tau) &= A_a(\sigma) - \Delta^{-1} \partial_a \partial^b A_b(\sigma)
\end{aligned}
$$

- complete observables are the transversal vector modes!

Relational observables: General idea

Problems

- interesting result on a formal level
- but there are some problems:
  - infinite sum
  - convergence properties will strongly depend on choice of clock-variables
  - have to invert infinite dimensional matrix $A_j^K(\sigma, \sigma') := \{T^K(\sigma), C_j(\sigma')\}$
- nevertheless, complete observables are an ideal starting point for perturbation theory

A perturbative approach

Perturbation theory around symmetry reduced sectors of the the theory

- split phase space into
  - symmetry reduced sector (0th order variables)
  - anything else (higher order variables)
- can be implemented by a projection operator $\mathcal{P}$ acting on a canonical basis
- insert split into $F_{[f,T^K]}(\tau^K)$ and sort by powers of "fluctuations"
- "perturbative complete observable": restrict series to the first $(n+1)$ terms in the series: $\{ {}^{[n]}F_{[f,T^K]}(\tau^K), C_j\} \simeq \mathcal{O}(n)$

A perturbative approach

Application to homogeneous, isotropic cosmologies

- phase space of GR given by connection $A_a(\sigma)$ and "electric" field $E^a(\sigma)$, Gauss–, Diffeo– and Hamiltonian constraint
- projection operator $\mathcal{P}$ averages over dynamical degrees of freedom:

$$\mathcal{P} := \frac{1}{3}\delta^i_a \int_{\Sigma_3} \delta^a_i (\dots) \qquad \beta\text{: Immirzi parameter}$$

$$A^i_a(\sigma) = \mathcal{P}\cdot A^i_a + (id - \mathcal{P})\cdot A^i_a =: \underbrace{A\beta\delta^i_a}_{\text{0th order}} + \underbrace{a_a{}^b(\sigma)\beta\delta^i_b}_{\text{1st order}}$$

- A describes the usual FRW-connection (homogeneous and isotropic), $a_a{}^b(\sigma)$ inhomogeneous and anisotropic fluctuations

Standard approach to cosmological perturbation theory

Homogeneous and isotropic universes

- FRW-solution (for vanishing spatial curvature) $ds^2 = a^2(\eta)(d\eta^2 - d\mathbf{x}^2)$
- put FRW-solution in Einstein's field equations and add matter (e.g. $T^\mu_\nu = \text{diag}(\rho, -p, -p - p)$)
  $\Rightarrow$ Friedman-equations: $(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho \quad \dot{\rho} = -3H(\rho + p)$

How to deal with inhomogeneities? [Brandenberger, Feldman, Mukhanov '92]

- perturbation theory: $g_{\mu\nu} = {}^{(\text{FRW})}g_{\mu\nu} + \delta g_{\mu\nu}$
- characterize 10 entries of $\delta g_{\mu\nu}$ according to their behaviour under $SO(3)$:
  - 4 scalar modes
  - 4 vector modes
  - 2 tensor modes
- What are the physical, i.e. gauge invariant degrees of freedom?

Standard approach to cosmological perturbation theory

Linear perturbation theory

- in linear order:
  - modes do not couple
  - can concentrate on scalar modes
  - gauge invariant modes are known: Bardeen Potentials [Bardeen '80]

    invariant under *small* gauge trafo's, commute with linearized constraints
- insert Bardeen Potentials and desired matter contribution (e.g. scalar fields) into Einstein's eq. and keep only terms linear in $\delta g_{\mu\nu}$ and δT^{μ}_{ν}
  $\Rightarrow$ system of DE's
  $\Rightarrow$ dynamical evolution of matter- and gravitational perturbations on a FRW–background
- most important tool to study physics of the early universe

Standard approach to cosmological perturbation theory

Problems with higher order perturbation theory

- modes couple
  - problematic to extract gauge-inv. observables already to 2nd order
  - difficult to distinguish physical perturbations from mere coordinate artefacts
  - cannot trust dynamical evolution
- backreactions
  - split into background and perturbations gets ambiguous in higher order
  - background dynamics gets influenced by perturbations starting at 2nd order

    i.e. FRW is only a 0th order approximation to the true dynamical behaviour of homogeneous and isotropic quantities
- messy calculations

Relational approach to cosmological perturbation theory

Outline

- based on Relational Observables
- "coordinates" are not put in by hand but defined in a purely dynamical way

  use physical fields to define what space and time are
- "background" emerges through an averaging procedure over dynamical degrees of freedom on the spatial hypersurfaces
- symmetry reduced sector of homogeneous and isotropic fields (background) stays entirely dynamical

  important for consistency in higher order perturbation theory, e.g. backreaction effects
- "Feynman-like interpretation" in terms of free propagation and interactions
  - free propagation: linear evolution of perturbations on FRW background
  - interactions: couplings between different modes

Relational approach to cosmological perturbation theory

projection and symplectic structure

- phase space of GR: $\{A_a^j(\sigma), E_k^b(\sigma')\} = \kappa\delta_a^b\delta_k^j\delta(\sigma,\sigma')$, Gauss–, Diffeo– and Hamiltonian constraint (use weight 2 version throughout this talk)
- projection operator onto homogeneous, isotropic subspace: $\mathcal{P} := \frac{1}{3}\delta_a^i \int\limits_{\Sigma_3} \delta_i^a(\dots)$

split canonical basis:

$$A_a^i(\sigma) = \mathcal{P}\cdot A_a^i + (id - \mathcal{P})\cdot A_a^i =: \underbrace{A\beta\delta_a^i}_{\text{0th order}} + \underbrace{a_a{}^b(\sigma)\beta\delta_b^i}_{\text{1st order}}$$

$$E_i^a(\sigma) = \mathcal{P}\cdot E_i^a + (id - \mathcal{P})\cdot E_i^a =: \underbrace{E\beta^{-1}\delta_i^a}_{\text{0th order}} + \underbrace{e^a{}_b(\sigma)\beta^{-1}\delta_i^b}_{\text{1st order}}$$

symplectic structure

$$\{A, E\} = \frac{\kappa}{3}$$

$$\{a_a{}^b(\sigma), e^c{}_d(\sigma')\} = \kappa\delta_a^c\delta_d^b\delta(\sigma,\sigma') - \frac{\kappa}{3}\delta_a^b\delta_d^c$$

Relational approach to cosmological perturbation theory

some details

- technical details:
  - assume compact spatial topology $\Sigma_3 = T^3$
  - work in Fourier–space
  - choose fixed value for Immirzi parameter: $\beta = \frac{i}{2}$

constraints for GR + scalar field

Gauss: $G_j = \partial_a E^a_j + \epsilon_{jkl} A^k_a E^a_l$

Diffeo: $V_a = \frac{1}{\kappa} F^j_{ab} E^b_j + \frac{1}{\gamma} \pi \partial_a \phi$

Hamiltonian: $C = \beta^2 \epsilon_{jkl} F^j_{ab} E^a_k E^b_l + \frac{1}{\gamma}(\frac{1}{2}\pi^2 + \frac{1}{2}\beta^2 E^a_j E^b_j \partial_a \phi \partial_b \phi + \beta^3 \text{det}(E) V(\phi))$

Relational approach to cosmological perturbation theory

some details

perturbative structure of the constraints

	$k = 0$	$k \neq 0$
Gauss	$\mathcal{O}(1) + \mathcal{O}(2) + \ldots$	$\mathcal{O}(1) + \mathcal{O}(2) + \ldots$
Diffeo	$\mathcal{O}(2) + \mathcal{O}(3) + \ldots$	$\mathcal{O}(1) + \mathcal{O}(2) + \ldots$
Hamiltonian	$\mathcal{O}(0) + \mathcal{O}(2) + \ldots$	$\mathcal{O}(1) + \mathcal{O}(2) + \ldots$

- ${}^{(0)}C(0)$ is the FRW Hamiltonian:

  $${}^{(0)}C(0) = \frac{1}{\kappa}6\beta^2 A^2 E^2 + \frac{1}{\gamma}(\frac{1}{2}\pi^2 + E^3 V(\Phi))$$

- integrated Diffeo-constraint starts at 2nd order
  $\Rightarrow$ linearization instabilities [see e.g. Fischer, Marsden '73 or Moncrief '75/'76]
  2 possibilities:
  - use massless scalar fields as clocks
  - one can show that one can deal with the linearization instabilities AFTER solving for the remaining constraints

Clock variables

4 scalar fields as clocks

- $$\varphi^0(\sigma) = \Phi^0 + \phi^0(\sigma) \qquad \pi_0 = \Pi_0 + \rho_0(\sigma)$$
  $$\varphi^A(\sigma) = \delta^A_a \sigma^a + \phi^A(\sigma) \qquad \pi_A(\sigma) = 0 + \rho_A(\sigma) \qquad A = 1, 2, 3$$
- clock functions:
  - $T^{D_a}(k) = \sum_A \varphi^A(k)\delta^a_A$ (associated to the Diffeomorphism constraint)
  - $T^C(k) = \varphi^0(k)$ (associated to the Hamiltonian constraint)
  - $T^{G_a} = \epsilon^{abc} e_{bc}(k)$ (associated to the Gauss constraint)
- $+$: no linearization instabilities
- $-$: hard to compare with standard approach

Clock variables

clock variables out of gravitational modes

- non–local (Δ^{-1}) functions containing certain tensorial modes of the inhomogeneous, anisotropic parts of the gravitational fields
- e.g. longitudinal clocks:
  - $T^{G_a} = \epsilon^{abc}(\, ^{AT}e_{bc} + \, ^{LT}e_{bc} + \, ^{TL}e_{bc})$
  - $T^{D_a} = \Delta^{-1}(-\partial^a \, ^{LL}e^d_{\ d} + \frac{1}{2}\partial^a \, ^{T}e^d_{\ d} - \partial_e \, ^{TL}e^{ae} - \partial_d \, ^{LT}e^{da})$
  - $T^C = -\Delta^{-1}(\frac{1}{2} \, ^{T}a^d_{\ d} - \, ^{LL}a^d_{\ d})$
  - plus homogeneous clock to measure "time"
- $+$: very close to the standard treatment
- $-$: non–local terms in higher orders

Linear perturbation theory

First order dynamics for perturbations on FRW

- calculate ${}^{[1]}F_{[f,T^K]}(\tau^K)$ for a first order quantity f (e.g. scalar, vector or tensor modes of the linearized gravitational field)
- using the longitudinal clocks we can reproduce the standard results of linear cosmological perturbation theory, but in a purely relational way
- $\Rightarrow$ system of DE's for ${}^{[1]}F_{[f,T^K]}(\tau^K)$
  $\Rightarrow$ generalized propagator functions on curved backgrounds, etc. ...

Higher order perturbation theory

Backreaction effects

- consider complete observables for a zeroth order quantity f
- $^{[0]}F_{[f,T^K]}(\tau^K) =: \alpha_{\text{free}}^{t-T^0(0)}(f)$ coincides with usual FRW dynamics for homogeneous and isotropic quantities f (e.g. scale factor, total spatial volume, ...)
- But there are higher order corrections!
- calculate $^{[2]}F_{[f,T^K]}(\tau^K)$ for homogeneous and isotropic f:

$$^{[2]}F_{[f,T^K]}(\tau^K) = \alpha_{\text{free}}^{t-T^0(0)}(f) + \int_0^{t-T^0(0)} ds\, \alpha_{\text{free}}^{t-T^0(0)}\left[\{\alpha_{\text{free}}^{s}(f),\ {}^{(2)}\tilde{C}\}\right]$$

+gauge inv. extension of these terms

An example: Bianchi-I as perturbation around FRW

The model

- homogeneous but anisotropic cosmologies, exactly solvable
- $A^j_a = \beta \text{diag}(A_1, A_2, A_3) \quad E^a_j = \beta^{-1} \text{diag}(E_1, E_2, E_3)$
  $\phi(x) = \phi \quad \pi(x) = \pi$
- Hamiltonian constraint
  $C = \frac{2\beta^2}{\kappa}(E_1 A_1 E_2 A_2 + E_1 A_1 E_3 A_3 + E_2 A_2 E_3 A_3) + \frac{1}{2\gamma}\pi^2$
- treat Bianchi-I as perturbation around FRW
  $A := \frac{1}{3}\sum_i A_i \quad E := \sum_i E_i$
  $a_i := A_i - A \quad e_i := E_i - E$

Observables

- consider complete observable associated to E and the clock $T := \frac{\phi}{\pi}$

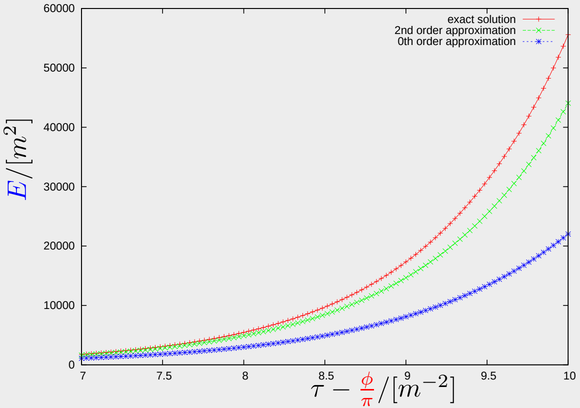
An example: Bianchi-I as perturbation around FRW

Observables

- exact solution
  $$F_{[E,T]}(\tau) = \frac{1}{3}\sum_i E_i \exp[2\beta^2 A_i E_i(\tau - \frac{\phi}{\pi})] \exp[-2\beta^2 \sum_j A_j E_j(\tau - \frac{\phi}{\pi})]$$
- 0th order approximation (FRW) $^{[0]}F_{[E,T]}(\tau) = E \exp[-\omega(\tau - \frac{\phi}{\pi})]$
- 2nd order approximation (backreaction effects) $\omega = 4\beta^2 AE$
  $$^{[2]}F_{[E,T]}(\tau) = \exp[-\omega(\tau - \frac{\phi}{\pi})] \left(E + \frac{2}{3}\beta^2 A(\tau - \frac{\phi}{\pi})\sum_i e_i e_i - \frac{2}{3}\beta^2 E(\tau - \frac{\phi}{\pi})\sum_i a_i e_i + \frac{1}{6}\beta^2 \omega A(\tau - \frac{\phi}{\pi})^2 \sum_i e_i e_i + \frac{1}{6}\beta^2 \omega \frac{E^2}{A}(\tau - \frac{\phi}{\pi})^2 \sum_i a_i a_i + \frac{1}{3}\beta^2 \omega E(\tau - \frac{\phi}{\pi})^2 \sum_i a_i e_i\right)$$

An example: Bianchi-I as perturbation around FRW

Observables



initial conditions: $A = 1,\ E = 1m^2,\ a_1 = a_2 = 0.1,\ a_3 = -0.2,\ e_1 = e_2 = 0.1m^2, e_3 = -0.2m^2$

An example: Bianchi-I as perturbation around FRW

What we learned from Bianchi-I:

- calculation of backreaction effects using relational observables is straight forward
- FRW is not a good approximation to the true dynamical behaviour of homogeneous and isotropic quantities for all times
- backreaction effects are physical effects (gauge invariance), not mere coordinate artefacts
- model is to simple to draw any conclusion for the "real world"
- $\Rightarrow$ use relational observables to calculate backreactions in a physically more realistic model
- for which range of initial conditions does perturbation theory make sense?

Outlook and Conclusions

- Relational observables are well suited to describe dynamics in GR
- Applying the relational framework to cosmology has several advantages compared to the standard approach:
  - issue of gauge invariance is very clear
  - one scheme for arbitrary orders
  - nice interpretation: "background" emerges through averaging over dynamical degrees of freedom
- ideal framework to calculate backreaction effects in cosmology
- can backreaction effects possibly resolve the dark energy problem? [see e.g. Buchert gr-qc/0612166]