Corrections to the Friedmann Equations from LQG:Mu-bar framework

Victor Taveras

ILQG Seminar
The Pennsylvania State University

February 19, 2008

Motivation: Quantum Corrections at Late Times

- In quantum theory we work with an infinite dimensional Hilbert space.
- We wish to know if we can find a finite dimensional submanifold (isomorphic to the classical phase space) on which we can approximate the quantum dynamics by effective equations.
- Effective action methods are unsuitable for our model which is written in the Hamiltonian framework.
- Use the geometric quantum mechanics framework to obtain these effective equations.
- Numerical work has been done in the full quantum theory and we wish to check if the states evolved in the numerical work do remain sharply peaked and follow the trajectories given by the effective equations.

Outline

- Classical Theory
  - Cosmological Model
  - Framework for Effective Theories
- Quantum Theory
  - State
  - Restrictions
  - Effective Equations
- Summary

Classical Theory

Hamiltonian Constraint (regime $p > 0$): $C = -\frac{3}{\kappa\gamma^2}c^2 p^{\frac{1}{2}} + \frac{1}{2}\frac{p_\phi^2}{p^{\frac{3}{2}}}$

$k = 0$, $\kappa = 8\pi G$, γ is the Barbero-Imirzi parameter, $V(\phi) = 0$

For convenience we switch variables to the volume $V = p^{\frac{3}{2}}$ and its conjugate momentum $\beta = c/p^{\frac{1}{2}}$

These are related to the geometrodynamical variables $V = a^3$, $\beta = \gamma\frac{\dot{a}}{a}$

$$C = -\frac{3}{\kappa\gamma^2}\beta^2 V + \frac{p_\phi^2}{2V} \quad (1)$$

The symplectic structure and Poisson bracket on the phase space are,

$$\Omega = \frac{2}{\kappa\gamma}d\beta \wedge dV + d\phi \wedge dp_\phi \quad (2)$$

$$\{f, g\} = \frac{\kappa\gamma}{2}\left(\frac{\partial f}{\partial \beta}\frac{\partial g}{\partial V} - \frac{\partial g}{\partial \beta}\frac{\partial f}{\partial V}\right) + \frac{\partial f}{\partial \phi}\frac{\partial g}{\partial p_\phi} - \frac{\partial g}{\partial \phi}\frac{\partial f}{\partial p_\phi} \quad (3)$$

Equations of Motion

To get the equations of motion we just take the Poisson brackets with the Hamiltonian

$$\dot{\beta} = -\frac{3}{2}\frac{\beta^2}{\gamma} - \frac{\kappa\gamma}{4}\frac{p_\phi^2}{V^2} \quad (4)$$

$$\dot{V} = 3\frac{\beta}{\gamma}V \quad (5)$$

$$\dot{\phi} = \frac{p_\phi}{V} \quad (6)$$

$$\dot{p_\phi} = 0 \quad (7)$$

- The 1st pair of equations are just the standard Friedmann equations in the β and V variables.
- The 2nd pair of equations are just the equations of motion for a free scalar field.

It is to these equations and to the Hamiltonian constraint that we wish to find corrections due to quantum geometry effects.

Framework for Effective Theories

- Showing that quantum mechanics has the correct semiclassical limit usually involves an appeal to Ehrenfests theorem (e.g. particle)

$$m\frac{\partial^2}{\partial t^2}\langle x\rangle = -\langle\frac{\partial V(x)}{\partial x}\rangle \tag{8}$$

- But this is true for ALL states, not just semiclassical ones. What we need is

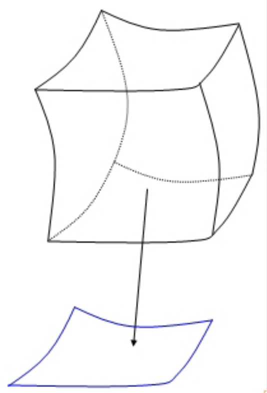
$$m\frac{\partial^2}{\partial t^2}\langle x\rangle \approx -\frac{\partial V(\langle x\rangle)}{\partial\langle x\rangle} \tag{9}$$

- Furthermore we would like to know if we could incorporate the corrections to the RHS into a small corrected potential:

$$m\frac{\partial^2}{\partial t^2}\langle x\rangle = -\frac{\partial}{\partial\langle x\rangle}\left[\langle V(\langle x\rangle)\rangle + \delta V(\langle x\rangle)\right] + \ldots \tag{10}$$

- The so-called geometric formulation of quantum mechanics gives us a framework in which we can answer this question.

Fiber Bundle Structure

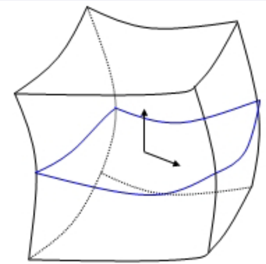


Framework (Cont.)

- Quantum phase space, i.e. Hilbert Space
- Taking expectation values of classical observables provides a natural projection from the quantum phase space to the classical phase space.
- So we can view the quantum phase space as a bundle over the classical phase space.
- Classical phase space

- T. Schilling, PhD Thesis, *Geometric Formulation of Quantum Mechanics*, Pennsylvania State University 1996.
- A. Ashtekar, T. Schilling, gr-qc/9706069

Fiber Bundle Structure



Framework (Cont.)

- In the geometric formulation of quantum mechanics $\mathcal{H}$ can be described as a symplectic space via the inner product
- $\langle\phi|\psi\rangle = \frac{1}{2\hbar}G(\phi,\psi) + \frac{i}{2\hbar}\Omega(\phi,\psi)$
- G a Riemannian metric
- Ω a symplectic form

- Vertical vectors are those with components in the direction where the expectation values dont change.
- Ω gives a notion of horizontal vectors, which we use to construct horizontal sections.

Framework (Cont.)

- Any horizontal section can be identified with the classical phase space.
- When we consider dynamics we can look for one preserved by the Hamiltonian flow.
- If we can find such a section then the quantum dynamics on such a section can be expressed in terms of an effective Hamiltonian which is simply the expectation value of the quantum Hamiltonian operator. The expectation value yields the classical term as the leading term and gets modified with corrections .
- This can be shown, e.g., for the harmonic oscillator.
- Can be done approximately for FRW with dust.
- Can also be done approximately for FRW with a free scalar field.

- J. Willis, PhD Thesis, *On the Low-Energy Ramifications and a Mathematical Extension of Loop Quantum Gravity*, Pennsylvania State University 2004.

Coherent State

- We need a state in which to evaluate expectation values...
- Gaussian coherent state peaked at the classical values β', V', ϕ', p'_ϕ
  $\langle \psi_{\beta', V', \phi', p'_\phi}| =$
  $\int dp_\phi \sum_v e^{-\frac{1}{2}\epsilon^2(v-v')^2 - \frac{i}{2}\sqrt{\Delta}\beta'(v-v')} e^{-\frac{1}{2}\epsilon_\phi^2(p_\phi - p'_\phi)^2} e^{\frac{i}{\hbar}\phi'(p_\phi - p'_\phi)} \langle v; p_\phi|$
- v and v' are the eigenvalues of the $\hat{V}$ operator
  $\hat{V}|v\rangle = \left(\frac{8\pi\gamma}{6}\right)^{\frac{3}{2}} \frac{l_p^3}{K} v|v\rangle$
- $K = \frac{2\sqrt{2}}{3\sqrt{3\sqrt{3}}}$
- We take this state and calculate all of expectation values in this state.
- The spread ϵ has to be a function of the phase space point.

Aside on ϵ

- We want the relative spreads $\frac{\Delta V}{V}$ and $\frac{\Delta\beta}{\beta}$ to be small.
- However, at e.g. at late times $V \to \infty$, $\beta \to 0$
- The ΔV and $\Delta\beta$ both depend on epsilon.
- Therefore ϵ can not be fixed a constant because then the relative spreads in V and β would go to 0 and diverge, respectively.
- We need to choose ϵ as a function of the phase space point in order for ψ to remain sharply peaked.

Operators in the Quantum Theory

- Recall that the Stone von Neumann theorem does not hold in LQG and thus the operator $\hat{c}$ and by extension $\hat{\beta}$ does not exist. We construct an approximate operator in terms of holonomy operators
- Action of β on the basis kets is given by
- $\hat{V}$ by multiplication
- Scalar field is quantized in the usual way

- $\beta \simeq \frac{1}{i\sqrt{\Delta}}\left(e^{\frac{i}{2}\sqrt{\Delta}\beta} - e^{-\frac{i}{2}\sqrt{\Delta}\beta}\right)$
- $\hat{\beta} = \frac{1}{i\sqrt{\Delta}}\left(\widehat{e^{\frac{i}{2}\sqrt{\Delta}\beta}} - \widehat{e^{-\frac{i}{2}\sqrt{\Delta}\beta}}\right)$
  Take $\sqrt{\Delta}\beta \ll 1$, where $\Delta = 2\sqrt{3}\pi\gamma l_p^2$
- $\hat{\beta}|v;p_\phi\rangle = \frac{1}{i\sqrt{\Delta}}\left(|v+1;p_\phi\rangle - |v-1;p_\phi\rangle\right)$
- $\hat{V}|v;p_\phi\rangle = \left(\frac{8\pi\gamma}{6}\right)^{\frac{3}{2}}\frac{l_p^3}{K}v|v;p_\phi\rangle$
- $\hat{p}_\phi = p_\phi$
- $\hat{\phi} = \frac{\hbar}{i}\frac{\partial}{\partial p_\phi}$

Restrictions on Parameters

Physically motivated

- $v' \gg 1$
- $\sqrt{\Delta}\beta' \ll 1$
- $v'\epsilon \gg 1$
- $\epsilon \ll \sqrt{\Delta}\beta'$
- $\phi' \gg \epsilon_\phi$
- $p'_\phi \epsilon_\phi \gg 1$

Corresponding to

- $V' \gg l_p^3$
- $\dot{a} \ll c_{light}$
- $\frac{\Delta V}{V} \ll 1$
- $\frac{\Delta\beta}{\beta} \ll 1$
- $\frac{\Delta\phi}{\phi} \ll 1$
- $\frac{\Delta p_\phi}{p_\phi} \ll 1$

Confirming that ψ is sharply peaked

- We calculate the expectation value of our basic variables
- $\langle \beta \rangle = \frac{2}{\sqrt{\Delta}} e^{-\frac{1}{4}\epsilon^2} \sin(\frac{1}{2}\sqrt{\Delta}\beta')$
- $\langle V \rangle = V'$
- $\langle \phi \rangle = \phi'$
- $\langle p_\phi \rangle = p'_\phi$
- Similarly, for the squares of these operators, which we use to obtain their spreads and confirm that ψ saturates the Heisenberg bound
- $\Delta\beta\Delta V = \frac{\kappa\gamma}{2}\frac{\hbar}{2}$
- $\Delta\phi\Delta p_\phi = \frac{\hbar}{2}$

Expectation Value of the Hamiltonian Constraint

Compute $\langle \frac{1}{16\pi G} C_{grav} + \frac{1}{2} C_{\phi} \rangle$

$$\langle \hat{C} \rangle = -\frac{3}{16\pi G \gamma^2 \bar{\mu}'^2} p^{\frac{1}{2}} \left[1 + e^{-4\epsilon^2} \left(2\sin^2(\sqrt{\Delta}\beta') - 1 \right) \right] + \frac{1}{2} \left(p_{\phi}'^2 + \frac{1}{2\epsilon_{\phi}^2} \right) \frac{1}{V'} \left[1 + O(V'^{-2}, V'^{-2}\epsilon^{-2}) \right] \tag{11}$$

Summary of calculation

- Using this, and the expansion for ψ we compute the expectation value of the Hamiltonian constraint.
- We end up with a very slowly converging summation over v, i.e. it is not clear what the behavior is to leading order.
- We do a re-summation of the slowly converging series in terms of a series that converges much faster, but at the cost that we can only obtain an asymptotic expansion for it.

Effective Equations from Exact Quantum Equations

- We compute the expectation value of the constraint to obtain the effective Hamiltonian that generates the dynamics on the section.
- To compute the time derivatives we compute the expectation value of the commutator with the Hamiltonian.
- We obtain asymptotic expansions which we examine at leading order and next to leading order.
- Recall that the coordinates on the phase space are the expectation values. We call these "barred variables"
- We compute the equations of motion and then express them in terms of these barred variables to obtain the effective equations.

- $\bar{\beta} = \langle\beta\rangle, \bar{V} = \langle V\rangle, \bar{\phi} = \langle\phi\rangle, \bar{p}_\phi = \langle p_\phi\rangle$

Effective Equations (cont.)

$$\begin{aligned}
\bar{C} &= -\frac{3}{\kappa\gamma^2}\bar{V}\bar{\beta}^2\left(1-\frac{1}{4}\Delta\bar{\beta}^2\right)-\frac{6\epsilon^2}{\kappa\gamma^2}\frac{\bar{V}}{\Delta} \\
&\quad +\frac{\bar{p}_\phi^2}{2\bar{V}}\left[1+O(\bar{V}^{-2})\right] \qquad (12) \\
\dot{\bar{\beta}} &= \frac{3}{4\gamma}\sqrt{1-\frac{1}{4}\Delta\bar{\beta}^2}\left[-2\bar{\beta}^2+\Delta\bar{\beta}^4\right] \\
&\quad -\frac{\kappa}{4}\sqrt{1-\frac{1}{4}\Delta\bar{\beta}^2}\frac{\bar{p}_\phi^2}{V'^2}[1+O(\bar{V}^{-2})] \qquad (13) \\
\dot{\bar{V}} &= 3\frac{\bar{\beta}}{\gamma}\bar{V}\sqrt{1-\frac{\Delta\bar{\beta}^2}{4}} \qquad (14) \\
\dot{\bar{\phi}} &= \frac{\bar{p}_\phi}{\bar{V}}+O(\bar{V}^{-3}) \qquad (15) \\
\dot{\bar{p}}_\phi &= 0 \qquad (16)
\end{aligned}$$

Modified Symplectic Structure

$$\Omega = \frac{2}{\kappa\gamma} \frac{1}{\sqrt{1 - \frac{1}{4}\Delta\bar{\beta}^2}} d\bar{\beta} \wedge \bar{V} + d\bar{\phi} \wedge d\bar{p}_\phi$$

$$\{f, g\} = \frac{\kappa\gamma}{2} \sqrt{1 - \frac{1}{4}\Delta\bar{\beta}^2} \left(\frac{\partial f}{\partial\bar{\beta}} \frac{\partial g}{\partial\bar{v}} - \frac{\partial g}{\partial\bar{\beta}} \frac{\partial f}{\partial\bar{v}} \right) + \frac{\partial f}{\partial\bar{\phi}} \frac{\partial g}{\partial\bar{p}_\phi} - \frac{\partial g}{\partial\bar{\phi}} \frac{\partial f}{\partial\bar{p}_\phi}$$

- $(\bar{\beta}, \bar{V}, \bar{\phi}, \bar{p}_\phi)$ are the coordinates on the horizontal section, Γ, and we wish to know whether they remain on the section, i.e. is the vector $(\dot{\bar{\beta}}, \dot{\bar{V}}, \dot{\bar{\phi}}, \dot{\bar{p}}_\phi)$ tangent to Γ or off it?
- If it is not tangent to Γ then is it approximately tangent? That is, are its components off Γ small?
- This is indeed the case, since to our order of approximation the equations of motion in terms of the barred variable hold. Indeed, it can be verified that the equations hold in terms of the pullback of the symplectic structure to the barred variables.

Corrected Friedmann Equation

We can compute the corrected Friedmann equation via

$$H^2 = \left(\frac{1}{3}\frac{\dot{V}}{V}\right)^2 = \frac{\kappa\rho}{3}\left(1 - \frac{\rho}{\rho_{crit}}\right) + O(\epsilon^2) \tag{17}$$

where

$$\rho_{crit} = \frac{\sqrt{3}}{16\pi^2\gamma^3 G^2\hbar} \simeq 0.82\rho_p \tag{18}$$

Even though we are looking at a semiclassical approximation we see that gravity is already becoming repulsive.

Summary

- We found an approximately horizontal section of the quantum phase space where the full quantum dynamics can be approximated by effective differential equations.
- That is, we found corrections to the Friedman equations which can be described as being generated by an effective Hamiltonian constraint on the section.
- The effective equations are consistent to within our order of approximation and include a leading correction due to quantum geometry effects.
- Surprisingly, even though our approximation was intended for late times numerical results show that these effective equations continue to be a good description up to small volumes as well.