Spectral dimension of quantum geometries

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work in collaboration with G. Calcagni & D. Oriti

Calcagni, Oriti, JT: CQG 30(2013)125006 [arXiv:1208.0354], arXiv:1311.3340 and wip

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Motivation: an "observable" of quantum geometry

Characterize geometric meaning of quantum gravity states/quantum histories in Loop Quantum Gravity (Spin Foams/Group Field Theory)

$$\left\langle \hat{P}(\tau) \right\rangle_{QG} = \left\langle \mathrm{Tr}\hat{K}(x,y;\tau) \right\rangle_{QG} = \left\langle \mathrm{Tr} e^{\tau\hat{\Delta}} \right\rangle_{QG} \propto \tau^{-\frac{d_s}{2}}$$

- indicator of topology and geometry
- tool to compare approaches; dimensional flow? **[Ambjorn *et al* 2005], [Lauscher, Reuter 2005], [Horava 2009], [Benedetti 2009], [Modesto *et al* 2008/09]**
- fract(ion)al QFT as effective theory in intermediate regimes **[Calcagni 2011]**

LQG: combination of discreteness & additional (pre)geometric data: heat kernel on spin networks in terms of discrete Laplacian **[Desbrun *et al* 2005]**

$$K(n_1, n_2; \tau) = \langle n_1 | e^{\tau\Delta} | n_2 \rangle$$

How do features of discreteness, geometry and quantum superposition interact?

Outline

1. Spectral dimension and Laplacian
   - Spectral dimension
   - Laplacian on discrete geometries
   - Quantum spectral dimension
2. Properties of classical spectral dimension
   - Topology and geometry
   - Discreteness effects
3. Analysis of quantum spectral dimension
   - Coherent states
   - Superpositions of geometry
   - Superpositions of combinatorics
4. Summary and outlook

Outline

Spectral dimension

Extract information about geometry from a scalar test particle, i.e. its heat kernel:

$$\partial_\tau K(x,y;\tau) - \Delta_y K(x,y;\tau) = 0$$

On Riemannian manifold: expansion around the flat case

$$K(x,y;\tau) = \langle x | e^{\tau\Delta} | y \rangle = \frac{e^{-\frac{D^2(x,y)}{4\tau}}}{(4\pi\tau)^{\frac{d}{2}}} \sum_{n=0}^{\infty} b_n(x,y)\tau^n$$

Use its trace (*return probability*)

$$P(\tau) = \mathrm{Tr} K(x,y;\tau) = \frac{1}{(4\pi\tau)^{\frac{d}{2}}} \sum_{n=0}^{\infty} a_n \tau^n$$

to define *spectral dimension* $d_S(\tau)$ as its scaling

$$P(\tau) \sim \tau^{-\frac{d_S}{2}} \quad \rightarrow \quad d_S(\tau) := -2\frac{\partial \ln P(\tau)}{\partial \ln \tau}$$

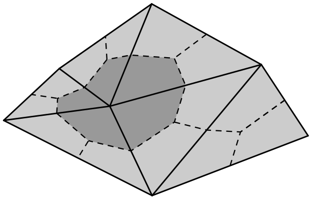
Definition for discrete geometries

- So far either purely combinatorial (CDT) or smooth setting (AS, HL, NCFT,...)
- LQG/SF/GFT built on discrete geometries
  $\rightarrow$ Use discrete (exterior) calculus (DEC) [Desbrun *et al* 2005]:

Definition of $\Delta = \mathbf{d}\delta + \delta\mathbf{d}$ acting on *p*-forms on abstract simplicial (or polyhedral) *d*-complexes with geometric interpretation (assignment of volumes to simplices).

On dual scalar fields ϕ:

$$-(\Delta\phi)_n = \frac{1}{V_n} \sum_{m\sim n} \frac{V_{nm}}{l_{nm}} (\phi_n - \phi_m)$$



Discrete Field Spaces

- *Discrete Space(time)*: Abstract finite simplicial d-complex K with manifold properties $\rightarrow \exists$ combinatorial dual $\star K$
- *Geometric Interpretation*: volumes V_{σ_p}, $V^{(d)}_{\sigma_p}$ associated with p-simplices σ_p
  - bra-ket formalism for p-form fields ϕ
  - orthonormal and complete position basis
  - $\langle \sigma_p | \sigma'_p \rangle = \frac{1}{V^{(d)}_{\sigma_p}} \delta_{\sigma\sigma'}$
  - $\sum_{\sigma_p} V^{(d)}_{\sigma_p} |\sigma_p\rangle\langle\sigma_p| = \mathbb{1}$

Field expansion in position space

$$\langle\phi| = \sum_{\sigma_p \in K} V^{(d)}_{\sigma_p} \phi_{\sigma_p} \langle\sigma_p| \overset{*}{\longleftrightarrow} |\phi\rangle = \sum_{\sigma_p \in K} V^{(d)}_{\sigma_p} \phi^*_{\sigma_p} |\sigma_p\rangle$$

Remark: Generalizable to polyhedral complexes and manifolds with boundaries

Definitions: Identifying Dualities

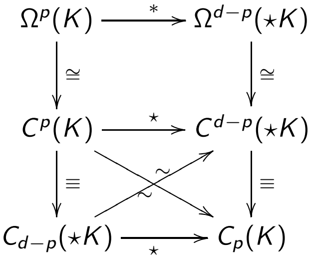
- fields are cochains

$$\langle \phi | \sigma_p \rangle := \phi_{\sigma_p} = \frac{1}{V_{\sigma_p}} \int_{\sigma_p} \phi_{cont}$$

- dual fields $\equiv$ fields on the dual $\star K$

$$\langle * \phi | \sigma_p \rangle := \langle \star \sigma_p | \phi \rangle := \langle \phi | \sigma_p \rangle^*$$

Field spaces



- finally: Identify bases $\langle \sigma_p | \equiv \langle \star \sigma_p |$
- consequence: Inner product like $\langle \phi, \psi \rangle = \int_M \phi \wedge * \psi$:

$$\langle \phi | \psi \rangle = \sum_{\sigma_p} V_{\sigma_p}^{(d)} \phi_{\sigma_p} \psi^*_{\star \sigma_p}$$

Calculus and Laplacian

Differential

Stokes theorem as definition:

$$V_{\sigma_p}\langle \mathbf{d}\phi|\sigma_p\rangle := \sum_{\sigma_{p-1}\in\partial\sigma_p} \mathrm{sgn}(\sigma_{p-1},\sigma_p) V_{\sigma_{p-1}}\langle\phi|\sigma_{p-1}\rangle$$

- analogous definition for dual differential δ
- p-volumes V_{σ_p} and $V_{\star\sigma_p}$ needed

Discrete Laplacian

$$\Delta = \mathbf{d}\delta + \delta\mathbf{d}$$

Dual Scalar Laplacian

$$-(\Delta\phi)_\sigma = \frac{1}{V_\sigma}\sum_{\sigma'\sim\sigma}\frac{V_{\sigma\cap\sigma'}}{V_{\star(\sigma\cap\sigma')}}(\phi_\sigma - \phi_{\sigma'})$$

- null condition: $(\Delta\phi) = 0 \Leftrightarrow \phi = cons$
- self-adjointness: $\langle\phi|\Delta\psi\rangle = \langle\Delta\phi|\psi\rangle$
  - symmetry of coefficients $w_{\sigma\sigma'} = \frac{V_{\sigma\cap\sigma'}}{\hat{l}_{\sigma\sigma'}}$
- locality:$(\Delta\phi)_\sigma$ depends only on neighboring $\phi_{\sigma'}$
  - (Δ 2nd order diff operator)

For cellular decompositions of smooth manifolds:

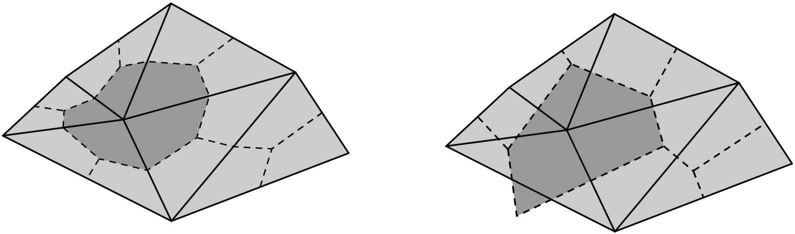
- convergence to continuum Laplacian under refinement

Geometric data

Volumes can be defined (motivated by simplicial setting) as functions of

- edge lengths
- $(d-1)$-face normals
- face bivectors/fluxes or area-angle variables (in $4d$)

Freedom for dual volumes: barycentric vs. circumcentric dual:



Positivity of Laplacian on generic geometries → barycentric dual preferred

Heat kernel in momentum space

Eigenfunctions of the Laplacian $e_\sigma^\lambda = \langle \lambda | \sigma \rangle$ form an orthonormal and complete momentum basis (for non-degenerate volumes):

$$\langle \lambda | \lambda' \rangle = \sum_\sigma V_\sigma^{(d)} e_\sigma^\lambda e_\sigma^{\lambda' *} = \frac{1}{V^\lambda} \delta_{\lambda\lambda'}$$

$$\sum_\lambda V^\lambda |\lambda\rangle\langle\lambda| = \mathbb{1}$$

Appropriate basis for functionals of Δ, e.g. the heat kernel and trace

$$K_{\sigma\sigma'}(\tau) = \left\langle \sigma' \middle| e^{\tau\Delta} \middle| \sigma \right\rangle = \sum_\lambda V^\lambda e^{-\tau\lambda} e_{\sigma'}^{\lambda *} e_\sigma^\lambda$$

$$P(\tau) = \mathrm{Tr} K_{\sigma\sigma'}(\tau) = \sum_\lambda e^{-\tau\lambda}$$

Quantum spectral dimension

$$d_S^{\psi}(\tau) := -2 \frac{\partial \ln \left\langle \widehat{P(\tau)} \right\rangle_{\psi}}{\partial \ln \tau}$$

Expectation value on the level of the heat trace (cf CDT)

$$\left\langle \widehat{P(\tau)} \right\rangle_{\psi} = \left\langle \psi | \mathrm{Tr} e^{\tau \widehat{\Delta}} | \psi \right\rangle = \sum_s |\psi(s)|^2 \left\langle s | \mathrm{Tr} e^{\tau \widehat{\Delta}} | s \right\rangle = \sum_s |\psi(s)|^2 \, \mathrm{Tr} e^{\tau \left\langle s | \widehat{\Delta} | s \right\rangle}$$

- alternative definition on the level of Laplacian (AS, NC-QFT, Horava etc)?

$$P^{\psi}(\tau) \propto \mathrm{Tr}\, e^{\tau \sum_s |\psi(s)|^2 \left\langle s | \widehat{\Delta} | s \right\rangle} = \mathrm{Tr} \prod_s e^{\tau |\psi(s)|^2 \left\langle s | \widehat{\Delta} | s \right\rangle}$$

$\rightarrow$ undefined for states $|s\rangle$ on different combinatorial manifolds

2+1 kinematical LQG states

Setting in the following

- $d = 2+1$ LQG restricted to $\mathcal{H}_{\mathrm{kin}} = \bigoplus_\Gamma \mathcal{H}_\Gamma$, Γ dual graph of combinatorial manifold
- in spin network basis $|s\rangle \in \mathcal{H}_{\mathrm{kin}}$

$$\widehat{l_i^2}|s\rangle = l_i^2|s\rangle = l^2(j_i)|s\rangle = l_\gamma^2 C_{j_i}|s\rangle = l_\gamma^2[j_i(j_i+1)+c]|s\rangle$$

- edge length Laplacian on spin networks $\widehat{\Delta} = \widehat{\Delta(l^2)} = \Delta(\widehat{l^2}) = \Delta(j)$

$\rightarrow$ Heat trace is indeed self-adjoint (either for operator ordering s.t. $c > 0$, or excluding degenerate states from $\mathcal{H}_{\mathrm{kin}}$

- reason for restriction to $d = 2+1$
  - feasibility of calculations: complexity grows exponentially with d
  - straightforward definition of Δ: full commuting set of necessary geometric operators

Outline

Circle S^1

Before analyzing d_S of LQG states/SF histories:

- understand classical features of underlying complexes

Ex.: Circle

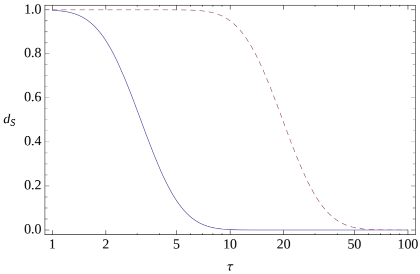
$$P_{S^1}(\tau) = \sum_{k\in\mathbb{Z}} e^{-(\frac{k}{R})^2\tau} = \theta_3\left(0, e^{-(\frac{1}{R})^2\tau}\right) = \theta_3\left(0|(\frac{1}{R})^2\frac{i\tau}{\pi}\right)$$

Topology

- compactness → fall-off to zero
- (important for finite spin networks)

Geometry

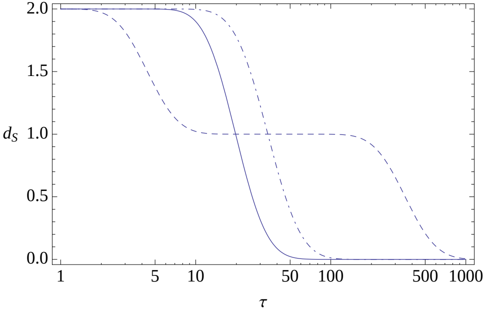
- scale of fall-off (effective rescaling of τ)
- shape of slope



$R = 1, \frac{1}{2\pi}$ (straight, dashed)

Torus T^d

$$P_{T^d}(\tau) = \sum_{\vec{k}\in\mathbb{Z}^d} e^{-\sum(\frac{k_i}{R_i})^2\tau} = \theta\left(0 \,\middle|\, \frac{i\tau}{\pi}\begin{pmatrix} R_1^{-2} & & \\ & \ddots & \\ & & R_d^{-2}\end{pmatrix}\right)$$



T^2 with $(R_1, R_2) = (1,1), (1, \frac{\sqrt{3}}{2}), (3, \frac{1}{3})$ (straight, dash-dotted, dashed)

- no difference to S^1 for radii all equal

$$P_{T^d}(\tau) = (P_{S^1}(\tau))^d$$

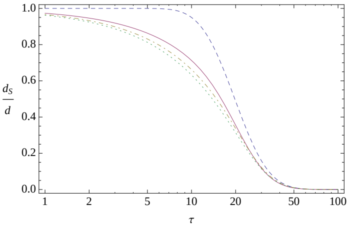
- radii of different order: Dimensional reduction in the usual sense (compactification)

Sphere S^d

$$P_{S^d}(\tau) = \sum_{j=0}^{\infty} \left[\binom{d+j}{d} - \binom{d+j-2}{d} \right] e^{-\frac{j(j+d-1)}{R^2}\tau}$$

for $d > 1$:

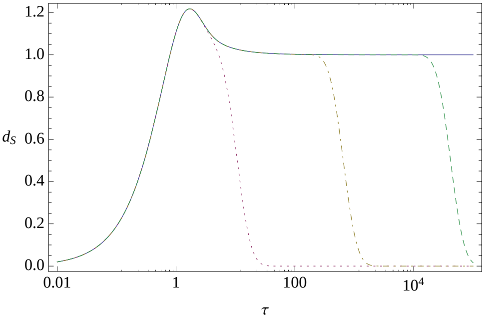
- no plateau at height d
- only limit $d_S \underset{\tau \to 0}{\longrightarrow} d$
- shape of slope depending on d



$d = 1, 2, 3, 4$ normed to $V_{S^d} = 1$ (dashed, straight, dash-dotted, dotted)

Hypercubic lattices

Discreteness artifact: fall-off at lattice scale



$\mathbb{Z}_p$ with $p = 8, 64, 512, \infty$ (dotted, dash-dotted, dashed, straight)

- finite lattices

$$P_{\mathbb{Z}_p}(\tau) = \sum_{j \in \mathbb{Z}_p} e^{-(1-\cos(2\pi j/p)\tau}$$

- infinite lattice: closed analytic solution

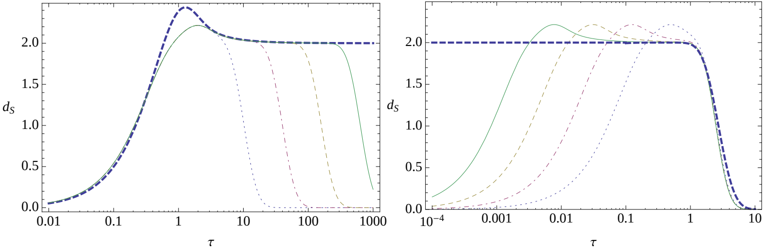
$$P_{\mathbb{Z}^d}(\tau) = \left(e^{-\tau} I_0(\tau)\right)^d$$

- convergence to the infinite lattice case

Triangulations

No analytic solutions known for equilateral triangulations → explicit calculations:

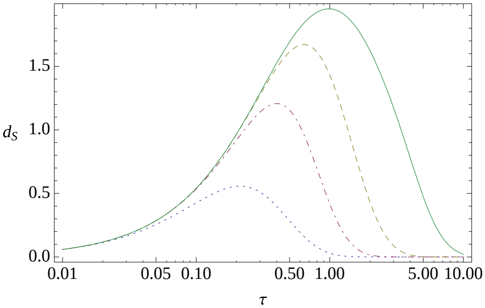
- define the abstract complex for standard triangulation of $\mathbb{Z}_p^d$ lattice
- compute Laplacian using DEC and diagonalize



T^2 triangulation with $N_0 = p^2 = (3 \cdot 2^k)^2$ vertices, $k = 0, 1, 2, 3$, compared to $\mathbb{Z}^2$ lattice (left), rescaled (right)

- only difference to hypercubic lattice: shape of discreteness artifact
- Rescaling edges to triangulate a given smooth T^2: “refinement limit”

Sphere triangulation



dipole, tetrahedron, octahedron, icosahedron

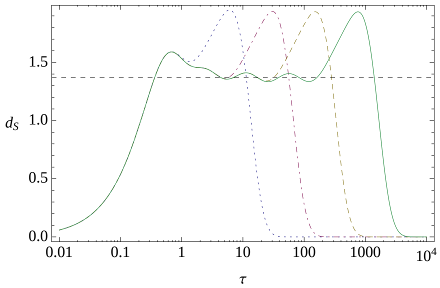
- only equilateral triangulations: boundary of platonic solids
- too small to even peak at the value of d
- dipole: *d*-independent analytic solution of d_S (max at ≈ 0.56)

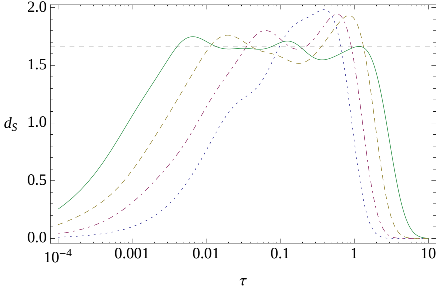
Lesson: for a concept of d_S as dimension at all, complexes must be large enough! (regime between discreteness and topological effect needed)

$\rightarrow$ only T^2 from now on

Torus subdivisions

Triangulation of T^2 obtained from $k = 1, 2, 3, 4$ subdivisions by Pachner 1-2 move on all triangles





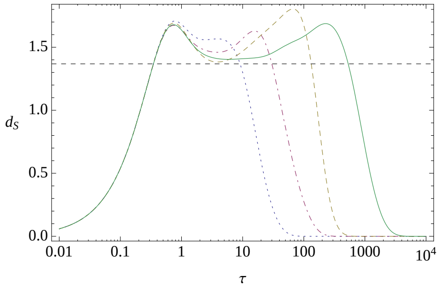
- equilateral → nontrivial geometry
- oscillations around $d_S \approx 1.37$ (effect of geometry or discreteness?)

- rescaled → triangulation of torus with constant curvature
- torus dimension still not reproduced

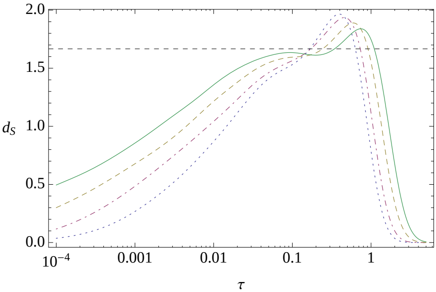
Combinatorics matters!

Random subdivisions

Same number of subdivisions, now applied randomly



- equilateral → nontrivial geometry
- no oscillations



- rescaled → triangulation of torus with constant curvature
- smaller plateau, slower fall-off

Combinatorics matters!

Conclusions for discrete geometries

- large scale (global curvatures) behaviour: topology effect
- small scale behavior (lattice scale): discreteness effect
- intermediate "geometric" regime needed for concept of dimension
- dimension depends on geometry *as well as* on the combinatorics

(Note: discrete geometries can already be seen as quantum in 2+1 LQG: pure spin network states)

Outline

Coherent states: Dependence on spins

Coherent state on Γ, peaked on spins J_l (and extrinsic curvature K_l) with spread σ

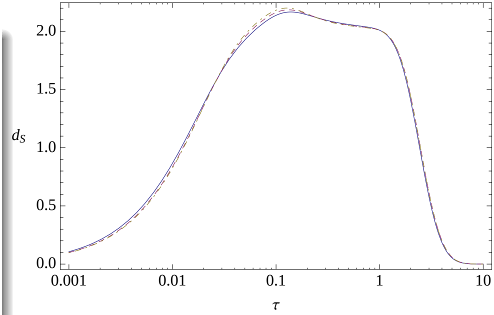
$$\left\langle \widehat{P(\tau)} \right\rangle_{\Gamma}^{\{J_l, K_l\}} \approx \sum_{\{j_l\}} \left(\prod_{l \in \Gamma} e^{-\frac{(J_l - j_l)^2}{\sigma^2}} \right) \mathrm{Tr}\, e^{\tau \Delta_\Gamma (j_l)}$$

Goal:

- Check semi-classicality
- Identify quantum corrections

Challenge:

- Even with cutoffs, exp. growth of sum with # of links, $\sim (j_{\max} - j_{\min})^L$
- implementation: approximation by sum over Gaussian samples



$l(J) = J + 1/2 = 16, 32, 64$ (straight, dashed, dash-dotted)

Coherent states: Dependence on spins

Coherent state on Γ, peaked on spins J_l (and extrinsic curvature K_l) with spread σ

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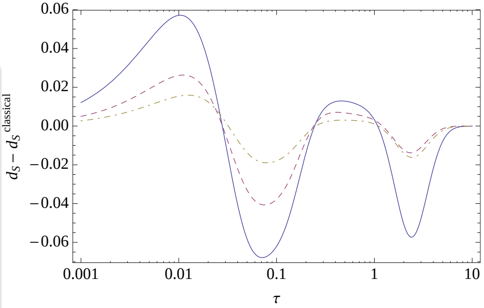
Goal:

- Check semi-classicality
- Identify quantum corrections

Result for dependence on J:

- Very close too discrete geometry peaked at
- Deviation only of order $\mathcal{O}(10^{-2})$

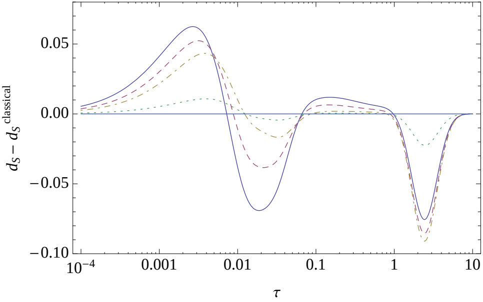


$l(J) = J + 1/2 = 16, 32, 64$ (straight, dashed, dash-dotted)

Dependence on the spread

Coherent state on Γ, peaked on spins J_l (and extrinsic curvature K_l) with spread σ

$$\left\langle \widehat{P(\tau)} \right\rangle_\Gamma^{\{J_l,K_l\}} \approx \sum_{\{j_l\}} \left(\prod_{l\in\Gamma} e^{-\frac{(J_l-j_l)^2}{\sigma^2}} \right) \mathrm{Tr}\, e^{\tau \Delta_\Gamma(j_l)}$$



$l(J) = J + 1/2 = 16,\ \sigma = 1, 2, 3, \sqrt{15}$

Goal:

- Check semi-classicality
- Identify quantum corrections

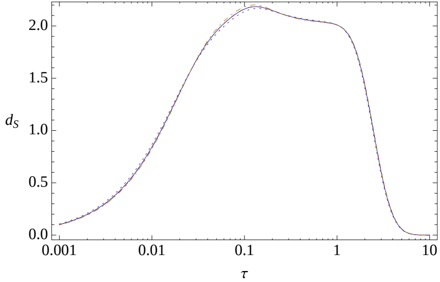
Result for dependence on σ:

- Very close too discrete geometry peaked at
- Deviation only of order $\mathcal{O}(10^{-2})$

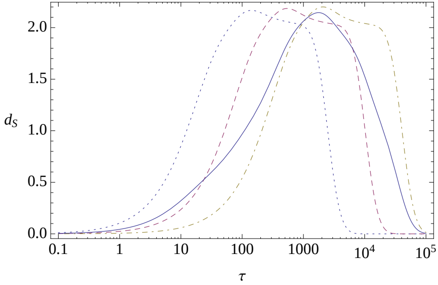
Superpositions of geometry

Sum over coherent states, peaked at various J

$$\left\langle \widehat{P(\tau)} \right\rangle_{\psi} \approx \sum_{J_l,\sigma} \sum_{\{j_l\}} |c_{J_l,\sigma}|^2 \left(\prod_{l\in\Gamma} e^{-\frac{(J_l - j_l)^2}{\sigma^2}} \right) \mathrm{Tr}\, e^{\tau \Delta_\Gamma(j_l)}$$



- interpreted as peaked on the same geometry, $l_* l(J) = cons$
- again only deviations $\mathcal{O}(10^{-2})$

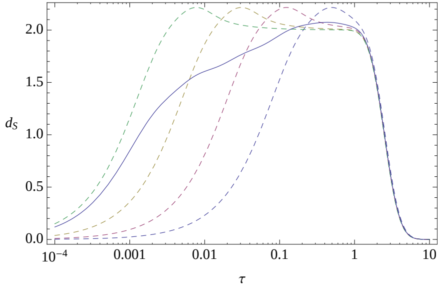


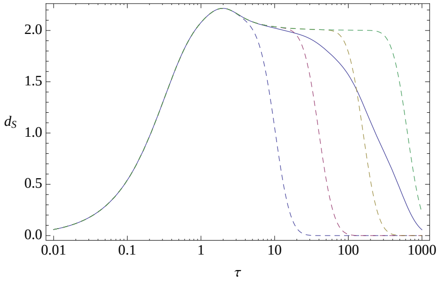
- interpreted as peaked on different scales
- "averaging" of d_S

Superpositions of regular complexes

Sum over graphs Γ (assuming orthogonality $\delta_{\Gamma,\Gamma'}$):

- sum over above triangulations with $N_0 = p^2 = (3 \cdot 2^k)^2$ vertices, $k = 0, 1, 2, 3$
- (for the sake of numerical feasibility: sharply peaked on J)





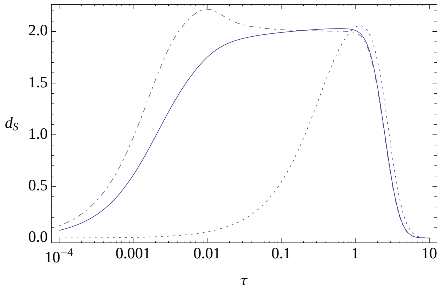
- interpreted as peaked on the same geometry
- interpreted as peaked on different scales

Again, some kind of "averaging"

Superpositions of regular complexes

Sum over more graphs Γ (assuming orthogonality $\delta_{\Gamma,\Gamma'}$):

- sum over triangulations with $N_0 = p^2$ vertices, $p = 3, 4, ..., 42$
- (for the sake of numerical feasibility: sharply peaked on J)



- interpreted as peaked on the same geometry

- "averaging" more interesting for larger sums
- convergence to topological dimension for infinite sum limit?
- (but larger triangulations not feasible...)

Conclusions for quantum geometries

- d_S of coherent states approximates classical geometry
- quantum corrections only of order $\mathcal{O}(10^{-2})$
- superposition of geometries peaked at/ superposition of graphs: larger effects due to nontrivial "averaging"

Work in progress:

- very large superpositions do not just result in convergence $d_S \to d$
- $d = 2$ is special here: for $d > 3$ there are certain superpositions with UV flow $d_S \to 2$
- spin foam dynamics: new features such as imaginary contributions (degenerate configurations)

Outline

Summary

- for discrete geometries new definition of Laplacian is needed $\rightarrow$ extension of discrete exterior calculus to discrete QG
- concept of dimension only for states on large complexes
- techniques developed to compute observables of states of geometry on large complexes
- semiclassical states provide good approximation to classical case, stronger quantum effects don't show up
- dependence on combinatorics seems dominant

Outlook

- spacetime dimension: Evaluation of insertion of heat trace in path integrals (SFs/GFT)
- generalize analytic results of random trees/ dynamical triangulations/ tensor models to include geometric data
- use discrete calculus for model building on the discrete level (in particular matter coupling)
- use methods developed for analysis of other observables to check geometric and continuum properties

Thank you for your attention!