Deriving LQC Dynamics from Diffeomorphism Invariance

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Motivation

- Loop Quantum Cosmology - quantum gravity in simplified symmetry-reduced setting
- Cosmology provides arena for testing predictions of quantum gravity
- Has seen a lot of development


How robust are these results?

Agullo, Ashtekar, Nelson, CQG30, 085014 (2013)
Uniqueness results

- There are choices in the quantization procedure
- In the full theory seminal work by Lewandowski, Okolow, Sahlmann, Thiemann (2006) used diffeomorphism invariance to select unique representation of quantum algebra
- In symmetry-reduced setting almost all diffeomorphism symmetry is fixed except for residual diffeos
- Can we use physical principles to also select unique dynamics?
Main Result

- Obtain Bianchi I Hamiltonian proposed by Ashtekar, Wilson-Ewing (2009) using invariance under residual diffeomorphisms:
  - Volume-preserving dilations
  - Parity transformations
  - Reflections
  - as well as a certain minimality principle and a planar loops assumption

- Obtain isotropic Hamiltonian proposed by Ashtekar, Pawlowski, Singh (2006) by projecting down from Bianchi I to isotropic model *without need for planar loops assumption*
Bianchi I model

\[ ds^2 = -N^2(t)dt^2 + a_1^2(t)dx_1^2 + a_2^2(t)dx_2^2 + a_3^2(t)dx_3^2, \]

Introduce fiducial cell \( V \) adapted to fiducial triads \( \hat{e}_i^a \) with side lengths \( L_1, L_2, L_3 \) and volume \( V_o \), fiducial metric \( \hat{q}_{ab} \)

Basic variables \( c_i, p^i \)

\[ A_a^i = c^i (L^i)^{-1} \hat{e}_a^i \quad E_i^a = p_i L_i V_o^{-1} \sqrt{\hat{q}} \hat{e}_i^a \]

Poisson bracket

\[ \{ c^i, p_j \} = 8\pi G \gamma \delta^i_j, \]
Classical Hamiltonian constraint

- Hamiltonian constraint
  \[ C_H = \int_{\mathcal{V}} N\mathcal{H} \, d^3 x, \]
  where the Hamiltonian density \( \mathcal{H} \) is
  \[ \mathcal{H} = \frac{E_i^a E_j^b}{16\pi G \sqrt{|q|}} (\epsilon^{ij}_k F_{ab}^k - 2(1 + \gamma^2) \epsilon^{ci} e^{dj} K_{c[a} K_{b]d}). \]

- Assume the lapse \( N(v) \) to be a function of the volume \( v := \sqrt{|p_1 p_2 p_3|} \) only, with the form \( N(v) = v^n \)

- Integrating over the fiducial cell we then obtain the constraint
  \[ C_H = -\frac{1}{8\pi G \gamma^2} v^{n-1} (p_1 p_2 c_1 c_2 + p_1 p_3 c_1 c_3 + p_2 p_3 c_2 c_3). \]
Hilbert space: almost periodic functions on $\mathbb{R}^3$.

Require $\hat{H}$ to preserve this Hilbert space

$\hat{H}|\vec{p}\rangle = \sum_i g_i(\vec{p})|\vec{F}_i(\vec{p})\rangle$

Define translation operator:

$T_F|\vec{p}\rangle := |\vec{F}(\vec{p})\rangle$

Impose self-adjointness. Can rewrite $\hat{H}$ as

$\hat{H} = \sum_i \left( T_{F_i} g_i(\vec{p}) + \overline{g_i(\vec{p})} T_{F_i}^\dagger \right)$
Existence of classical analogue

- Want to take the classical limit in the state-independent way.
- Assume $\vec{F}_i$ is generated as the flow, evaluated at unit time, of some vector field $8\pi\gamma G\hbar \vec{f}_i(\vec{p}) \cdot \nabla$ on $\mathbb{R}^3$.
- Get

$$\hat{H} = \sum_i \left( e^{i\vec{f}_i(\vec{p}) \cdot \vec{c}} g_i(\vec{p}) + g_i(\vec{p}) e^{-i\vec{f}_i(\vec{p}) \cdot \vec{c}} \right)$$
Volume-preserving positive rescalings $\Lambda(\vec{\lambda})$, with $\lambda_1 + \lambda_2 + \lambda_3 = 0$, act on the variables $c_i, p^i$:

$$\Lambda(\vec{\lambda})p^i = e^{-\lambda_i}p^i \quad \Lambda(\vec{\lambda})c_i = e^{\lambda_i}c_i.$$ 

The invariance leads to condition:

$$e^{\lambda_k}f^k_i(e^{-\lambda_1}p_1, e^{-\lambda_2}p_2, e^{\lambda_1+\lambda_2}p_3) = f^k_i(\vec{p}) \quad g_i(e^{-\lambda_1}p_1, e^{-\lambda_2}p_2, e^{\lambda_1+\lambda_2}p_3) = g_i(\vec{p}).$$

Obtain

$$f^k_i(\vec{p}) = p^k \tilde{f}^k_i(v, \text{sgn}\vec{p}) \quad g_i(\vec{p}) = g_i(v, \text{sgn}\vec{p}),$$

where $\text{sgn}\vec{p} = (\text{sgn}p_1, \text{sgn}p_2, \text{sgn}p_3)$

Therefore,

$$\hat{H} = \sum_{i=1}^{N} \left( e^i \sum_k \tilde{f}^k_i(v, \text{sgn}\vec{p})p^k c_k g_i(v, \text{sgn}\vec{p}) + \text{h.c.} \right).$$
Invariance under parity

- Invariance under parity implies that
  - Either $\tilde{f}^k_i$, $g_i$ independent of $\text{sgn} p$
  - Or $\hat{H}$ includes all the terms generated by parity so that for example

$$
\hat{H} = \sum_i \left( e^{i\tilde{f}^k_i(v,-\text{sgn} p_1,\text{sgn} p_2,\text{sgn} p_3)} p^k c_k g_i(v,-\text{sgn} p_1,\text{sgn} p_2,\text{sgn} p_3) + 
\right.
\left. + e^{i\tilde{f}^k_i(v,\text{sgn} p_1,\text{sgn} p_2,\text{sgn} p_3)} p^k c_k g_i(v,\text{sgn} p_1,\text{sgn} p_2,\text{sgn} p_3) + \text{rest of terms} \right) = 
\sum_i \left( e^{i\tilde{f}^k_i(v,-1,\text{sgn} p_2,\text{sgn} p_3)} p^k c_k g_i(v,-1,\text{sgn} p_2,\text{sgn} p_3) + 
\right.
\left. + e^{i\tilde{f}^k_i(v,1,\text{sgn} p_2,\text{sgn} p_3)} p^k c_k g_i(v,1,\text{sgn} p_2,\text{sgn} p_3) + \text{rest of terms} \right).
$$

- Parity invariance leads to

$$
\hat{H} = \sum_i \left( e^{i \sum_k \tilde{f}^k_i(v)} p^k c_k g_i(v) + \text{h.c.} \right).
$$
Reflections about the $x = y$, $x = z$ or $y = z$ planes and combinations thereof act on $c_i, p^i$ like permutations of labels.

$\hat{H}$ is invariant if includes all the terms generated by such permutations

$$\hat{H} = \sum_i \sum_{\sigma \in S_3} \left( e^{i \sum_k (\sigma \tilde{f}_i)^k(v) p^k c_k g_i(v)} + \text{h.c.} \right).$$
Imposition of the classical limit

- Require that the Hamiltonian reduces to the classical constraint in the classical limit
- Introduce a classicality parameter and consider the classical analogue of $\hat{H}$

$$H = \sum_i \sum_{\sigma \in S_3} \left( e^{i \sum_k (\sigma \tilde{f}_i)^k(v, \ell_p) p^k c_k} g_i(v, \ell_p) + \text{c.c.} \right)$$

- Require that the only length is the Planck length

$$\tilde{f}_i^k(v, \ell_p) = \frac{1}{\ell_p^2} \tilde{h}_i^k \left( \frac{\ell_p^3}{v} \right) = \frac{1}{\ell_p^2} \left( \tilde{h}_i^k(0) + (\tilde{h}_i^k)'(0) \frac{\ell_p^3}{v} + O \left( \frac{\ell_p^6}{v^2} \right) \right)$$

- Also use dimensional arguments for $g_i$

$$g_i(v, \ell_p) = \frac{\ell_p^{3n+1}}{G} \left( \sum_{j=0}^{\infty} \tilde{B}_i^j \frac{\ell_p^3 j}{v^j} \right)$$
Classical limit

- Let $\tilde{h}^k_i(0) = 0$ for all $i \in I$. Then matching the classical limit leads to

$$H = \sum_{i \in I} \sum_{\sigma \in S_3} \left( \frac{\ell p^{3n+1}}{G} \left( \sum_{j=j_0}^{\infty} \tilde{B}^i_j \frac{\ell p^3 j}{\nu^j} \right) e^{i \left( \frac{\ell p}{\nu} \sum_k (\sigma \tilde{A}^k_i)^k p_k c_k + O(\ell^4 p) \right)} + \text{c.c.} \right) + O(\ell^6 p)$$

- Can simplify

$$H = \sum_{i=1}^{N'} \frac{\ell p^{3n+1}}{G} \left( \sum_{j=j_0}^{\infty} B^i_j \frac{\ell p^3 j}{\nu^j} \right) e^{i \left( \frac{\ell p}{\nu} \sum_k A^k_i p_k c_k + O(\ell^4 p) \right)} + O(\ell^6 p)$$

- Combine exponentials (for simplification) so that

$$\tilde{A}^i_i = \tilde{A}^j_j \text{ implies } i = j.$$
Conditions for matching classical limit

- Conditions obtained by matching

\[ \sum_i \text{Re} \, B_i = 0 \]
\[ \sum_i A_{ij} \text{Im} \, B_i = 0 \]
\[ \sum_i A_{ij} (\text{Re} \, B_i) A_{ik} = M_{jk}. \]

where

\[ M := \frac{1}{8\pi\gamma^2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \]

- Minimality Assumption: smallest number of terms so that all the conditions on \( \hat{H} \) are satisfied
- Consider up to 12 rows in matrix \( A \) which corresponds to AW
Possibilities

- Eight rows
  \[
  \begin{pmatrix}
  a_1 & a_1 & b_1 \\
  \end{pmatrix}
  \]
  2 permutations
  3 reflections
  \[
  \begin{pmatrix}
  a_2 & a_2 & a_2 \\
  \end{pmatrix}
  \]
  1 reflection
  2 permutations
  3 reflections
  
- Ten rows
  \[
  \begin{pmatrix}
  a_1 & a_1 & b_1 \\
  \end{pmatrix}
  \]
  2 permutations
  3 reflections
  \[
  \begin{pmatrix}
  a_2 & a_2 & a_2 \\
  \end{pmatrix}
  \]
  1 reflection
  \[
  \begin{pmatrix}
  a_3 & a_3 & a_3 \\
  \end{pmatrix}
  \]
  1 reflection
  2 permutations
  3 reflections
  \[
  \begin{pmatrix}
  a_2 & a_2 & a_2 \\
  \end{pmatrix}
  \]
  1 reflection
  \[
  \begin{pmatrix}
  a_3 & a_3 & a_3 \\
  \end{pmatrix}
  \]
  1 reflection
Twelve rows

\[
\begin{pmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \\ a_4 & a_4 & a_4 \end{pmatrix}
\]

2 permutations
3 reflections
1 reflection
1 reflection

\[
\begin{pmatrix} a_1 & -a_1 & 0 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \\ a_4 & a_4 & a_4 \end{pmatrix}
\]

2 permutations
3 reflections
1 reflection
1 reflection
Twelve rows (plus some conditions on $a_1, b_1, a_2, b_2$)

\[
\begin{pmatrix}
  a_1 & a_1 & b_1 \\
\end{pmatrix}
\]

2 permutations
3 reflections

\[
\begin{pmatrix}
  a_2 & a_2 & b_2 \\
\end{pmatrix}
\]

2 permutations
3 reflections

Twelve rows

\[
\begin{pmatrix}
  a_1 & -a_1 & 0 \\
\end{pmatrix}
\]

2 permutations
3 reflections

\[
\begin{pmatrix}
  a_2 & a_2 & b_2 \\
\end{pmatrix}
\]

2 permutations
3 reflections
Selection of AW Hamiltonian

- Impose planar loops

\[ \text{depends on } e^{i \frac{\ell_p}{\gamma} A^k p^k c_k} + \mathcal{O}(\ell_p^c) \text{ one of } A^k \text{ is zero} \]

- Matrix $A$ is ($\Delta \ell_p^2$ the area gap)

\[
\begin{pmatrix}
\sqrt{\Delta} & -\sqrt{\Delta} & 0 \\
\sqrt{\Delta} & \sqrt{\Delta} & 0
\end{pmatrix}
\]

- Obtain AW Hamiltonian

\[
H_{AW} = \frac{1}{32\pi G \gamma^2 \Delta \ell_p^2} \nu^2 \left( e^{i \frac{\sqrt{\Delta} \ell_p}{\nu} (p_1 c_1 + p_2 c_2)} - e^{i \frac{\sqrt{\Delta} \ell_p}{\nu} (p_1 c_1 - p_2 c_2)} + e^{i \frac{\sqrt{\Delta} \ell_p}{\nu} (p_2 c_2 + p_3 c_3)} - e^{i \frac{\sqrt{\Delta} \ell_p}{\nu} (p_2 c_2 - p_3 c_3)} + e^{i \frac{\sqrt{\Delta} \ell_p}{\nu} (p_1 c_1 + p_3 c_3)} - e^{i \frac{\sqrt{\Delta} \ell_p}{\nu} (p_1 c_1 - p_3 c_3)} + \text{h.c.} \right) + \mathcal{O}(\ell_p^2)
\]
Selection of isotropic Hamiltonian

- Do not need planar loops assumption
- Use AW projector from Bianchi I states to isotropic states:
  \[
  (\hat{P}\Psi)(v) := \sum_{p_1, p_2} \Psi(p_1, p_2, v) \equiv \psi(v).
  \]
- Use the minimum number of terms to obtain
  \[
  A = \begin{pmatrix} 0 \\ a \\ -a \end{pmatrix}
  \]
- Obtain the APS Hamiltonian
  \[
  H_{APS} = k\ell_p^{-2}v \left(1 + e^{i(\frac{\ell_p}{v}2\sqrt{\Delta p}c)} + e^{-i(\frac{\ell_p}{v}2\sqrt{\Delta p}c)}\right) + O(\ell_p^2)
  \]
Derived Bianchi I Hamiltonian from residual diffeomorphism invariance and minimality principle as well as a planar loops assumption.

Obtained isotropic Hamiltonian without recourse to the planar loops assumption.

Our results increase confidence in phenomenological predictions of LQC as coming from the use of the holonomy-flux algebra.

By relaxing the minimality assumption have parametrization of ambiguities in the quantum Hamiltonian.