Covariant loop quantum gravity as a topological field theory with defects

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Basic idea

There are hints that LQG can be understood as a topological field theory with defects. Which kind of defects: Curvature defects? Area defects? Defects that carry torsion as well?

Goal of this talk: *Study a class of models for discretised gravity in terms of connection variables, where all gravitational degrees of freedom are carried by a finite number of topological defects.*

LQG à la Aharonov Bohm [Bianchi], [Freidel, Geiller, Ziprick], [ww]
Three-manifold $\Sigma_\Gamma^* = \Sigma - \Gamma_1^*$ with defects on one-skeleton of cellular decomposition of Σ.

$$\int_{\mathscr{A}_\Sigma} d\mu_{\mathrm{AL}}(A)\overline{\Psi_\Gamma[A]}\Phi_\Gamma[A] = \int_{\mathscr{A}_{\Sigma_\Gamma^*}} \prod_{\vec{x}\in\Sigma_\Gamma^*} \Big[dA(\vec{x})\Big]\,\delta\big(F[A]\big)\cdot$$
$$\delta\big(\Psi[A]\big)\Delta^{\Psi}_{\mathrm{FP}}[A]\;\overline{\Psi_\Gamma[A]}\Phi_\Gamma[A].$$

Area defects in LQG Electric field around a link γ dual to a face f

$$\tilde{E}_i{}^a(\vec{x}) = \beta {\ell_{\mathrm{P}}}^2\sqrt{j_f(j_f+1)}n_i^f \int_\gamma \mathrm{d}t\,\tilde{\delta}\big(\vec{x},\vec{X}(t)\big)\frac{\partial X^a(t)}{\partial t}.$$

DG BF distribution [Geiller, Dittrich] Distributional states as excitations over $F=0$.

Isolated Horizons Spinnetwork functions create punctures on an isolated horizon, which is a null surface of topology $S^2\times\mathbb{R}$. The spin network intertwiners are blown up and turn into punctured two-spheres.

*M Geiller and B Dittrich, A new vacuum for Loop Quantum Gravity Class. Quantum Grav. 32 (2015), arXiv:1401.6441.
*E Bianchi, Loop Quantum Gravity a la Aharonov-Bohm , Gen. Rel. Grav. 46 (2014), arXiv:0907.4388.
*ww, One-dimensional action for simplicial gravity in three dimensions, Phys. Rev. D 90 (2014), arXiv:1402.6708.
*L Freidel, M Geiller, J Ziprick, Continuous formulation of the Loop Quantum Gravity phase space ,
Class. Quant. Grav. 30 (2013), arXiv:1110.4833.

LQG as a TFT with defects — which defects?

Roughly speaking, two different strategies in the literature

(i) covariant theory with spacetime defects

Start from cellular (simplicial) decomposition Δ of spacetime M (discretization), remove the two-dimensional faces (triangles) and define $M^* := M - \Delta_2$. The curvature vanishes in M^*, classical configuration space: moduli space $\mathrm{Hom}(\pi_1(M^* : G))/G$ of flat connections.

Idea: *Identify the non-contractible cycles in M^* with worldlines of LQG area defects. Write down the simplest action and study the coupled system: area defects (auxiliary particles) coupled to a topological field theory.*

(ii) Canonical theory with defects on spatial slices [Dittrich's program]

Similar idea, but one dimension lower. The Dittrich–Geiller BF distribution on a spatial slice Σ is seen as the unique ground state of a certain Hamiltonian. Excitations over this ground state are topological defects and carry curvature and torsion.

Challenge: *Find the correct gapped Hamiltonian, which should have a vastly degenerate vacuum containing all physical states of quantum gravity.*

Outline of the talk

Main message: *BF action with simplicity constraints can be written as a wordline integral for area defects. These defects carry charges for curvature and torsion that can be naturally coupled to a topological field theory.*

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Worldline action for LQG area defects

The BF action is topological, and determines the symplectic structure of the theory:

$$S_{\mathrm{BF}}[\Sigma, A] = \int_M \underbrace{\frac{1}{16\pi G}\left({}^*\Sigma_{\alpha\beta} - \frac{1}{\beta}\Sigma_{\alpha\beta}\right)}_{\Pi_{\alpha\beta}} \wedge F^{\alpha\beta}[A]. \tag{1}$$

General relativity follows from the simplicity constraints added to the action:

$$\Sigma^{\alpha\beta} \wedge \Sigma^{\mu\nu} \propto \epsilon^{\alpha\beta\mu\nu}. \tag{2}$$

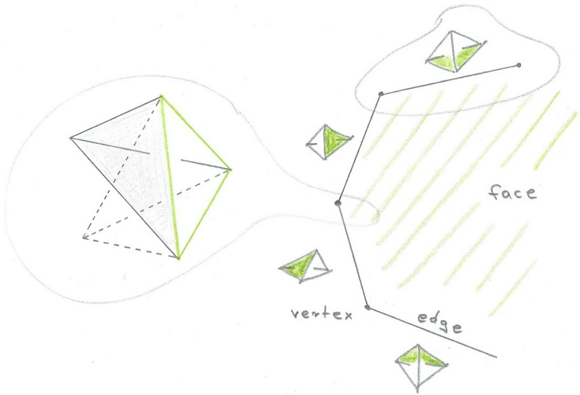
With the solutions:

$$\Sigma^{\alpha\beta} = \begin{cases} \pm e_\alpha \wedge e_\beta, \\ \pm^*(e_\alpha \wedge e_\beta). \end{cases} \tag{3}$$

Notation:

- $\alpha, \beta, \gamma \dots$ are internal Lorentz indices.
- $\Sigma^\alpha{}_\beta$ is an $\mathfrak{so}(1,3)$-valued two-form.
- $A^\alpha{}_\beta$ is an $SO(1,3)$ connection, with $F^\alpha{}_\beta = \mathrm{d}A^\alpha{}_\beta + A^\alpha{}_\mu \wedge A^\mu{}_\beta$ denoting its curvature.
- e^α is the tetrad, diagonalizing the four-dimensional metric $g = \eta_{\alpha\beta} e^\alpha \otimes e^\beta$.

Simplicial discretization



Spinorial action for discretised BF theory

Discretise $S_{\mathrm{BF}}[\Pi, A] = \int_M \Pi_{\alpha\beta} \wedge F^{\alpha\beta}$ as a sum over faces f.

Worldline action for discretised BF

$$S_{\mathrm{BF\ discrete}}[Z, A, a, b] = \sum_{f:\mathrm{faces}} \left[\oint_{\partial f} \pi_A (D - a) \omega^A + \int_f b \, \mathrm{d}a \right] + \mathrm{cc.} \qquad (4)$$

Geometric interpretation:

- the spinors $Z = (\bar{\pi}_{\bar{A}}, \omega^A)$ diagonalise the simplicial fluxes.
- $D\pi^A = \mathrm{d}\pi^A + A^A{}_B \pi^B$ is the selfdual Ashtekar conenction.
- a is a $\mathbb{C} - 0$ connection from the $(\bar{\pi}_{\bar{A}}, \omega^A) \to (\mathrm{e}^{-\bar{z}} \bar{\pi}_{\bar{A}}, \mathrm{e}^{z} \omega^A)$ gauge symmetry.
- the Lagrange multiplier b imposes that a is flat in f, $\mathrm{d}a = 0$

*L Freidel and S Speziale, From twistors to twisted geometries, Phys. Rev. D 82 (2010), arXiv:1006.0199.
*S Speziale and ww, Twistorial structure of loop-gravity transition amplitudes, Phys. Rev. D 86 (2012), arXiv:1207.6348.
*ww, Hamiltonian spinfoam gravity, Class.Quant.Grav. 31 (2014), arXiv:1301.5859.
*ww, New action for simplicial gravity in four dimensions, Class. Quant. Grav. 32 (2015), arXiv:1407.0025.
*L Freidel, M Geiller, J Ziprick, Continuous formulation of the Loop Quantum Gravity phase space ,
Class. Quant. Grav. 30 (2013), arXiv:1110.4833.

- $SO(1,3)$ BF-action

$$S_{\mathrm{BF}}[\Pi, A] = \int_M \Pi_{\alpha\beta} \wedge F^{\alpha\beta} \tag{5}$$

- $SO(1,3)$ fluxes over triangles $\triangle_f$ dual to the faces

$$[\Pi_f]^{\alpha\beta} = \int_{\triangle_f} \left[h_f[A]\right]^{\alpha}{}_{\mu} \left[h_f[A]\right]^{\beta}{}_{\nu} \Pi^{\mu\nu} \tag{6}$$

- Spinors are eigenvectors of the selfdual component Π_f^{AB} of $\Pi_f^{\alpha\beta}$.

$$\Pi^{AB} := \frac{1}{4}\sigma^{A}{}_{\bar{C}[\alpha}\bar{\sigma}^{\bar{C}B}{}_{\beta]}\Pi^{\alpha\beta} = -\frac{1}{2}\omega^{(A}\pi^{B)} \tag{7}$$

- $\mathbb{C} - \{0\}$ gauge symmetry (requires gauging $D \to (D - a)$ of covariant derivative.

$$(\pi^A, \omega^A) \longrightarrow (\mathrm{e}^z \pi^A, \mathrm{e}^{-z}\omega^A) \tag{8}$$

Instead of discretizing the quadratic simplicity constraints

$$\Sigma_{\alpha\beta} \wedge \Sigma_{\mu\nu} \propto \epsilon_{\alpha\beta\mu\nu}, \tag{9}$$

we will use the linear simplicity constraints:

For a tetrahedron T_e (dual to an edge e) there exist an internal future-oriented four-vector n_e^α such that the fluxes through the four bounding triangles $\triangle_f$ (dual to a face f: $e \subset \partial f$) annihilate n_e^α:

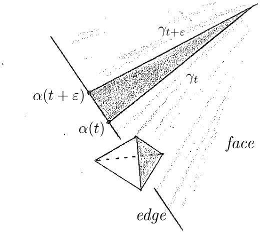
$$\int_{\triangle_f} \Sigma_{\alpha\beta} n_e^\beta = 0. \tag{10}$$

The spinorial parametrization turns the simplicity constraints into the following complex conditions:

$$V_f = \frac{\mathrm{i}}{\beta + \mathrm{i}} \pi^f_A \omega^A_f + \mathrm{cc.} \stackrel{!}{=} 0, \tag{11a}$$

$$W_{ef} = n_e^{A\bar{A}} \pi^f_A \bar{\omega}^f_{\bar{A}} \stackrel{!}{=} 0. \tag{11b}$$

What is the geometric interpretation of $n^{A\bar{A}}$?

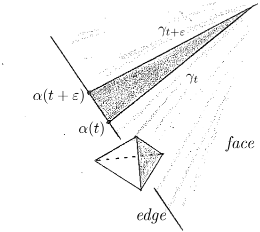


We interpret the normal $n^{A\bar{A}} = n^{\alpha}$ as the pull back of a one-form $e^{\alpha}{}_{a}$ to the boundary of f.

$$n^{A\bar{A}} = e^{A\bar{A}}{}_{a} t^{a}.$$

We add the constraints and arrive at the following action:

$$S_f[Z|\zeta,\varphi|A,e,a,b] = \oint_{\partial f}\Big[\pi_A(D-a)\omega^A - \zeta e_{A\bar{A}}\pi^A\omega^{\bar{A}} + \mathrm{cc.}\Big] + \\ + \int_f \Big[b\,\mathrm{d}a + \bar{b}\,\mathrm{d}\bar{a} - \varphi\left(\frac{\mathrm{i}}{\beta+\mathrm{i}}b + \mathrm{cc.}\right)\Big].$$



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discretised BF

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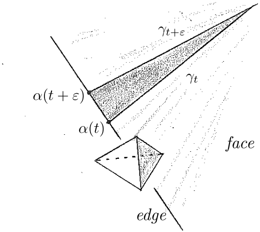

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simplicity constraints

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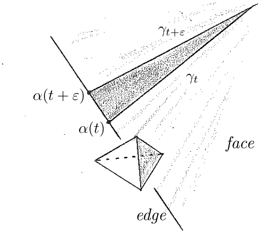

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Lagrange multipliers

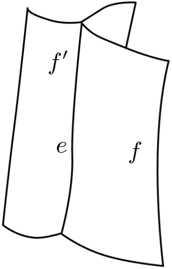
Coupling to a three-dimensional field theory on the horizon

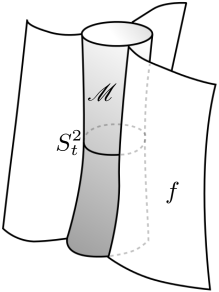
The action depends on background fields.

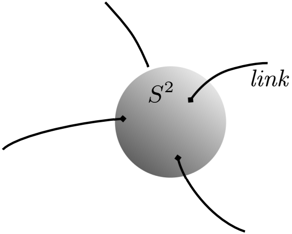
$$S_f[Z|\zeta,\varphi|A,e,a,b]=\oint_{\partial f}\Big[\pi_A(D-a)\omega^A-\zeta e_{A\bar{A}}\pi^A\omega^{\bar{A}}+\mathrm{cc.}\Big]+\\+\int_f\Big[b\,\mathrm{d}a+\bar{b}\,\mathrm{d}\bar{a}-\varphi\Big(\frac{\mathrm{i}}{\beta+\mathrm{i}}b+\mathrm{cc.}\Big)\Big].$$

It seems natural to turn the one-forms $A^{A}{}_{Ba}$ and $e^{A\bar{A}}{}_{a}$ into dynamical fields as well. We propose an additional field theory for $A^{A}{}_{Ba}$ and $e^{A\bar{A}}{}_{a}$.

Smearing out the discretisation







Smear out the system of edges e of the discretisation and replace them by a three-manifold $\mathscr{M}$.

Similar topological structure appears in the isolated horizon framework: Punctured two-spheres connected by spin network links

Advantage: Removal of point-like interactions — spinfoam vertices.

Simplest action on $\mathscr{M}$: Chern–Simons action for de-Sitter connection

Propose the following action for a topological field theory on $\mathscr{M}$

$$S_{\mathscr{M}}[A,e] = \frac{\alpha}{2}\int_{\mathscr{M}}\Big[A^{\alpha}{}_{\beta}\wedge \mathrm{d}A^{\beta}{}_{\alpha} + \frac{2}{3}A^{\alpha}{}_{\mu}\wedge A^{\mu}{}_{\nu}\wedge A^{\nu}{}_{\alpha}\Big] - \lambda\int_{\mathscr{M}} e_{\alpha}\wedge De^{\alpha}.$$

Equations of motion

$$\textit{vanishing of torsion:}\quad De^{\alpha} = 0 \tag{12a}$$

$$\textit{local de Sitter curvature:}\quad F^{\alpha}{}_{\beta} - \frac{\lambda}{\alpha}e^{\alpha}\wedge e_{\beta} = 0 \tag{12b}$$

Comments:

- "cosmological constant" $\Lambda = 3\lambda/\alpha$ on $\mathscr{M}$.
- $S_{\mathscr{M}}$ is an $SO(1,4)$ resp. $SO(2,3)$ Chern–Simons action depending on the sign $\Lambda > 0$ resp. $\Lambda < 0$ of the cosmological constant.
- Local $SO(p,q)$ gauge symmetries are on-shell equivalent to diffeomorphisms$\ltimes$Lorentz transformations.
- The limit $\alpha, \zeta, \varphi, \beta \to 0$ yields a recent model of Perez and Freidel.

*L Freidel and A Perez, Quantum gravity at the corner (2015), `arXiv:1507.02573`.
*W Donnelly and L Freidel, Local subsystems in gauge theory and gravity (2015), `arXiv:1601.04744`.
*ww, Complex Ashtekar variables, the Kodama state and spinfoam gravity (2011), `arXiv:1105.2330`.
*H Haggard, M Han, W Kaminski, A Riello, Four-dimensional Quantum Gravity with a Cosmological Constant from Three-dimensional Holomorphic Blocks, Phys Lett. B 752 (2016), `arXiv:1509.00458`.

Integrability conditions
and the value of the cosmological constant

The N defects on S^2 are sources for curvature and torsion

$$T^{A\bar{A}} = De^{A\bar{A}} = +\frac{1}{2\lambda}\sum_{i=1}^{N}\big(\zeta^i\pi_i^A\bar{\omega}_i^{\bar{A}} - \mathrm{cc.}\big)\tilde{\delta}(\vec{x}_i), \tag{13a}$$

$$F^{AB} - \frac{\Lambda}{6}e^{A\bar{C}}\wedge e^{B}{}_{\bar{C}} = -\frac{1}{2\alpha}\sum_{i=1}^{N}\pi_i^{(A}\omega_i^{B)}\tilde{\delta}(\vec{x}_i). \tag{13b}$$

The spinors have a non-trivial dynamics along their trajectory

$$\frac{D}{\mathrm{d}t}\omega^A = \big\{t\llcorner H, \omega^A\big\}, \quad \frac{D}{\mathrm{d}t}\pi^A = \big\{t\llcorner H, \pi^A\big\}. \tag{14}$$

with Hamiltonian

$$t\llcorner H = t^b a_b\,\pi_A\omega^A + t^b e^{A\bar{A}}{}_b\,\zeta\pi_A\bar{\omega}_{\bar{A}} + \mathrm{cc.} \tag{15}$$

and Poisson brackets

$$\big\{\pi_A, \omega^B\big\} = \delta_A^B, \quad \big\{\bar{\pi}_{\bar{A}}, \bar{\omega}^{\bar{B}}\big\} = \delta_{\bar{A}}^{\bar{B}}. \tag{16}$$

Integrability conditions

Bianchi identities

$$DF^{\alpha\beta} = 0, \qquad DT^{\alpha} = D^2 e^{\alpha} = F^{\alpha}{}_{\beta} \wedge e^{\beta}. \tag{17}$$

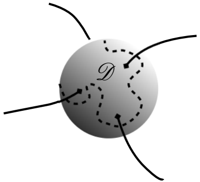
Imply integrability conditions (secondary constraints)

Definite sign of Λ

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta = t \llcorner (a - \bar{a})\zeta, \tag{18a}$$

$$\Lambda = -6\zeta\bar{\zeta} < 0. \tag{18b}$$

The N defects on S^2 are sources for curvature and torsion



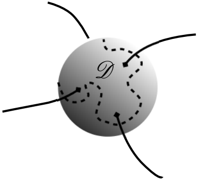
$$T^{A\bar{A}} = De^{A\bar{A}} = \sum_{i=1}^{N} J_i^{A\bar{A}} \tilde{\delta}(\vec{x}_i),$$

$$F^{AB} - \frac{\Lambda}{6} e^{A\bar{C}} \wedge e^{B}{}_{\bar{C}} = \sum_{i=1}^{N} J_i^{AB} \tilde{\delta}(\vec{x}_i).$$

$SO(1,4)$ De Sitter connection

$$A^{IJ}{}_{a} = \frac{1}{2} A^{\mu\nu}{}_{a} M^{IJ}{}_{\mu\nu} + e^{\mu}{}_{a} P^{IJ}{}_{\mu}\,, \quad \begin{cases} \left[M_{\alpha\beta}, M_{\mu\nu}\right] = +4\delta^{\rho}_{[\alpha}\delta^{\rho'}_{\beta]}\eta_{\rho'\sigma'}\delta^{\sigma'}_{[\mu}\delta^{\sigma}_{\nu]} M_{\rho\sigma}\,, \\ \left[P_{\mu}, P_{\nu}\right] = -\frac{\Lambda}{3} M_{\mu\nu}, \\ \left[M_{\mu\nu}, P_{\alpha}\right] = -2\eta_{\alpha[\mu} P_{\nu]}. \end{cases}$$

The N defects on S^2 are sources for curvature and torsion



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Deformed Gauss law from nonabelian Stokes' theorem

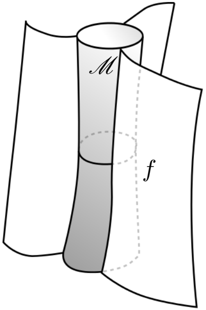
$$\begin{aligned} \mathbb{1} &= \mathrm{Pexp}\Big(-\frac{1}{2} \oint_{\partial\mathscr{D}} (A^{\alpha\beta} M_{\alpha\beta} + 2e^{\alpha} P_{\alpha})\Big) = \\ &= \mathrm{PPexp}\Big(-\frac{1}{2} \int_{\mathscr{D}} (F^{\alpha\beta} M_{\alpha\beta} + 2T^{\alpha} P_{\alpha})\Big) = \\ &= \mathrm{P} \prod_{i=1}^{N} \exp\Big(-\frac{1}{2} J_i^{\alpha\beta} M_{\alpha\beta} - J_i^{\alpha} P_{\alpha}\Big). \end{aligned} \tag{21}$$

Conclusion

Spin-foam amplitudes = TFT with defects

four-dimensional amplitudes through three-dimensional path integral

$$\mathscr{Z} = \sum_{\substack{\text{3-geometries}\\ \mathscr{M}}} \int \mathscr{D}[A,e]\mathrm{e}^{\mathrm{i}S_{\mathscr{M}}[A,e]} \int \mathscr{D}[Z|\zeta,\varphi|a,b] \prod_{f:\text{faces}} \mathrm{e}^{\mathrm{i}S_f[Z|\zeta,\varphi|A,e,a,b]}.$$



SO(1,4) Chern–Simons action

Spin-foam amplitudes = TFT with defects

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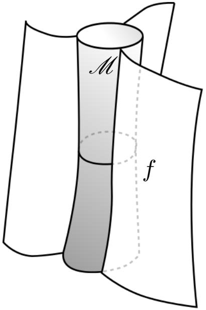

insertion of charges along worldlines $\gamma_i = \partial f_i$

- The discretised BF action can be written as a worldline action for spinors propagating along the edges of the discretisation.
- Gravity is a constrained topological theory (BF+simplicity) — we assume the same holds for the discretised theory and add the discretised simplicity constraints to the worldline model.
- We proposed to blew up the one-dimensional edges and replaced them by an oriented three-manifold $\mathscr{M}$. This has a number of advantages: (i) removal of point-like interactions (spinfoam vertices), (ii) same topological structure as in isolated horizon framework, (iii) the spinors couple naturally to a three-dimensional field theory on $\mathscr{M}$. The simplest field theory that we can build out of the background fields e^α and $A^\alpha{}_\beta$ on $\mathscr{M}$ is an $G = SO(p,q)$ Chern–Simons theory for the de Sitter group.
- Integrability conditions fix the gauge group to $SO(2,3)$ rather than $SO(1,4)$. AdS/CFT?