Covariant loop quantum gravity as a topological field theory with defects

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There are hints that LQG can be understood as a topological field theory with defects. Which kind of defects: Curvature defects? Area defects? Defects that carry torsion as well?

**Goal of this talk:** Study a class of models for discretised gravity in terms of connection variables, where all gravitational degrees of freedom are carried by a finite number of topological defects.
Hints from Loop Quantum Gravity

LQG à la Aharonov Bohm [Bianchi], [Freidel, Geiller, Ziprick], [ww]
Three-manifold $\Sigma^*_\Gamma = \Sigma - \Gamma^*_1$ with defects on one-skeleton of cellular decomposition of $\Sigma$.

$$\int_{\mathcal{A}_\Sigma} d\mu_{\text{AL}}(A) \overline{\Psi_\Gamma[A]} \Phi_\Gamma[A] = \int_{\mathcal{A}_\Sigma^*} \prod_{\vec{x} \in \Sigma^*_\Gamma} \left[ dA(\vec{x}) \right] \delta(F[A]) \cdot \delta(\Psi[A]) \Delta_{\text{FP}}^\Psi[A] \overline{\Psi_\Gamma[A]} \Phi_\Gamma[A].$$

Area defects in LQG
Electric field around a link $\gamma$ dual to a face $f$

$$\tilde{E}^a_i(\vec{x}) = \beta \ell_P^2 \sqrt{j_f(j_f + 1)n_f^i} \int_{\gamma} dt \tilde{\delta}(\vec{x}, \tilde{X}(t)) \frac{\partial X^a(t)}{\partial t}.$$

DG BF distribution [Geiller, Dittrich]
Distributional states as excitations over $F = 0$.

Isolated Horizons
Spinnetwork functions create punctures on an isolated horizon, which is a null surface of topology $S^2 \times \mathbb{R}$. The spin network intertwiners are blown up and turn into punctured two-spheres.

LQG as a TFT with defects — which defects?

Roughly speaking, two different strategies in the literature

(i) covariant theory with spacetime defects

Start from cellular (simplicial) decomposition $\Delta$ of spacetime $M$ (discretization), remove the two-dimensional faces (triangles) and define $M^* := M - \Delta_2$. The curvature vanishes in $M^*$, classical configuration space: moduli space $\text{Hom}(\pi_1(M^*: G))/G$ of flat connections.

**Idea:** Identify the non-contractible cycles in $M^*$ with worldlines of LQG area defects. Write down the simplest action and study the coupled system: area defects (auxiliary particles) coupled to a topological field theory.

(ii) Canonical theory with defects on spatial slices [Dittrich’s program]

Similar idea, but one dimension lower. The Dittrich–Geiller BF distribution on a spatial slice $\Sigma$ is seen as the unique ground state of a certain Hamiltonian. Excitations over this ground state are topological defects and carry curvature and torsion.

**Challenge:** Find the correct gapped Hamiltonian, which should have a vastly degenerate vacuum containing all physical states of quantum gravity.
Main message: BF action with simplicity constraints can be written as a wordline integral for area defects. These defects carry charges for curvature and torsion that can be naturally coupled to a topological field theory.

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Worldline action for LQG area defects
The BF action is topological, and determines the symplectic structure of the theory:

\[
S_{BF}[\Sigma, A] = \int_M \frac{1}{16\pi G} \left( *\Sigma_{\alpha\beta} - \frac{1}{\beta} \Sigma_{\alpha\beta} \right) \wedge F^{\alpha\beta}[A].
\] (1)

General relativity follows from the simplicity constraints added to the action:

\[
\Sigma^{\alpha\beta} \wedge \Sigma^{\mu\nu} \propto \epsilon^{\alpha\beta\mu\nu}.
\] (2)

With the solutions:

\[
\Sigma^{\alpha\beta} = \begin{cases} 
\pm e_\alpha \wedge e_\beta, \\
\pm^* (e_\alpha \wedge e_\beta).
\end{cases}
\] (3)

Notation:

- \(\alpha, \beta, \gamma \ldots\) are internal Lorentz indices.
- \(\Sigma^{\alpha\beta}\) is an \(\mathfrak{so}(1, 3)\)-valued two-form.
- \(A^\alpha_\beta\) is an \(SO(1, 3)\) connection, with \(F^{\alpha\beta} = dA^\alpha_\beta + A^\alpha_\mu \wedge A^\mu_\beta\) denoting its curvature.
- \(e^\alpha\) is the tetrad, diagonalizing the four-dimensional metric \(g = \eta_{\alpha\beta} e^\alpha \otimes e^\beta\).
Simplicial discretization
Spinorial action for discretised BF theory

Discretise $S_{BF}[\Pi, A] = \int_M \Pi_{\alpha\beta} \wedge F^{\alpha\beta}$ as a sum over faces $f$.

Worldline action for discretised BF

$$S_{BF \text{ discrete}}[Z, A, a, b] = \sum_{f:\text{faces}} \left[ \oint_{\partial f} \pi_A (D - a) \omega^A + \int_f b \, da \right] + \text{cc.}$$

(4)

Geometric interpretation:

- the spinors $Z = (\bar{\pi}_A, \omega^A)$ diagonalise the simplicial fluxes.
- $D \pi^A = d\pi^A + A^A_B \pi^B$ is the selfdual Ashtekar conection.
- $a$ is a $\mathbb{C} - 0$ connection from the $(\bar{\pi}_A, \omega^A) \to (e^{-\bar{z}} \bar{\pi}_A, e^{z} \omega^A)$ gauge symmetry.
- the Lagrange multiplier $b$ imposes that $a$ is flat in $f$, $da = 0$

Geometric interpretation of the spinors

- **$SO(1, 3)$ BF-action**

  \[ S_{BF}[\Pi, A] = \int_M \Pi_{\alpha\beta} \wedge F^{\alpha\beta} \quad (5) \]

- **$SO(1, 3)$ fluxes over triangles $\triangle_f$ dual to the faces**

  \[ [\Pi_f]^{\alpha\beta} = \int_{\triangle_f} [h_f [A]]^{\alpha}_\mu [h_f [A]]^{\beta}_\nu \Pi^{\mu\nu} \quad (6) \]

- **Spinors are eigenvectors of the selfdual component $\Pi_{fAB}^A$ of $\Pi_f^{\alpha\beta}$**.

  \[ \Pi^{AB} := \frac{1}{4} \sigma^A \tilde{C}[\alpha \tilde{\sigma}^{CB}_\beta] \Pi^{\alpha\beta} = -\frac{1}{2} \omega(A, \pi B) \quad (7) \]

- **$\mathbb{C} - \{0\}$ gauge symmetry (requires gauging $D \rightarrow (D - a)$ of covariant derivative.**

  \[ (\pi^A, \omega^A) \rightarrow (e^{-z} \pi^A, e^{-z} \omega^A) \quad (8) \]
Instead of discretizing the quadratic simplicity constraints

\[ \Sigma_{\alpha\beta} \wedge \Sigma_{\mu\nu} \propto \epsilon_{\alpha\beta\mu\nu}, \]  

(9)

we will use the linear simplicity constraints:

For a tetrahedron \( T_e \) (dual to an edge \( e \)) there exist an internal future-oriented four-vector \( n^\alpha_e \) such that the fluxes through the four bounding triangles \( \triangle_f \) (dual to a face \( f \): \( e \subset \partial f \)) annihilate \( n^\alpha_e \):

\[ \int_{\triangle_f} \Sigma_{\alpha\beta} n^\beta_e = 0. \]  

(10)

The spinorial parametrization turns the simplicity constraints into the following complex conditions:

\[ V_f = \frac{i}{\beta + i} \pi^f_A \omega^A_f + \text{cc.} = 0, \]  

(11a)

\[ W_{ef} = n^A_{e \bar{A}} \pi^f_A \bar{\omega}^f_{\bar{A}} = 0. \]  

(11b)
What is the geometric interpretation of $n^{A\bar{A}}$?

We interpret the normal $n^{A\bar{A}} = n^\alpha$ as the pull back of a one-form $e^\alpha_a$ to the boundary of $f$.

$$n^{A\bar{A}} = e^{A\bar{A}}_a t^a.$$ 

We add the constraints and arrive at the following action:

$$S_f [Z|\zeta, \varphi|A, e, a, b] = \oint_{\partial f} \left[ \pi_A (D - a) \omega^A - \zeta e_{A\bar{A}} \pi^A \omega^{\bar{A}} + \text{cc.} \right] +$$

$$+ \int_f \left[ b \, da + \bar{b} \, d\bar{a} - \varphi \left( \frac{i}{\beta + i b} + \text{cc.} \right) \right].$$
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discretised BF
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simplicity constraints
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$$\quad + \int_f \left[ b da + \bar{b} d\bar{a} - \varphi \left( \frac{i}{\beta + i} b + cc. \right) \right].$$

Lagrange multipliers
Coupling to a three-dimensional field theory on the horizon
The action depends on background fields.

\[ S_f[Z|\zeta, \varphi|A, e, a, b] = \oint_{\partial f} \left[ \pi_A (D - a) \omega^A - \zeta e_{A\bar{A}} \pi^A \omega^{\bar{A}} + cc \right] + \int_f \left[ b \, da + \bar{b} \, d\bar{a} - \varphi \left( \frac{i}{\beta + i} b + cc \right) \right]. \]

*It seems natural to turn the one-forms $A^A_{\bar{B}a}$ and $e^{A\bar{A}}_a$ into dynamical fields as well. We propose an additional field theory for $A^A_{\bar{B}a}$ and $e^{A\bar{A}}_a$.*
Smearing out the discretisation and replace them by a three-manifold $\mathcal{M}$.

Similar topological structure appears in the isolated horizon framework: Punctured two-spheres connected by spin network links

**Advantage:** Removal of point-like interactions — spin foam vertices.
Propose the following action for a topological field theory on $\mathcal{M}$

$$S_{\mathcal{M}}[A, e] = \frac{\alpha}{2} \int_{\mathcal{M}} \left[ A_{\beta}^\alpha \wedge dA^\alpha_\beta + \frac{2}{3} A^\alpha_\mu \wedge A^\mu_\nu \wedge A^\nu_\alpha \right] - \lambda \int_{\mathcal{M}} e_\alpha \wedge De^\alpha.$$ 

Equations of motion

\begin{align*}
\text{vanishing of torsion:} & \quad De^\alpha = 0 \quad (12a) \\
\text{local de Sitter curvature:} & \quad F_{\alpha}^{\alpha} - \frac{\lambda}{\alpha} e_\alpha \wedge e_\beta = 0 \quad (12b)
\end{align*}

Comments:

- “cosmological constant” $\Lambda = 3\lambda/\alpha$ on $\mathcal{M}$.
- $S_{\mathcal{M}}$ is an $SO(1, 4)$ resp. $SO(2, 3)$ Chern–Simons action depending on the sign $\Lambda > 0$ resp. $\Lambda < 0$ of the cosmological constant.
- Local $SO(p, q)$ gauge symmetries are on-shell equivalent to diffeomorphisms $\times$ Lorentz transformations.
- The limit $\alpha, \zeta, \varphi, \beta \to 0$ yields a recent model of Perez and Freidel.

*L Freidel and A Perez, Quantum gravity at the corner (2015), arXiv:1507.02573.
*ww, Complex Ashtekar variables, the Kodama state and spinfoam gravity (2011), arXiv:1105.2330.
Integrability conditions and the value of the cosmological constant
The $N$ defects on $S^2$ are sources for curvature and torsion

\[ T^{A\bar{A}} = De^{A\bar{A}} = + \frac{1}{2\lambda} \sum_{i=1}^{N} (\zeta_i \pi_i^A \bar{\omega}_i^\bar{A} - \text{cc.}) \tilde{\delta}(\vec{x}_i), \]  

(13a)

\[ F^{AB} - \frac{\Lambda}{6} e^{A\bar{C}} \wedge e^B_{\bar{C}} = - \frac{1}{2\alpha} \sum_{i=1}^{N} \pi_i^A \omega_i^B \tilde{\delta}(\vec{x}_i). \]  

(13b)

The spinors have a non-trivial dynamics along their trajectory

\[ \frac{D}{dt} \omega^A = \{t \mathcal{H}, \omega^A\}, \quad \frac{D}{dt} \pi^A = \{t \mathcal{H}, \pi^A\}. \]  

(14)

with Hamiltonian

\[ t \mathcal{H} = t^b a_b \pi_A \omega^A + t^b e^{A\bar{A}}_b \zeta \pi_A \bar{\omega}_{\bar{A}} + \text{cc.} \]  

(15)

and Poisson brackets

\[ \{\pi_A, \omega^B\} = \delta_A^B, \quad \{\bar{\pi}_{\bar{A}}, \bar{\omega}^\bar{B}\} = \delta_{\bar{A}}^\bar{B}. \]  

(16)
Bianchi identities

\[ DF^{\alpha\beta} = 0, \quad DT^\alpha = D^2 e^\alpha = F^{\alpha\beta} \wedge e^\beta. \]  

(17)

Imply integrability conditions (secondary constraints)

Definite sign of \( \Lambda \)

\[ \frac{d}{dt} \zeta = t_L (a - \bar{a}) \zeta, \quad (18a) \]

\[ \Lambda = -6 \zeta \bar{\zeta} < 0. \quad (18b) \]
Deformed Gauss law

The $N$ defects on $S^2$ are sources for curvature and torsion

\[ T^{A\bar{A}} = De^{A\bar{A}} = \sum_{i=1}^{N} J_{i}^{A\bar{A}} \tilde{\delta}(\vec{x}_i), \]

\[ F^{AB} - \frac{\Lambda}{6} e^{A\bar{C}} \wedge e^{B\bar{C}} = \sum_{i=1}^{N} J_{i}^{AB} \tilde{\delta}(\vec{x}_i). \]

$SO(1, 4)$ De Sitter connection

\[ A^{IJ}_a = \frac{1}{2} A^{\mu\nu} a M^{IJ}_{\mu\nu} + e^\mu a P^{IJ}_\mu, \]

\[
\begin{align*}
[M_{\alpha\beta}, M_{\mu\nu}] &= +4\delta^\rho_{[\alpha} \delta^\rho'_{\beta]} \eta_{\rho'\sigma'} \delta^\sigma_{[\mu} \delta^\sigma_{\nu]} M_{\rho\sigma}, \\
[P_\mu, P_\nu] &= -\frac{\Lambda}{3} M_{\mu\nu}, \\
[M_{\mu\nu}, P_\alpha] &= -2\eta_{\alpha[\mu} P_{\nu]}. \end{align*}
\]
The $N$ defects on $S^2$ are sources for curvature and torsion

\[ T^{A\bar{A}} = De^{A\bar{A}} = \sum_{i=1}^{N} J^{A\bar{A}}_i \delta(\vec{x}_i), \]

\[ F^{AB} - \frac{\Lambda}{6} e^{A\bar{C}} \wedge e^{B\bar{C}} = \sum_{i=1}^{N} J^{AB}_i \tilde{\delta}(\vec{x}_i). \]

Deformed Gauss law from nonabelian Stokes' theorem

\[ 1 = \text{Pexp}\left(-\frac{1}{2} \oint_{\partial \mathcal{D}} (A^{\alpha\beta} M_{\alpha\beta} + 2 e^\alpha P_\alpha) \right) = \]

\[ = \text{P} \text{Pexp}\left(-\frac{1}{2} \int_{\mathcal{D}} (F^{\alpha\beta} M_{\alpha\beta} + 2 T^\alpha P_\alpha) \right) = \]

\[ = \text{P} \prod_{i=1}^{N} \exp\left(-\frac{1}{2} J^{\alpha\beta}_i M_{\alpha\beta} - J^\alpha_i P_\alpha \right). \quad (21) \]
Conclusion
Spin-foam amplitudes = TFT with defects

four-dimensional amplitudes through three-dimensional path integral

\[ \mathcal{L} = \sum_{\mathcal{M}} \int \mathcal{D}[A, e] e^{iS_{\mathcal{M}}[A, e]} \int \mathcal{D}[Z|\zeta, \varphi|a, b] \prod_{f:\text{faces}} e^{iS_f[Z|\zeta, \varphi|A, e, a, b]} \].

SO(1,4) Chern–Simons action
Spin-foam amplitudes = TFT with defects

four-dimensional amplitudes through three-dimensional path integral

\[ \mathcal{L} = \sum_{\text{3-geometries}} \int \mathcal{D}[A, e] e^{iS_M[A, e]} \int \mathcal{D}[Z|\zeta, \varphi|a, b] \prod_{f:\text{faces}} e^{iS_f[Z|\zeta, \varphi|A, e, a, b]} \]

insertion of charges along worldlines \( \gamma_i = \partial f_i \)
The discretised BF action can be written as a worldline action for spinors propagating along the edges of the discretisation.

Gravity is a constrained topological theory (BF+simplicity) — we assume the same holds for the discretised theory and add the discretised simplicity constraints to the worldline model.

We proposed to blew up the one-dimensional edges and replaced them by an oriented three-manifold $\mathcal{M}$. This has a number of advantages: (i) removal of point-like interactions (spinfoam vertices), (ii) same topological structure as in isolated horizon framework, (iii) the spinors couple naturally to a three-dimensional field theory on $\mathcal{M}$. The simplest field theory that we can build out of the background fields $e^\alpha$ and $A^\alpha_\beta$ on $\mathcal{M}$ is an $G = SO(p, q)$ Chern–Simons theory for the de Sitter group.

Integrability conditions fix the gauge group to $SO(2, 3)$ rather than $SO(1, 4)$. AdS/CFT?