Questions & Answers about Perturbative Quantum Gravity

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Seven Questions

1. Why is QGR so bad & CGR so good?
2. Why must we quantize gravity?
3. Why do QFT’s have ∞’s?
4. Why are those of QGR worse?
5. How bad is the problem?
6. What are the main approaches to it?
7. What would we do with QGR?
1. Gravitational Field: $g_{\mu\nu}(t,x)$
   - $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$

2. $g_{\mu\nu}$ affects fields thru Minimal Coupling
   - $-(\partial/\partial ct)^2 + \nabla^2 \rightarrow (-g)^{1/2} \partial_\mu [(-g)^{1/2}g_{\mu\nu}\partial_\nu]$

3. Other fields affect $g_{\mu\nu}$ thru Einstein Eqn
   - $G_{\mu\nu} = 8\pi G T_{\mu\nu}$
   - Key Principle: Energy gravitates

4. Eqn also predicts gravitational radiation
   - Purely grav. DOF’s not fixed by matter
A Solution versus the GENERAL Solution

QM eqns same as in CM, eg \( \ddot{x} + \omega^2 x = 0 \)
- \( x(t) = x_0 \cos(\omega t) + \dot{x}_0/\omega \sin(\omega t) \) BUT
- \( x_0 = 0 \) & \( \dot{x}_0 = 0 \) OK in CM
- \( x_0 = 0 \) & \( \dot{x}_0 = 0 \) not OK in QM

Classic CGR tests have most IVD = 0
- \( ds^2 = -[1-2GM/r]dt^2 + dr^2/[1-2GM/r] + r^2d\Omega \)
- Need \( g_{\mu\nu}(t,x) \) for general IVD
- Lots of DOF’s with 0-pt motion
Some Quantization Unavoidable

**Dual role of force fields:**
- Mediate interactions
- Harbor new quanta

Cf. EM

\[
\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \\
\vec{\nabla} \cdot \vec{B} = 0 \\
\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \frac{1}{\epsilon_0} \frac{\partial \vec{E}}{\partial t} = \vec{J} \\
\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0
\]

\[
\vec{B}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \int_0^t dt' \frac{\sin[ck(t - t')]}{ck\epsilon_0} i\vec{k} \times \tilde{\vec{J}}(t', \vec{k}) \\
+ \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \left\{ \tilde{\vec{B}}_0(\vec{k}) \cos(ckt) - \frac{i}{ck} \bar{\vec{k}} \times \tilde{\vec{E}}_0(\vec{k}) \sin(ckt) \right\}
\]

- 1\textsuperscript{st} term quantized because of matter, whether or not there are photons
Matter quantized whether or not photons & gravitons are!

- Cf Hydrogen: \[ H = \frac{p^2}{2m} - e \Phi(q) \]
  \[ \Phi(q) = \frac{e}{4\pi\epsilon_0 q} \] an operator from q
- Same for General Relativity
  \[ g_{\mu\nu} = (\text{functional of } T_{\mu\nu}) + (\text{gravitons}) \]
  Fields in \( T_{\mu\nu} \) are certainly quantum!
- Allowing quantum matter to interact gravitationally causes problems with or without gravitons
Asymptotic Series Are Your Friends

- Impossible to find general solution ➔ MUST approximate

- Typical Asymptotic series
  \[ E_1(x) = \int_x^\infty \frac{e^{-t}}{t} \, dt \to e^{-x}/x \sum_{n=0}^\infty (-1/x)^n n! \]
  - Great for small 1/x at fixed n
  - But diverges for large n at fixed x
  - Hence use out to n ~ x and no further
  - Not exact but often good enough
Should Be Great for QGR

- **QED:** \[ \alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137} \]
  
  Results = (0th order) \[ 1 + a_1\alpha + a_2\alpha^2 + \ldots \]
  
  - Begins diverging at \( L \approx 430 \)
  
  - Best experiments sensitive to \( L \approx 4 \)

- **QGR:** \[ \kappa = \frac{G\varepsilon^2}{\hbar c^5} \approx \left( \frac{E}{10^{19} \text{Gev}} \right)^2 \]
  
  Results = (0th order) \[ 1 + b_1\kappa + b_2\kappa^2 + \ldots \]
  
  - Same factorials & \( E = 1 \text{ TeV} \) \( \Rightarrow \kappa \approx 10^{-32} \)
  
  - But the coefficients \( b_n \) diverge!
Physics behind the ∞’s: Recall the QM Harmonic Osc.

- $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$
  \[ \Rightarrow q(t) = q_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t) \]

- $\cos(\omega t) = \frac{1}{2} (e^{i\omega t} + e^{-i\omega t})$
  \[ \Rightarrow q(t) = \frac{1}{2}(q_0 + ip_0/m\omega)e^{-i\omega t} + \frac{1}{2}(q_0 - ip_0/m\omega) e^{i\omega t} \]
  \[ = [a e^{-i\omega t} + a^\dagger e^{i\omega t}]/(2\omega)^{1/2} \]

- Mode coordinate: $a = (\omega/2)^{1/2} (q_0 + ip_0/m\omega)$

- Mode function: $\varepsilon(t) = e^{-i\omega t}/(2\omega)^{1/2}$
  \[ \Rightarrow q(t) = a \varepsilon(t) + a^\dagger \varepsilon^*(t) \]
Field Theories have $\infty$ Modes

E.g., EM for $0 \leq x_i \leq L$

\[ E_1(t,x) = \sum_k [a_k \, \varepsilon_k(t,x) + a_k^\dagger \, \varepsilon_k^*(t,x)] \]

where $k = \pi/L \ (n_1,n_2,n_3)$ & $\omega = \pi n c/L$

\[ \varepsilon_k = [\pi^3/2\omega L^3]^{1/2} \sin(k_1 x_1) \cos(k_2 x_2) \cos(k_3 x_3) e^{-i\omega t} \]

\[ a_k = \int_L^L dx_1 \int_L^L dx_2 \int_L^L dx_3 \, \varepsilon_k^*(0,x) \{ \omega \, E_1(0,x) \]

\[ + ic \, [\partial_2 B_3(0,x) - \partial_3 B_2(0,x)] \}

Two modes for every $(n_1,n_2,n_3)$
Conspiracy of Four Principles

1. Continuum Field Theory $\rightarrow \infty$ Modes
2. Q. Mechanics $\rightarrow$ Can’t have $q_0 = p_0 = 0$
   - Each mode has $\frac{1}{2}\hbar \omega$ + interactions
   - Changes shift energies (Casimir & Lamb)
3. General Relativity $\rightarrow$ Energy gravitates
4. Perturbation theory $\rightarrow$ shifts add
   “Too many modes interacting too strongly”
Renormalization 1:
Classical EM in a medium

- **Dumb way:** \( \varepsilon_0 \text{div}(E) = \rho_{free} + \rho_{bnd} \)

- \( \rho_{bnd} = \sum_{atm} q \left[ \delta^3(x - x_{atm}) - \delta^3(x - x_{atm} - \Delta x) \right] \)
  \( \Rightarrow -\Delta \varepsilon \text{div}(E) + O(\Delta x) \)

- **Smart way:**
  \( [\varepsilon_0 + \Delta \varepsilon]\text{div}(E) = \rho_{free} + O(\Delta x) \)
Renormalization 2:
\[ \epsilon_0 \text{ div}(E) = \rho_{\text{exc}} + \rho_{\text{opt}} \text{ in QED} \]

\[ e^+ \text{ w. } \epsilon(k) = [ (\hbar c k)^2 + m^2 c^4 ]^{1/2} \text{ & } e^- \text{ w. } \epsilon(p-k) \]

live \( \Delta t \sim \frac{\hbar}{[\epsilon(k) + \epsilon(p-k)]} \)

They polarize \( e \Delta x \), where
\[
\frac{d}{dt} \left[ \frac{\epsilon}{c^2} \Delta x \right] = eE \Rightarrow \Delta x \sim c^2 \Delta t^2 \ eE/2\epsilon
\]

Sum over \( k \) to get total polarization
\[
\vec{P}(\vec{k}) = \frac{8}{3} \int \frac{d^3k}{(2\pi)^3} \frac{e^2 \hbar^2 c^2 \vec{E}(\vec{k})}{[\epsilon(\vec{k}) + \epsilon(\vec{p} - \vec{k})]^3}
\]

Renormalization: \( \Delta \epsilon(p) = \Delta \epsilon(0) + \text{finite} \)
Renorm 3: $c^4/8\pi G \left[ G_{\mu\nu} + \Lambda g_{\mu\nu} \right] = (T_{\mu\nu})_{\text{exc}} + (T_{\mu\nu})_{\text{0pt}}$

Cf. $-\varepsilon_0 \nabla^2 \Phi = \rho_{\text{exc}} + \rho_{\text{0pt}}$

\[
\left( \rho(\vec{x}) \right)_{\text{opt}} = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}.\vec{x}} \int \frac{d^3k}{(2\pi)^3} \frac{\frac{8}{3} e^2\hbar^2 c^2 p^2 \Phi(\vec{p})}{[\mathcal{E}(\vec{k}) + \mathcal{E}(\vec{p} - \vec{k})]^3}
= A\alpha\varepsilon_0 \ln(K^2)\Phi(\vec{x}) + \text{Finite}
\]

\[
\left( T_{\mu\nu}(\vec{x}) \right)_{\text{opt}} = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}.\vec{x}} \int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{E}^4 h(\vec{p})}{[\mathcal{E}(\vec{k}) + \mathcal{E}(\vec{p} - \vec{k})]^3}
= \hbar c \left\{ AK^4 h(\vec{x}) + BK^2 \nabla^2 h(\vec{x}) + C \ln(K^2) \nabla^4 h(\vec{x}) + \text{Finite} \right\}
\]

$A K^4 \rightarrow \Lambda, B K^2 \rightarrow G$ but $C \ln(K^2)$ new
What’s wrong with higher ∂’s?

- **Newton**: \( m\ddot{x} = F(x,\dot{x}) \)
  - \( L = L(x,\dot{x}) \) \( \Rightarrow \) \( Q = x \) and \( P = \frac{\partial L}{\partial \dot{x}} \)
  - \( H(Q,P) = P\dot{x}(Q,P) - L(Q,\dot{x}(Q,P)) \)

- **Ostrogradsky**: \( \frac{d^4x}{dt^4} = f(x,\dot{x},\ddot{x},\frac{d^3x}{dt^3}) \)
  - \( L = L(x,\dot{x},\ddot{x}) \) \( \Rightarrow \) \( Q_1 = x, \) \( Q_2 = \dot{x}, \) \( P_1 = \frac{\partial L}{\partial \dot{x}}, \) \( P_2 = \frac{\partial L}{\partial \ddot{x}}, \)
  - \( P_1 = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \)
  - \( H(Q_1,Q_2,P_1,P_2) = P_1Q_2 + P_2\ddot{x}(Q_1,Q_2,P_2) - \)
  - \( L(Q_1,Q_2,\ddot{x}(Q_1,Q_2,P_2)) \)

- Why physics is based on 2\(^{nd}\) order eqns!
Perturbative QGR differs

- Other forces couple the same $\forall$ modes
- GR couples more strongly to large $k$ modes
- This requires 4th order counterterms which would make the universe blow up instantly
Divergent Opinions

1. Relativists (love General Relativity)
   - “Perturbation theory is wrong!”
   - Nonlinear grav. ints cancel the ∞’s

2. Particle Theorists (love Pert. Theory)
   - “General Relativity is wrong!”
   - Superstrings have ± 0-point energies and interact more weakly at large k
Repartee with Relativists

**Fact:** GR is weak & QGR unobs. small

**Question:** How can PT be wrong?

1. Correct series nonanalytic in G
   - Eg (0\(^{th}\)) \{1 + GE^2/\hbar c^5 \ln(GE^2/\hbar c^5) + \ldots\}

2. 0\(^{th}\) order may diverge for G \(\rightarrow 0\)
Charged shell of radius $R \to 0$  
(ADM 1960)

- Without GR: $mc^2 = m_0c^2 + q^2/8\pi\varepsilon_0 R$
  "renormalize" with $m_0c^2 = m_{\text{obs}}c^2 - q^2/8\pi\varepsilon_0 R$
- With GR: $mc^2 = m_0c^2 + q^2/8\pi\varepsilon_0 R - Gm^2/2R$

$m = \frac{Rc^2}{G} \left[-1 + \sqrt{1 + \frac{2Gm_0}{Rc^2} + \frac{Gq^2}{4\pi\varepsilon_0 R^2 c^2}}\right] \rightarrow \sqrt{\frac{q^2}{4\pi\varepsilon_0 G}}$

- Perturbative Result:
  ➞ Oscillating series of ever-higher $\infty$'s
All Proposed Fixes Involve $E < 0$

1. Relativist’s dream: gravity regulates
   ➔ Negative grav. int. energy cancels $\infty$’s nonperturbatively

2. Particle Theorist’s dream: superstrings
   ➔ SUSY adds $E > 0$ fermions which contribute $\frac{1}{2}\hbar\omega$

3. Pert. Gravity’s wish: induce higher $\partial$’s
   ➔ $E < 0$ particles
The Other Problem ➔ Experiments!

- Theoretical theory (Wizards dueling)
- Study weak ints using unique features
  - $\mu^- \rightarrow e^- e^+ \nu^\mu \nu_e \nu^\mu$ doesn’t occur in QED
- Unique features of gravity
  - Negative interaction energy
  - M=0 gravitons without conformal invariance
    - Photons can’t distinguish $g_{\mu\nu}(x)$ from $\Omega^2(x) g_{\mu\nu}(x)$
- Cosmology is a natural venue
  - $ds^2 = -dt^2 + a^2(t) dx^2 = a^2 [-d\eta^2 + dx^2]$
  - Gravity is long range and knows about $a(t)$
Seven Questions Answered

1. CGR works when almost all DOF’s off, QGR hurt by 0-point motion of DOF’s
2. Must quantize because part of $g_{\mu\nu}$ from quantum matter
   - This has been seen in CMB anisotropies!
3. QFT $\propto$’s from infinite modes
4. QGR is worse because it couples to energy instead of charge
Seven Questions Answered

5. GR + most matter fails a one loop, pure GR fails at two loops

6. Divergent Opinions
   - Relativists: “GR is right & PT is wrong!”
   - Particle Theorists: “PT is right & GR is wrong!”

7. Phenomenology of QGR exploits unique
   - Negative interaction energy
   - $M = 0$ gravitons w/o conformal invariance