Multisymplectic effective
General Boundary Field Theory

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What is presented, its origin and close relatives

**Discrete spacetime multisymplectic GBFT**
- Geometry of covariant 1st order filed theory
- Decimation for scalar 1st order models
- The action and its variation
- The multisymplectic formula
- Symmetries and conserved quantities
- The canonical framework
- Examples

**Gauge theories and modified BF theories**

**Regularization, coarse graining and continuum limit**
What is presented

“Multisymplectic Effective General Boundary Field Theory”
M. Arjang and J. A. Zapata (to appear)
A continuation of:

- J. A. Zapata (2004),
  - Loop quantization from a LGT perspective–
  - Loop quantization as a continuum limit–
- M. P. Reisenberger (1997, 1994),
  - Classical discrete gauge theory and gravity / S F models–
- R. Oeckl (2003- ),
  - General Boundary Field Theory–
- A. P. Veselov (1988),
  - On discrete time hamiltonian systems–
- J. E. Marsden, G. W. Patrick, S. Shkoller (1998),
  - On discrete spacetime multisymplectic field theory–
Closely related to:

  -Consistent discretizations–

- Gambini, Pullin, Di Bartolo, Campiglia (2006)  
  -Uniform discretizations–

- Dittrich, Bahr, Hoehn (2009)  
  -Variational integrators and improved actions–

- Halvorsen, Sørensen, Christiansen (2011)  
  -Noether’s theorem for spacetime simplicial gauge theory–

- Kogut, Susskind (1975)  
  -Hamiltonian Formulation of Lattice Gauge Theories–

For regularization see

- Calcagni, Oriti, Thurigen (2013)  
  -Structure for discrete Hodge star and Laplacian–
Geometry of covariant 1st order filed theory

Continuum histories are local sections 
\( M \supset U \xrightarrow{\phi} E \) of a bundle with standard fibre \( \mathcal{F} \).

Physical motions are selected by Hamilton’s principle with action

\[
S(\phi) = \int_U L(j^1 \phi), \quad \text{where } j^1 \phi(x) = (x, \phi(x), D\phi(x)).
\]

Its variation can be written as

\[
dS(\phi) \cdot \delta \phi = -\int_U (j^1 \phi)^* (j^1 (\delta \phi) \eta) + \int_{\partial U} (j^1 \phi)^* (j^1 (\delta \phi) \Theta_L)
\]

with \( \Theta_L, \eta \) differential forms on \( J^1 Y \).

If we further define \( \Omega_L = -d\Theta_L \) then

** \( \phi \) solution, \( v, w \) first var. \( \implies \) \( \int_{\partial U} (j^1 \phi)^* (v \eta \wedge \Omega_L) = 0 \)**

* \( S \) \( G \)-inv., \( \xi \in \mathfrak{g} \) gives \( v_\xi \) first var. \( \implies \) \( \int_{\partial U} (j^1 \phi)^* (v_\xi \eta \wedge \Theta_L) = 0 \)*
Decimation for scalar 1st order models

Decimated local record of a history in 1st order format

\[ \nu \xrightarrow{\tilde{\phi}} (\nu, \phi_\nu \in \mathcal{F}, \{\phi_\tau \in \mathcal{F}\}_{\tau \subset \partial \nu}) \]

A variation \( \delta \tilde{\phi}(\nu) = \tilde{v}(\nu) = (v_\nu \in T_{\phi_\nu} \mathcal{F}, \{v_\tau \in T_{\phi_\tau} \mathcal{F}\}_{\tau \subset \partial \nu}) \)
Scalar field’s action and its variation

Our starting point is a variational principle with action

\[ S(\phi) = \sum_{\nu \in U_\Delta} L(\tilde{\phi}(\nu)) \, . \]

Its variation yields: (i) eqs. of motion and (ii) geometric structure

\[ dS(\phi)[\nu] = - \sum_{U-\partial U} \tilde{\phi}^*(\tilde{v} \cdot \hat{\Omega}_L) + \sum_{\partial U} \tilde{\phi}^*(\tilde{v} \cdot \Theta_L) \]

where

\[ \Theta_L(\cdot, \phi(\nu)) = \frac{\partial L}{\partial \phi_T}(\phi(\nu))d\phi_T \, , \]

\[ \hat{\Omega}_L(\cdot, \phi(\nu)) = - \frac{\partial L}{\partial \phi}(\phi(\nu))d\phi_{\nu} - \sum_{\tau \in (\partial \nu)^{n-1}} \frac{\partial L}{\partial \phi_T}(\phi(\nu))d\phi_T \, . \]
The multisymplectic formula

We define

\[ \Omega_L(\tilde{v}(\nu), \tilde{w}(\nu), \tilde{\phi}(\tau_\nu)) = -d(\Theta_L|_{\tilde{\phi}(\tau_\nu)})(\tilde{v}(\nu), \tilde{w}(\nu)). \]

The geometrical structure arises when only solutions and first variations are considered. \(dS(\phi)[v] = \sum_{\partial U} \tilde{\phi}^*(\tilde{v} \llcorner \Theta_L)\) implies \(0 = -ddS = \sum_{\partial U} \tilde{\phi}^*(\Omega_L)\). More precisely,

\[ \sum_{\partial U} \tilde{\phi}^*(\tilde{w} \llcorner \tilde{v} \llcorner \Omega_L) = 0 \]

for any first variations \(v, w\) of any solution \(\phi\). This is the multisymplectic formula.
(Sols[U(∂U)] = \times_i Sols[U(B_i)], \text{diag}(\omega_{L,1}, \omega_{L,m})) is not trivial.

Solutions in U correlate the spaces of the boundary components.

I_U : Sols[U] \rightarrow Sols[U(\partial U)] = \times_i Sols[U(B_i)]

The multisymplectic formula implies

I_U^* \text{diag}(\omega_{L,1}, \ldots, \omega_{L,m}) = 0.

*** In particular, U \approx \Sigma \times [0, 1] \subset M with \partial \Sigma = \emptyset leads to ... ***
Symmetries and conserved quantities

The Lie group $G$ acts on $\mathcal{F}$ and on histories. In 1st order format

$$(g\tilde{\phi})(\nu) = (\nu, g(\phi_{\nu}), \{g(\phi_{\tau})\}_{\tau \subset \partial \nu})$$

$L(g\tilde{\phi}(\nu)) = L(\tilde{\phi}(\nu)) \quad \forall \quad \nu, \phi, g \implies S$ and Sols$[U]$ are $G$ inv.

$\xi \in g$ induces a first variation $\nu_{\xi}$ of any solution $\phi$. Then

$$0 = dS(\phi)[\nu_{\xi}] = \sum_{\partial \tilde{\phi}(U)} \tilde{\nu}_{\xi} \cdot \Theta_L.$$
The canonical framework for $\dim M = 1$

$L(\tilde{q}(\nu)) = L(\nu, q^-, q, q^+) \text{ provides momentum 1-forms}$

$$(q^-, \frac{\partial L}{\partial q^-}(\tilde{q}(\nu)))dq^- \in T^*Q_{\nu^-},$$

$$(q^+, \frac{\partial L}{\partial q^+}(\tilde{q}(\nu)))dq^+ \in T^*Q_{\nu^+}$$

that can be used to define “Legendre transformations”

$f_L^- : Q_{\nu^-} \times Q_{\nu} \rightarrow T^*Q_{\nu^-}$ and $f_L^+ : Q_{\nu} \times Q_{\nu^+} \rightarrow T^*Q_{\nu^+}$.

The relation between the described lagrangian structure and

$(T^*Q, \theta = pdq)$ is simply

$$\Theta_L(\cdot, \tilde{q}(\nu)^-) = -(f_L^-)^*\theta, \quad \Theta_L(\cdot, \tilde{q}(\nu)^+) = (f_L^+)^*\theta.$$

Dynamics arises as evolution in the form of canonical

transformations with $L$ as their generating function [Veselov].
The canonical framework for $\dim \mathcal{M} > 1$

In covariant field theory, $J^1 Y^*$ hosts canonical kinematics $(\Theta, \Omega = -d\Theta)$ and dynamics is placed over it. This canonical str. could be pulled back to the discrete framework. In a sense, this is what we do.

Another way to see it is that we use

- predetermined codim 1 surfaces
- endowed with a notion of vertical variation (constructed from each face of the surface)

...to extract a $1\dim$ system and proceed as in the previous slide.
Example: Particle in euclidean space with a potential

1. \( L(\nu, q^-_\nu, q_\nu, q^+_\nu) = \frac{m}{2} (\frac{q_\nu - q^-_\nu}{a})^2 + \frac{m}{2} (\frac{q^+_\nu - q_\nu}{a})^2 - V(q_\nu) \)

2. \( dS(q)[\nu] = \sum_{\nu \in U_{\text{disc}}} \tilde{v}[L](\bar{q}(\nu)) = \sum_{\nu \in U_{\text{disc}}} \frac{\partial L}{\partial \bar{q}}(\bar{q}(\nu)) \cdot \nu_\nu \\
+ \sum_{\nu, \nu+1 \in U_{\text{disc}}} \left( \frac{\partial L}{\partial \bar{q}^+}(\bar{q}(\nu)) + \frac{\partial L}{\partial \bar{q}^-}(\bar{q}(\nu + 1)) \right) \cdot \nu^+_{\nu} \\
+ \left[ \frac{\partial L}{\partial \bar{q}^-}(\bar{q}(1)) \cdot \nu^-_1 + \frac{\partial L}{\partial \bar{q}^+}(\bar{q}(n)) \cdot \nu^+_n \right] \)

3. \( \Theta_L(\cdot, \bar{q}(\nu)^-) = mg_{AB} \frac{(q_\nu - q^-_\nu)^B}{a} dq^-A, \Theta_L(\cdot, \bar{q}(\nu)^+), \hat{\Omega}_L(\cdot, \bar{q}(\nu)) \)

4. For solutions and first variations:
   (i) ** Conserv. of symplectic structure (indep. of \( V \)) **
   \( \Omega_L(\cdot, \cdot, \bar{q}(\nu)^-) \dot{=} - d(\Theta_L|_{\bar{q}(\nu)^-}) = \Omega_L(\cdot, \cdot, \bar{q}(\nu)^+) \)
   (ii) ** Symmetries lead to conserved quantities **
   \( V = 0 \Rightarrow \text{transl. symm.} \Rightarrow 0 = dS(q)[\nu_\xi] = \tilde{v}_\xi \cdot \Theta_L|_{\partial \bar{q}(\nu)} \)
   \( \Rightarrow \text{conserv. of} \ p \Rightarrow q_1 - q^-_1 = q^+_n - q_n \)

5. \( L \) time indep. \( \not\Rightarrow \) energy conserv.: most systems are chaotic
Example: Veselov's top \( Q = SO(n) \)

1. \( L(\nu, g^-, g, g^+) = \frac{1}{2} \text{Tr}(g^{-1}Ig^-)a + \frac{1}{2} \text{Tr}((g^+)^{-1}Ig^+)a \)
2. Veselov: It has a complete set of first integrals in involution.
3. \( \tilde{v}_\xi(\nu) = (\xi^-_\nu g^-_\nu, \xi^\nu g^\nu, \xi^+\nu g^+\nu) \)
4. \( \Theta_L(\tilde{v}_\xi, \tilde{g}^-_\nu) \doteq \tilde{v}_\xi[L](\tilde{g}(\nu))|_{\nu^-} = -\frac{1}{2} \text{Tr}(\xi^-_\nu g^-_\nu Ig^T_\nu) a \)
5. \( \Omega_L(\tilde{v}_\xi, \tilde{w}_\eta, \tilde{g}^-_\nu) = -\frac{1}{2} \text{Tr}((\xi^-_\nu \eta^- - \eta^-_\nu \xi^-_\nu)g^-_\nu Ig^T_\nu) a \)
6. Conserv. of the symplectic str. (\( q \) solution, \( \tilde{v}_\xi, \tilde{w}_\eta \) first var.)
   \( -\frac{1}{2} \text{Tr}((\xi^-_1 \eta^-_1 - \eta^-_1 \xi^-_1)q^-_1 Iq^T_1) = -\frac{1}{2} \text{Tr}((\xi^-_n \eta^-_n - \eta^-_n \xi^-_n)q^-_n Iq^T_n) \)
7. Conserv. of ang. momentum (\( q \) solution)
   \( m(q) = \frac{1}{2}(q^-_\nu Iq^T_\nu - q^\nu Iq^-_\nu T) = \frac{1}{2}(q^-_\nu Iq^+T_\nu - q^+ Iq^-_\nu) \)
Example: Non linear 2d waves (see Marsden et al)

In a hyper cubical discr. \( L = \sum_c L^c \) with \( L^{++}(\tilde{\phi}(\nu)) = \)

\[
\left\{ \frac{1}{2} \left[ \left( \frac{\phi_0+ - \phi_\nu}{h} \right)^2 - \left( \frac{\phi_1+ - \phi_\nu}{k} \right)^2 \right] + N(\phi_\nu) \right\} \cdot h k , \text{etc}
\]

Even with a non linear term, \( \sum_{\partial U} \tilde{\phi}^*(\Omega_L) = 0. \)
Our framework yields \( \Theta_L, \Omega_L \) (and gluing eqs.) indep. of \( N \), e.g.

\[
\Omega_L(\cdot, \cdot, \tilde{\phi}_{1+\nu}) = -\frac{2h}{k} d\phi_\nu \wedge d\phi_{1+}.
\]
Decimation for gauge theories and modified BF theories

Reisenberger’s discretization:

Record of a history in 1st order format

\[ \nu \xrightarrow{\tilde{A}} (\nu, \{ h_l \}_\nu, \{ k_r \}_\nu) \]

Our record of the curvature at \( \nu \) is

\[ \{ g_{\partial s} = h_{l'}^{-1} \circ k_{r'}^{-1} \circ k_r \circ h_l \}_\nu \]

A variation in 1st order format

\[ \tilde{v}(\nu) = (\{ v_l \in T_{h_l} G \}_\nu, \{ v_r \in T_{k_r} \}_\nu) \]
Gauge field's action and its variation

\[ S(A) = \sum_{\nu \in U_\Delta^n} L(\tilde{A}(\nu)) = \sum_{\nu \in U_\Delta^n} L(\nu, \{ g_{\partial s} \}) \]

\[ dS(A)[\nu] = - \sum_{U - \partial U} \tilde{A}^*(\tilde{v}_\perp \hat{\Omega}_L) + \sum_{\partial U} \tilde{A}^*(\tilde{v}_\perp \Theta_L). \]

The resulting geometric structure is, then,

\[ \Theta_L(\cdot, \tilde{A}(\tau_\nu)) = \sum_{r \subset \tau} \frac{\partial L}{\partial k_r} (\tilde{A}(\nu)) dk_r, \]

\[ \hat{\Omega}_L(\cdot, \tilde{A}(\nu)) = - \sum_{l \subset \nu} \frac{\partial L}{\partial h_l} (\tilde{A}(\nu)) dh_l - \sum_{r \subset \nu} \frac{\partial L}{\partial k_r} (\tilde{A}(\nu)) dk_r. \]
Internal gauge symmetries

Internal gauge transfs. are local symms. of the action. Then:

- For internal gauge transformations over interior points of $U$
  $\implies$ Redundancies of equations of motion.
- For internal gauge transformations over points $C_\tau \in \partial U$
  $\implies$ Constraints on possible boundary data for solutions.
- For internal gauge transformations over points $C_\sigma \in \partial U$
  $\implies$ The dynamical content of boundary data can be captured by the reduced variables $k_r^{-1}k_{r1}$.
Example: Lattice gauge theory without fermions

1. \( S_{\text{Euc}}(A) = \beta \sum_{\nu \subset U} \sum_{s < \nu} [1 - \frac{1}{n} \text{Re } \text{Tr}(g_{\partial s})] \)

2. \( \tilde{\nu}_\xi(\nu) = (\{h_l \xi_l \in T_h G\}_\nu, \{\xi_r k_r \in T_{kr}\}_\nu) \)

3. \( dS(A)[\nu] = - \sum_{U - \partial U} \tilde{A}^* (\tilde{\nu} \bot \hat{\Omega}_L) + \sum_{\partial U} \tilde{A}^* (\tilde{\nu} \bot \Theta_L) \) with

\[
\Theta_L(\tilde{\nu}_\xi, \tilde{A}(\tau_\nu)) = \tilde{\nu}_\xi L(\tilde{A}(\nu))|_\tau = -\frac{\beta}{n} \text{Re } \sum_{r \subset \tau} \text{Tr}(h_{l'}^{-1} k_{r'}^{-1} \xi_r k_r h_l)
\]

\[
\hat{\Omega}_L(\tilde{\nu}, \tilde{A}(\nu)) = \tilde{\nu}_\xi L(\tilde{A}(\nu)) = -\frac{\beta}{n} \text{Re } \sum_{l \subset \nu} \text{Tr}(h_{l'}^{-1} k_{r'}^{-1} k_r h_l \xi_l)
\]

\[
+ \frac{\beta}{n} \text{Re } \sum_{r \subset \partial \nu} \text{Tr}(h_{l'}^{-1} k_{r'}^{-1} \xi_r k_r h_l)
\]

4. \( 0 = -\frac{\beta}{n} \text{Re } \sum_{\partial U} \sum_{r \subset \tau} \text{Tr}(\{\hat{\xi}_r (\eta_l - \eta_{l'}) - \hat{\eta}_r (\xi_l - \xi_{l'})\} g_{\partial s}) \)

where \( \xi_r = k_r h_l \hat{\xi}_r h_l^{-1} k_r^{-1} \)
Example: Reisenberger’s model

\[ S(A, e, \psi) = \sum_{\nu \subset U} \left[ \sum_{s < \nu} e_s \, \text{Tr}(J^i g_{\partial s}) - \frac{1}{60} \sum_{s, \bar{s} < \nu} \psi_{ij} e_s \, e_s \, \text{sgn}(s, \bar{s}) \right] \]

The equations of motion were written by Reisenberger. Notice that at each \( \nu \) the boundary dof are purely connection dof.

** \( \Theta_L = \Theta_{L,BF} ** \) Consider \( \tilde{\nu}_\xi(\nu) \) with \( \{ \nu_{\xi r} = \xi k_r \in T_{kr} SU(2) \} \)

\[ \Theta_L(\tilde{\nu}_\xi, \tilde{\alpha}(\tau_{\nu})) = \tilde{\nu}_\xi L(\tilde{\alpha}(\nu))|_{\tau} = \sum_{r \subset \tau} e_{sr} \, \text{Tr}(J^i h_l^{-1} k_r^{-1} \xi_k k_r h_l) \]

Thus, the multisymplectic formula, \( \sum_{\partial U} \tilde{\alpha}^* \Omega_L = 0 \), is independent of the constraint term.
Regularization

Decimated Histories$_\Delta \overset{i_\Delta}{\rightarrow}$ Histories Continuum

e.g. selecting closest sols. to “the free theory” (if it makes sense)

$$S_\Delta = i_\Delta^* S$$

Reisenberger’s discretization can be obtained by intersecting a ‘prime’ simplicial decomposition with its dual.
Having two intersecting discretizations allows for a notion of **Hodge dual for cochains** that sets up a rich structure for regularization; see [Calcagni et al].
We have this structure plus the **simple gluing** already mentioned.
Coarse graining

Hists. at finer scale $\Delta' \geq \Delta$ are coarse grained decimating further:

$$\phi_{\Delta'} \xrightarrow{\tilde{\pi}} \phi_{\Delta}$$

Hists. and vars. in 1st order format are coarse grained likewise.

Notice the difference with directly working on $T^*Q$ or $J^1Y^*$. 
Coarse graining: Correction of a model at $\Delta$

Coarse graining hists. and vars. from $\Delta' > \Delta$ can be used to import solutions and first variations, but we can do better ...

Measures in the space of histories are also coarse grained. (In QFT and SFT there is a straightforward interp. of coarse gr.) They can be corrected by coarse graining from finer scales. ** A probabilistic perfect measure at scale $\Delta$ is one that agrees with all its corrections. **

(i) $\Delta$-Effective eqs. of motion are not exactly satisfied, and
(ii) cons. laws are not exactly satisfied either. (See next slide.)

In $\dim(M) = 1$ we have access to Hamilton’s function at any scale. In $\dim(M) > 1$ knowledge of bdary. cond. is only partial; thus, we have no access to it.
Continuum limit

Fix a history \( \phi \) and a variation \( \delta \phi \) from the continuum, and let the measuring (decimation) scale get finer

\[
\lim_{\Delta \to M} dS_\Delta(\phi_\Delta) \cdot \delta \phi_\Delta
\]

\( \mapsto \) Eqs. of motion and geo. str. may converge [Veselov, Marsden et al].
This also gives the error at scale \( \Delta \).

– Should we? –

Specify a solution \( \phi \) in the continuum as a sequence \( \{ \phi_\Delta \} \) of approx. solutions to effective eqs. of motion(\( \Delta \)) s.t.
(i) \( \{ \phi_\Delta \} \to \phi \), and (ii) \( dS_\Delta \) converges (see [Gambini et al]),
\( \implies \) Symmetries are a limit of “approx. symmetries(\( \Delta \)” (discussion Dittrich, Rovelli).