From coarse graining for quantum gravity to topological charges in discretized gauge theories

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Strategy of the talk

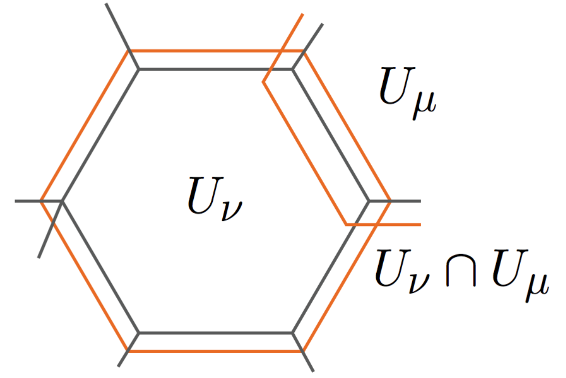
- The case of 2d gravity (2d abelian gauge theory)
- Four dimensional $SU(2)$ gauge theory (relevant for LQG)
- Non-abelian gauge theories in arbitrary dimension

Ideas / subjects presented

- Parallel transport $\leftrightarrow$ bundle and connection (mod. gauge)
- Truncation (real space renormalization):
  LGT truncation + homotopy data
  - Reconstruct bundle + connection
    up to a given accuracy (mod. gauge)
- Exact regularization of topological charges
  - 2d quantum gravity
  - Boundary terms in the presence of horizons in 4d LQG
- Remarks on this "revision" of loop quantization
  - Canonical kinematical $SU(2)$ LQG is unchanged
  - The LQG generalized projector, calculated as a (renormalized) sum over histories, can be refined to include extra local data encoding gluing information.

In collaboration with Claudio Meneses
`arXiv:1701.00775` and ... to appear

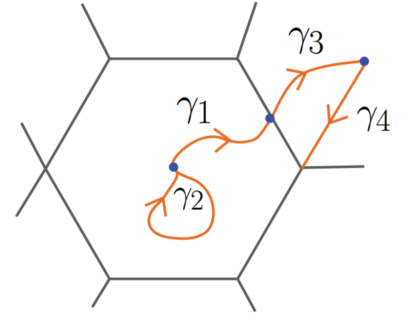
Parallel transport bundles and connections



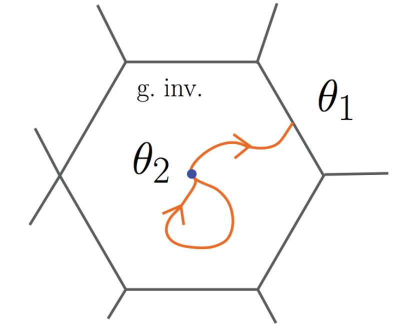
Consider a cellular decomposition $\mathcal{C}$ of the base space M and an open cover covering the closed n dim cells.

Parallel transport bundles and connections

We choose a base point b_ν for each k-cell c_ν (with $\partial c_\nu \approx S^{k-1}$) and consider paths contained in $\bar{c}_\nu$ with end points in any of the base points contained in $\bar{c}_\nu$.



Parallel transport bundles and connections



We consider (i) a parallel transport map[†]

$$\mathrm{PT} : \mathsf{Paths}_\nu \to G, \qquad \text{e.g. } \mathrm{PT}(\gamma_1) = \theta_1 \in SO(2)$$

and (ii) a path system

$\bar{c}_\nu \ni x \mapsto \gamma_\nu(x)$ with $s[\gamma_\nu(x)] = b_\nu, t[\gamma_\nu(x)] = x$ for every k-cell.

Parallel transport bundles and connections

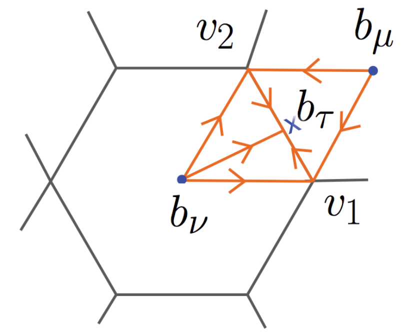
Together (i) and (ii) determine:

- a local trivialization of a G-bundle over each $\bar{c}_\nu$ (or U_ν),
- gluing maps

$$\bar{c}_\nu \supset \bar{c}_\tau \ni x \quad \longmapsto \quad \varphi_{\nu\tau}(x) = \mathrm{PT}((\gamma_\tau(x))^{-1} \circ \gamma_\nu(x))$$

  mapping bdary cells $c_\tau = c_\nu \cap c_\mu$ to G,
- a connection (mod. gauge).

Truncation (real space renormalization) $2d$ $SO(2)$ case



From the gluing functions in the continuum
$\varphi_{\nu\tau} : \bar{c}_\tau \to G = SO(2) \approx S^1$
the truncation records only

- $\varphi_{\nu\tau}(x) \in S^1$ for $x = v_1, x = v_2$ and
- $[\varphi_{\nu\tau}]$ the homotopy class of paths in S^1 with fixed end points.

Remarks on the truncation (2 dim)

- $\{\varphi_{\nu\tau}(v)\}$ can be calculated from LGT data.
- The truncated gauge field (LGT data $+$ homotopy data) determines the transition functions up to homotopy.
  Reconstruct bundle + connection up to a given accuracy.
  *** Exact regularization of topological charges ***.
- If $\dim M = 2$ and $G = SU(2) \approx S^3$
  the homotopy data $\{[\varphi_{\nu\tau}]\}$ is trivial.
  Hom. data about gluing a 2-face to its bdary faces
  is non trivial only if $\pi_1(G)$ is not trivial
  (e.g. $\pi_1(U(1)) = \mathbb{Z}$, $\pi_1(SO(3,1)) = \mathbb{Z}/2\mathbb{Z}$, $\pi_1(U(r)) = \mathbb{Z}$.

Remarks on the truncation (n dim)

LGT data determines the evaluation of gluing functions (gluing a bdle over a k-cell to a bdle over an r-cell in its bdary) $\{\varphi_{\nu\tau}(v)\}$ on a discrete set of points.

- Record the homotopy type of functions gluing bundles over 2-cells to their closed boundary 1-cells

  rel. to fixed end pts. and subject to compat. conditions.

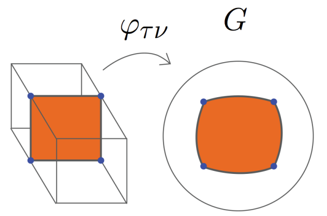
  Characterize the bdle over the 2-skeleton (up to eq.).

- Rec. the h. type of funs. gluing 3-cells to closed bdary 1-cells and bdary 2-cells **relative to fixed pts. and fixed homotopy types of curves and subject to compatibility conditions **:

  cocycle and extendibility conds. (Fig. next slide)

  Characterize the bdle over the 3-skeleton up to equivalence. (However, $\pi_2(G)$ is trivial for every Lie group.)

Remarks on the truncation (n dim)



- Rec. the h. type of funs. gluing 4-cells to closed bdary 1-cells, 2-cells and 3-cells * subject to compat. conds. *
- *** The gluing is "airtight" and known up to homotopy. ***
- For any Lie group $\pi_3(G) = \mathbb{Z}^m$ for some m and, in particular $\pi_3(SU(2)) = \mathbb{Z}$. (cell. dec. of M^4 with two 4-cells.)
- Coarse graining LGT data is naturally done by the pullback of Emb : $L_{\text{coarse}} \to L_{\text{fine}}$.
- Coarse graining hom. data is done by gluing k-surfaces with bdary in G (that we know only up to relative homotopy class). The data needed for this gluing is part of the LGT data.

Exact regularization of topological charges / 2d LQG

$\dim M = 2,\ G = SO(2) \approx S^1$
2d gravity in a spacetime region $U \subset M$
e dyad, A connection 1-form such that $de + A \wedge e = 0$,
$F = dA = f\tau$, $f = d\omega$, $\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$S_U(e, A) = \frac{1}{8\pi G} \int_U \mathrm{sgn}(\det e) f = \frac{\mathrm{sgn}(\det e)}{8\pi G} \int_{\partial U} w$$

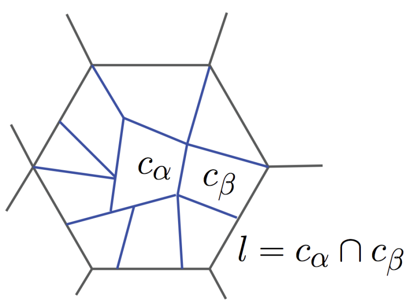
** S_U is inv. under variations of the fields in the interior of U **
Any (e, A) is an extremum of S_U.

Exercise: Loop quantize 2d gravity (S_U is a pure bdary term)

Essential ingredients: (i) $\mathcal{H}^{\mathrm{kin}}_{\Sigma_\alpha}$,
(ii) $\mathcal{A}_{U,\mathcal{C}}$ *space of truncated gauge fields*,

(iii) exact regularization of $\int_U f$, ...

Exact regularization of topological charges 2d, $G = SO(2)$



$$\int_U f = \sum_\alpha \int_{\partial c_\alpha} \omega_\alpha = \sum_{l \subset U^\circ} \int_{l = c_\alpha \cap c_\beta} (\omega_\alpha - \omega_\beta) + \sum_{l \subset \partial U} \int_l \omega_\alpha$$

Each of these integrals is easily calculated from (LGT + homotopy) data

Exact regularization of topological charges 2d, $G = SO(2)$

$$\int_{l=c_\nu \cap c_\mu} (\omega_\nu - \omega_\mu) = \frac{1}{2\pi} \int_{l=c_\nu \cap c_\mu} \psi^*_{\nu\mu} d\theta = \frac{1}{2\pi} \int_{\psi_{\nu\mu}(l) \subset SO(2)} d\theta,$$

where $\psi_{\nu\mu} = \varphi^{-1}_{\mu l} \cdot \varphi_{\nu l} : l \to SO(2)$ are the transition functions. It is determined by the boundary points of $\psi_{\nu\mu}(l)$ and a winding number.
Similarly,

$$\int_l \omega_\nu = \frac{1}{2\pi} \int_l \varphi^*_{\nu l} d\theta$$

is determined by the boundary points of $\varphi_{\nu l}(l)$ and the homotopy type of the curve.

If $U = M$ the collection of open curves $\{\psi_{\nu\mu}(l) \subset SO(2)\}_{l \subset M}$ may be tied together to obtain a single closed curve in $SO(2)$; the integral is determined by its winding number.

Exact regularization of topological charges 4d, $G = SU(2)$

This case involves similar ideas

$$\begin{aligned} \int_M \mathrm{Tr}(F \wedge F) &= \sum_\nu \int_{\partial c_\nu} \mathrm{Tr}(F \wedge A + 1/3 A \wedge A \wedge A) \\ &= \frac{1}{8\pi} \sum_{\tau \subset M} \int_{\psi_{\nu\mu}(\tau = c_\nu \cap c_\mu)} \mathrm{vol}_{SU(2)} \end{aligned}$$

- One important difference is that we can evaluate $\int \mathrm{Tr}(F \wedge F)$ only for closed manifolds and not the contribution from each of its pieces.
- We could also regularize a "boundary action" (add a term only to some components of the spacetime boundary) corresponding to the Chern-Simons form (‡). In the last formula the transition functions would be replaced by the gluing functions $\varphi_{\nu\tau}$.
- We could <u>not</u> give a regularization of the Chern-Simons action in three dimensions.

Remarks on this "revision" of loop quantization

- Canonical kinematical $SU(2)$ LQG is unchanged, but parallel transport data at each scale acquires the interpretation of being a homotopy class of continuum gauge fields.
- The LQG generalized projector, calculated as a (renormalized) sum over histories, can be refined to include extra local data encoding gluing information.
- The partition function (calculated on a closed manifold) would involve a sum of terms corresponding to "monopole contributions"; meaning that the space of gauge fields is composed by a collection of connected components each of which corresponds to connections over inequivalent bundles.
- C-S "boundary terms" relevant in the presence of certain types of horizons in 4d LQG can be regularized exactly.

Thank you for your attention!