From coarse graining for quantum gravity to topological charges in discretized gauge theories

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Strategy of the talk

- The case of 2d gravity (2d abelian gauge theory)
- Four dimensional $SU(2)$ gauge theory (relevant for LQG)
- Non-abelian gauge theories in arbitrary dimension
Ideas / subjects presented

- Parallel transport ↔ bundle and connection (mod. gauge)
- Truncation (real space renormalization):
  - LGT truncation + homotopy data
    - Reconstruct bundle + connection up to a given accuracy (mod. gauge)
- Exact regularization of topological charges
  - 2d quantum gravity
  - Boundary terms in the presence of horizons in 4d LQG
- Remarks on this “revision” of loop quantization
  - Canonical kinematical \( SU(2) \) LQG is unchanged
  - The LQG generalized projector, calculated as a (renormalized) sum over histories, can be refined to include extra local data encoding gluing information.

In collaboration with Claudio Meneses
arXiv:1701.00775 and ... to appear
Consider a cellular decomposition $\mathcal{C}$ of the base space $M$ and an open cover covering the closed $n$ dim cells.
We choose a base point $b_\nu$ for each $k$-cell $c_\nu$ (with $\partial c_\nu \approx S^{k-1}$) and consider paths contained in $\bar{c}_\nu$ with end points in any of the base points contained in $\bar{c}_\nu$. 
Parallel transport bundles and connections

We consider (i) a parallel transport map\(^\dagger\)

\[
\text{PT} : \text{Paths}_\nu \to G, \quad \text{e.g. } \text{PT}(\gamma_1) = \theta_1 \in SO(2)
\]

and (ii) a path system

\[
\bar{c}_\nu \ni x \mapsto \gamma_\nu(x) \quad \text{with } s[\gamma_\nu(x)] = b_\nu, \ t[\gamma_\nu(x)] = x \quad \text{for every } k\text{-cell}.
\]
Together (i) and (ii) determine:

- a local trivialization of a $G$-bundle over each $\bar{c}_\nu$ (or $U_\nu$),
- gluing maps

$$
\bar{c}_\nu \supset \bar{c}_\tau \ni x \quad \mapsto \quad \varphi_{\nu\tau}(x) = \text{PT}((\gamma_\tau(x))^{-1} \circ \gamma_\nu(x))
$$

mapping bdary cells $c_\tau = c_\nu \cap c_\mu$ to $G$,
- a connection (mod. gauge).
Truncation (real space renormalization) $2d$ $SO(2)$ case

From the gluing functions in the continuum

$\varphi_{\nu\tau} : \bar{c}_\tau \rightarrow G = SO(2) \approx S^1$

the truncation records only

$\varphi_{\nu\tau}(x) \in S^1$ for $x = v_1, x = v_2$ and

$[\varphi_{\nu\tau}]$ the homotopy class of paths in $S^1$ with fixed end points.
Remarks on the truncation (2 dim)

- $\{\varphi_{\nu \tau}(v)\}$ can be calculated from LGT data.
- The truncated gauge field (LGT data $+$ homotopy data) determines the transition functions up to homotopy. Reconstruct bundle $+$ connection up to a given accuracy.
- *** Exact regularization of topological charges ***.
- If $\dim M = 2$ and $G = SU(2) \approx S^3$ the homotopy data $\{[\varphi_{\nu \tau}]\}$ is trivial.
- Hom. data about gluing a 2-face to its bdary faces is non trivial only if $\pi_1(G)$ is not trivial (e.g. $\pi_1(U(1)) = \mathbb{Z}$, $\pi_1(SO(3, 1)) = \mathbb{Z}/2\mathbb{Z}$, $\pi_1(U(r)) = \mathbb{Z}$.)
Remarks on the truncation (n dim)

LGT data determines the evaluation of gluing functions 
(gluing a bdle over a \( k \)-cell to a bdle over an \( r \)-cell in its bdary) 
\( \{ \varphi_{\nu \tau}(v) \} \) on a discrete set of points.

- Record the homotopy type of functions gluing bundles over 2-cells to their closed boundary 1-cells
  
  **rel. to fixed end pts. and subject to compat. conditions**.

  Characterize the bdle over the 2-skeleton (up to eq.).

- Rec. the h. type of funs. gluing 3-cells to closed bdary 1-cells and bdary 2-cells **relative to fixed pts. and fixed homotopy types of curves and subject to compatibility conditions**:

  cocycle and extendibility conds. (Fig. next slide)

  Characterize the bdle over the 3-skeleton up to equivalence.

(However, \( \pi_2(G) \) is trivial for every Lie group.)
Remarks on the truncation (n dim)

- Rec. the h. type of funs. gluing 4-cells to closed bdary 1-cells, 2-cells and 3-cells * subject to compat. conds. *
- *** The gluing is “airtight” and known up to homotopy. ***
- For any Lie group $\pi_3(G) = \mathbb{Z}^m$ for some $m$ and, in particular $\pi_3(SU(2)) = \mathbb{Z}$. (cell. dec. of $M^4$ with two 4-cells.)
- Coarse graining LGT data is naturally done by the pullback of $\text{Emb} : L_{\text{coarse}} \rightarrow L_{\text{fine}}$.
- Coarse graining hom. data is done by gluing $k$-surfaces with bdary in $G$ (that we know only up to relative homotopy class). The data needed for this gluing is part of the LGT data.
Exact regularization of topological charges / 2d LQG

\[ \dim M = 2, \ G = SO(2) \approx S^1 \]

2d gravity in a spacetime region \( U \subset M \)

\( e \) dyad, \( A \) connection 1-form such that \( de + A \wedge e = 0 \),

\[ F = dA = f \tau, \ f = d\omega, \ \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

\[ S_U(e, A) = \frac{1}{8\pi G} \int_U \text{sgn}(\det e) f = \frac{\text{sgn}(\det e)}{8\pi G} \int_{\partial U} w \]

** \( S_U \) is inv. under variations of the fields in the interior of \( U \)**

Any \( (e, A) \) is an extremum of \( S_U \).

Exercise: Loop quantize 2d gravity \( (S_U \text{ is a pure bdary term}) \)

Essential ingredients: \( \text{(i) } \mathcal{H}_{\Sigma_\alpha}^{\text{kin}} \),

\( \text{(ii) } \mathcal{A}_{U,C} \text{ *space of truncated gauge fields*} \),

\( \text{(iii) } \text{exact regularization of } \int_U f \text{ , } ... \)
Exact regularization of topological charges 2d, $G = SO(2)$

\[
\int_U f = \sum_{\alpha} \int_{\partial c_\alpha} \omega_\alpha = \sum_{l \subset U^o} \int_{l= c_\alpha \cap c_\beta} (\omega_\alpha - \omega_\beta) + \sum_{l \subset \partial U} \int_l \omega_\alpha
\]

Each of these integrals is easily calculated from (LGT + homotopy) data
Exact regularization of topological charges $2d$, $G = SO(2)$

$$\int_{l=c_\nu \cap c_\mu} (\omega_\nu - \omega_\mu) = \frac{1}{2\pi} \int_{l=c_\nu \cap c_\mu} \psi_{\nu\mu}^* d\theta = \frac{1}{2\pi} \int_{\psi_{\nu\mu}(l) \subset SO(2)} d\theta,$$

where $\psi_{\nu\mu} = \varphi^{-1}_{\mu l} \cdot \varphi_{\nu l} : l \rightarrow SO(2)$ are the transition functions.

It is determined by the boundary points of $\psi_{\nu\mu}(l)$ and a winding number.

Similarly,

$$\int_{l} \omega_\nu = \frac{1}{2\pi} \int_{l} \varphi_{\nu l}^* d\theta$$

is determined by the boundary points of $\varphi_{\nu l}(l)$ and the homotopy type of the curve.

If $U = M$ the collection of open curves $\{\psi_{\nu\mu}(l) \subset SO(2)\}_{l \subset M}$ may be tied together to obtain a single closed curve in $SO(2)$; the integral is determined by its winding number.
Exact regularization of topological charges $4d, \ G = SU(2)$

This case involves similar ideas

$$\int_M \text{Tr}(F \wedge F) = \sum_{\nu} \int_{\partial c_\nu} \text{Tr}(F \wedge A + 1/3 A \wedge A \wedge A)$$

$$= \frac{1}{8\pi} \sum_{\tau \subset M} \int_{\psi_{\nu\mu}(\tau = c_\nu \cap c_\mu)} \text{vol}_{SU(2)}$$

- One important difference is that we can evaluate $\int \text{Tr}(F \wedge F)$ only for closed manifolds and not the contribution from each of its pieces.
- We could also regularize a “boundary action” (add a term only to some components of the spacetime boundary) corresponding to the Chern-Simons form (‡). In the last formula the transition functions would be replaced by the gluing functions $\varphi_{\nu\tau}$.
- We could not give a regularization of the Chern-Simons action in three dimensions.
Remarks on this “revision” of loop quantization

- Canonical kinematical $SU(2)$ LQG is unchanged, but parallel transport data at each scale acquires the interpretation of being a homotopy class of continuum gauge fields.
- The LQG generalized projector, calculated as a (renormalized) sum over histories, can be refined to include extra local data encoding gluing information.
- The partition function (calculated on a closed manifold) would involve a sum of terms corresponding to “monopole contributions”; meaning that the space of gauge fields is composed by a collection of connected components each of which corresponds to connections over inequivalent bundles.
- C-S “boundary terms” relevant in the presence of certain types of horizons in 4d LQG can be regularized exactly.
Thank you for your attention!