Some analytical results of the Hamiltonian operator in LQG

Cong Zhang

January 22, 2018

Outline

Deparametrized model of LQG coupled to scalar field

General works about the physical Hamiltonian

Restriction on a special case

A toy model of LQG cosmology

Conclusion and outlook

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- Quantize this system.

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What I do is to study this (physical) Hamiltonian operator.

General works about the physical Hamiltonian

- Classical expression of $\sqrt{|\det E(x)|}C^{\mathrm{gr}}(x)$

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\begin{aligned}
&-\sqrt{|\det E(x)|}C^{\mathrm{gr}}(x)\\
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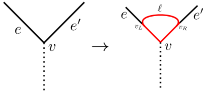
  - It is possible to start from the simplest case of 2-valent graph.

General works about the physical Hamiltonian

- The chosen loop to quantize F_{ab} (Thiemann, 2007, Yang and Ma, 2015)



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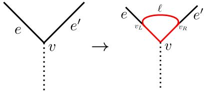

- Expression of the Hamiltonian operator(Alesci et al., 2015)

$$\hat{H} = \frac{1}{\sqrt{16\pi G\beta^2}} \sum_{v\in V(\gamma)} \sqrt{\sum_{e,e' \text{ at } v} \epsilon(e,e')(H^E_{v,ee'} + (H^E_{v,ee'})^\dagger) + (1+\beta^2)H^L_{v,ee'}}$$

$$H^E_{v,ee'} = \epsilon_{ijk}\mathrm{Tr}^{(l)}(h_{\alpha_{ee'}}\tau^i)J^j_{v,e}J^k_{v,e'}$$

$$H^L_{ee'} := \sqrt{\delta_{ii'}\left(\epsilon_{ijk}J^j_{v,e}J^k_{v,e'}\right)\left(\epsilon_{i'j'k'}J^{j'}_{v,e}J^{k'}_{v,e'}\right)}\left(\frac{2\pi}{\alpha} - \pi + \arccos\left[\frac{\delta_{kl}J^k_{v,e}J^l_{v,e'}}{\sqrt{\delta_{kk'}J^k_{v,e}J^{k'}_{v,e}}\sqrt{\delta_{kk'}J^k_{v,e'}J^{k'}_{v,e'}}}\right]\right)$$

General works about the physical Hamiltonian

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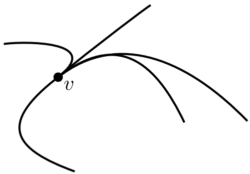
- Degenerate vertex: 

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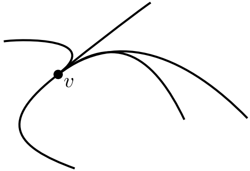
- Degenerate vertex: 

- $V_{\mathrm{nd}}(\gamma)$: non-degenerate vertices.
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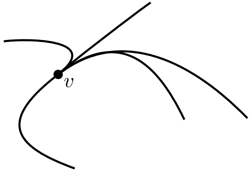
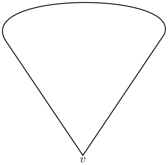
- Degenerate vertex: 

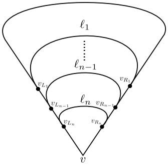
- $V_{\mathrm{nd}}(\gamma)$: non-degenerate vertices.
  $\mathrm{Diff}_{\mathrm{V_{nd}}}$:diffeomorphisms preserving V_{nd}.
  $\mathrm{Diff}(\gamma)_{\mathrm{Tr}}$: diffeomorphisms acting trivially on γ
- Physical state from $|\psi_\gamma\rangle$

$$(\psi_\gamma| = \mathcal{N}_\gamma \sum_{\phi\in \mathrm{Diff}_{\mathrm{V_{nd}}}/\mathrm{Diff}(\gamma)_{\mathrm{Tr}}} U_\phi|\psi_\gamma\rangle := \eta|\psi\rangle$$

Restriction on a special case

- **The graphs**:

$(H^E)^n : \gamma_0 :=$ 

$\to$ 

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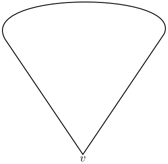

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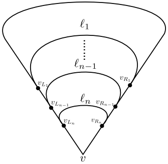

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- **The Hilbert space**: Spin networks on γ_n.

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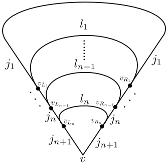
- Spin networks on γ_n: $|\gamma_n, \vec{j}, \vec{l}\,\rangle \sim$ 

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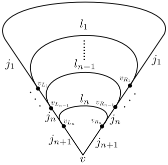
- Spin networks on γ_n: $|\gamma_n, \vec{j}, \vec{l}\,\rangle \sim$ 

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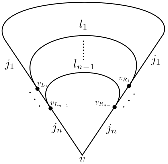
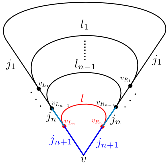
$\epsilon_{ijk}\mathrm{Tr}^{(l)}(h_{\alpha_{ee'}}, \tau^i) J^j_{v,e} J^k_{v,e'}$: 

$\rightarrow \sum_{j_{n+1}} C_{j_{n+1}}$  .

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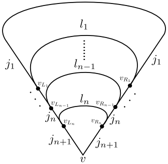
- Spin networks on γ_n: $|\gamma_n, \vec{j}, \vec{l}\,\rangle \sim$ 

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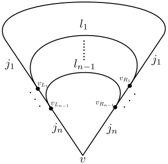
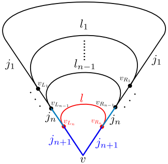
$$\epsilon_{ijk}\mathrm{Tr}^{(l)}(h_{\alpha_{ee'}}\tau^i)J^j_{v,e}J^k_{v,e'}:$$ 

$$\rightarrow \sum_{j_{n+1}} C_{j_{n+1}}$$ 

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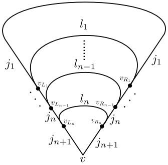
- Spin networks on γ_n: $|\gamma_n, \vec{j}, \vec{l}\,\rangle \sim$ 

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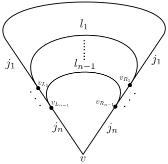
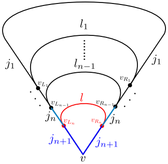
$\epsilon_{ijk}\mathrm{Tr}^{(l)}(h_{\alpha_{ee'}}\tau^i)J^j_{v,e}J^k_{v,e'}$: 

$\to \sum_{j_{n+1}} C_{j_{n+1}}$ 

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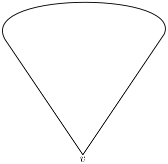
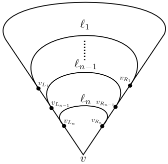
$(H^E)^n : \gamma_0 :=$  $\rightarrow$  $=: \gamma_n$

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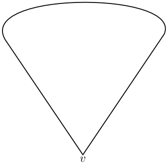
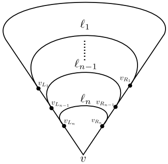
$(H^E)^n : \gamma_0 :=$ 

$\rightarrow$ 

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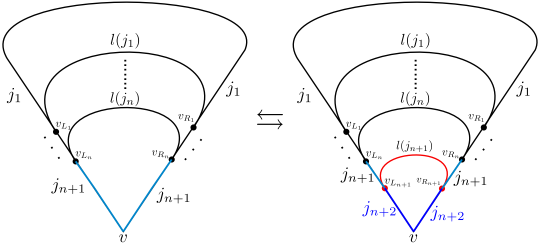

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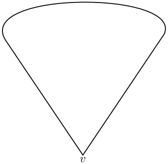

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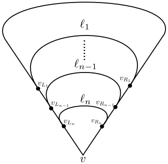

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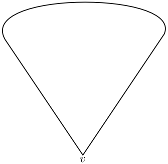
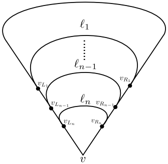
$(H^E)^n : \gamma_0 :=$ 

$\rightarrow$ 

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- **Operator**: Hamiltonian operator restricted on the Hilbert space, $H|_{\mathcal{H}_{\rm phy}}$.
- **Question**: Self-adjointness of the restricted H.

Self-adjointness of the Hamiltonian operator

Theorem

Let N be a self-adjoint operator with $N \geq 1$. Let H be a symmetric operator with domain D which is a core for N. Suppose that:

i *For some c and all $\psi \in D$,*

$$||H\psi|| \leq c||N\psi||.$$

ii *For some d and all $\psi \in D$,*

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We can choose the operator N, diagonalized under the basis, that is $\left([\gamma_n], \vec{j}\right| N = \left([\gamma_n], \vec{j}\right| N(j_{n+1})$, such that $N(j) \cong j^n$ with $n \geq 1$.

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- Turn to the graph preserving version.
  - Consider the coherent state peaking at the cosmology phase space.
  - Calculate the semiclassical dynamics with the coherent state as the initial data.

The heat kernel coherent state

- Classical phase space of the cosmology phase space:

$$A_a^i = cV_o^{-1/3}\mathring{\omega}_a^i, E_i^a = pV_0^{-2/3}\sqrt{q_o}\mathring{e}_i^a$$

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- Coherent state for $p \gg 1$ (Bahr and Thiemann, 2009, Bianchi et al., 2010):

$$\Psi_{c,p}(\vec{g}\ ) \cong \prod_{e\in E(\gamma)}\left(\sum_{j_e}(2j_e+1)e^{-tj_e(j_e+1)+\nu pj_e-i\mu cj_e}D^{j_e}_{j_e,j_e}(n_e^{-1}g_e n_e)\right)$$

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- The factor $e^{-tj(j+1)+\nu pj-i\mu cj}$ is a Gaussian function on j peaking at $j_0 \cong \frac{\nu p}{2t}$, so we focus on large j limit when calculating.

Action of holonomy and flux operators on $\mathcal{H}_{\cos}$

- For the holonomy operator:

$$
\begin{aligned}
& h_e^{1/2} D^{j_e}_{n_e n_e}(g) \\
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- The Hilbert space $\mathcal{H}_{\cos}$ is preserved approximately for large j.

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- For large j,

$$H_{v,e} \cong i \sqrt{j_e} \frac{d}{dj_e} \sqrt{j_e} =: H^c_e$$

which is a self-adjoint in the Hilbert space $L^2(\mathbb{R}^+)$ with "eigenvector" $\varphi_\omega(j_e) = \frac{e^{-i\omega \ln(j_e)}}{\sqrt{j_e}}$.

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- Does the evolution operator e^{iHt} preserve this condition?

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- The Hamiltonian operator $H = \sum_v \sqrt{\sum_{e,e'} H^E_{v,ee'}} \to H = \sqrt{|\sum_{ee'} H^c_e H^c_{e'}|} = \sqrt{\sum_{ee'} \sqrt{|x_e x_{e'}} \frac{\partial^2}{\partial x_e \partial x_{e'}} \sqrt{x_e x_{e'}}|}$

Semiclassical analysis:

- Coherent state $\Psi(\vec{x}) = \prod_{e\in\gamma} \psi(x_e)$

$$\psi(x) := \int_{-\infty}^{\infty} d\omega e^{-\frac{(\omega-\omega_0)^2}{2\sigma^2}+i\xi_0\omega}\phi_\omega(x) = \sqrt{\frac{\sigma^2}{x}}e^{-\frac{\sigma^2}{2}(\xi_0-\ln(x))^2-i\omega_0(\xi_0-\ln(x))}$$

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- The trajectory where the state peaks:

$$\ln x_i - \xi_0 = \frac{N-1}{2\sqrt{C_N^2}}\tau$$

Conclusion and outlook

- Semiclassically, the quantum dynamic gives us an expanding universe.
- The peak satisfies the Minkowski condition.
- Future works: the quantum phenomenon, generalization to general graph, same problem with graph changing Hamiltonian......

Thanks!