The S-matrix of the Faddeev-Reshetikhin Model

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Outline

- Introduction
  - AdS/CFT Correspondence $\surd$
  - Integrability in gauge theory $\surd$
- Strings in $R \times S^3$
- The Faddeed-Reshetikhin Model
- Hamiltonian diagonalization
- Conclusions

AdS/CFT Correspondence

- String theory in $AdS_5 \times S^5$ is supposed to be dual to $\mathcal{N} = 4$ super-YM theory. This conjecture can be extended to other situations as well.
- To prove the correspondence we should be able to build all correlation functions of the gauge theory and all states of string theory and show that they are the same. Very hard!
- However, in many cases it is possible to show that some sectors of both theories do indeed coincide.
- In the last few years integrable structures have been found on both sides of the correspondence. It provides one more step for a proof of the correspondence!

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- However, in many cases it is possible to show that some sectors of both theories do indeed coincide.
- In the last few years integrable structures have been found on both sides of the correspondence. It provides one more step for a proof of the correspondence!
- The classical string action in $AdS_5 \times S^5$ is integrable: there is an infinite set of nonlocal currents (Benna, Polchinski and Roiban, hep-th/0305116). Quantum integrability is an open issue!
- The anomalous dimension of gauge invariant operators of the gauge theory satisfy equations similar to those of a spin chain (Minaham and Zarembo, hep-th/0212208).
- The spectrum of the anomalous dimension corresponds to the energy spectrum of the string. This has been verified in several limits where they can be compared.

Strings in $R \times S^3$

- String action in $AdS_5 \times S^5$

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma (G^{(AdS_5)}_{\mu\nu} \partial^\alpha X^\mu \partial_\alpha X^\nu + G^{(S^5)}_{ab} \partial^\alpha Y^a \partial_\alpha Y^b) + \dots$$

$$-(X^0)^2 + (X^1)^2 + \cdots + (X^4)^2 - (X^5)^2 = R^2$$

$$(Y^1)^2 + \cdots + (Y^6)^2 = R^2$$

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- To look for integrability reduce $AdS_5 \times S^5$ to $R \times S^3$.
- Reduction to $R \times S^3$: $X^0 \neq 0$ and $Y^5 = Y^6 = 0$
- Virassoro constraints

$$\Lambda_{\alpha\beta} = \delta_{ab} \partial_\alpha Y^a \partial_\beta Y^b - \partial_\alpha X^0 \partial_\beta X^0, \quad a, b = 1 \dots 4$$

$$\Lambda_{\alpha\beta} + \frac{1}{2} \eta_{\alpha\beta} \Lambda^\gamma_\gamma = 0$$

Strings $R \times S^3$

- Make use of classical integrability: define the $SU(2)$ map:

$$g = \begin{pmatrix} Y^1 + iY^2 & Y^3 + iY^4 \\ -Y^3 + iY^4 & Y^1 - iY^2 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

  with $\det g = 1$
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- Virassoro constraints: $\vec{S}_\pm^2 = 1$
- Eqs. of motion: $\partial_\pm S^i_\mp \pm \kappa \epsilon^{ijk} S^j_- S^k_+ = 0$ (Faddeev-Reshetikhin model, 1986)

Strings in $R \times S^3$

- These eqs. of motion can be obtained from the action

$$S = \int d^2x[C_+(\vec{S}_-) + C_-(\vec{S}_+) + \frac{\kappa}{2}\vec{S}_+ \cdot \vec{S}_-]$$

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$$S = \int d^2x [\bar{\phi} i\gamma^\mu D_\mu \phi - m\bar{\phi}\phi - g(\bar{\phi}\gamma^\mu\phi)^2 + \mathcal{O}(\phi^6)], \quad \phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}$$

  where $D_0 = \partial_0 - im - ig\bar{\phi}\phi$, and $D_1 = \partial_1$; $m = \kappa$ and $g = \kappa/2$

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- Quantization:
  - Inverse scattering method (Faddeev and Reshetikhin, 1986).
  - Field theoretical methods (Klose and Zarembo, hep-th/0603039).
  - Hamiltonian diagonalization (Das, Melikyan and VOR, arXiv:0707.0511).

Diagonalization of the Hamiltonian

- To get the 2-particle S-matrix we need terms up to ϕ^4 in the action.
- The Hamiltonian is then:

$$H = \int dx[-\overline{\phi} i\gamma^1 \partial_x \phi + m\overline{\phi}\phi + g(\overline{\phi}\gamma^0\phi\ \overline{\phi}\phi - (\overline{\phi}\gamma^\mu\phi)^2)]$$

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$$[\phi_\alpha(x), \phi^\dagger_\beta(y)] = \delta_{\alpha\beta}\delta(x-y)$$

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$$|k_1, k_2 >= \int dx_1\ dx_2\ \chi_{\alpha\beta}(x_1, x_2, k_1, k_2)\ \phi^\dagger_\alpha(x_1)\phi^\dagger_\alpha(x_2)|0>$$

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- The free theory has the (positive energy) eigenfunctions

$$\chi^{(\pm)}_{\alpha\beta}(x_1, x_2, k_1, k_2) = e^{i(k_1x_1+k_2x_2)}u_\alpha(k_1)u_\beta(k_2) \pm e^{i(k_1x_2+k_2x_1)}u_\alpha(k_2)u_\beta(k_1)$$

  where $u(k)$ are the usual 2-component solutions of the momentum space Dirac equation $(\not{k} - m)u(k) = 0$ with positive energy

$$u(k) = \sqrt{m}\begin{pmatrix} e^{\beta/2} \\ e^{-\beta/2} \end{pmatrix} \qquad v(k) = \sqrt{m}\begin{pmatrix} -e^{\beta/2} \\ e^{-\beta/2} \end{pmatrix} \tag{1}$$

  where β is the rapidity $k = m\sinh\beta$, $E = m\cosh\beta$.

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- $\chi^{(\pm)}_{\alpha\beta}$ are eigenfunctions of the momentum with eigenvalue $k_1 + k_2$ and eigenfunctions of H with eigenvalue $E = \sqrt{m^2 + k_1^2} + \sqrt{m^2 + k_2^2}$

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$$\chi_{\alpha\beta}(x_1, x_2) = \chi_{\alpha\beta}^{(+)}(x_1, x_2) + \lambda(k_1, k_2)\ \epsilon(x_1 - x_2)\ \chi_{\alpha\beta}^{(-)}(x_1, x_2)$$

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- Another way of viewing this is by imposing matching conditions for the discontinuity due to the delta interaction. The result is the same!
- Then, a diagonalizable H leads to a S-matrix but having a S-matrix does not automatically imply that H is diagonalizable!

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- Therefore, in the presence of a Lorentz violating term, the Hamiltonian can not be diagonalized: the kinetic term is relativistic and positive and negative energy states must all be regarded. We have project out only the relevant sector.
- If the potential is relativistic this is already taken into account and the Hamiltonian is diagonalizable.

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- The Hamiltonian is not hermitian for real β! But the S-matrix is unitary!

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$$\begin{aligned} P: \quad & \phi(x,t) \rightarrow \eta_P \gamma^0 \phi(-x,t) \\ T: \quad & \phi(x,t) \rightarrow \eta_T C \gamma^0 \gamma^1 \phi(x,-t) \end{aligned}$$

and the Hamiltonian is PT symmetric if α and β are real!

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  and the Hamiltonian is PT symmetric if α and β are real!
- This is possibly the first example of an integrable field theory which has a non-hermitian Hamiltonian, is PT symmetric, and has a unitary S-matrix!

Conclusions

- The S-matrix for the FR model can be obtained by the inverse scattering method or by a field theoretic calculation.
- Its Hamiltonian is not diagonalizable because the potential breaks the Lorentz invariance of the kinetic term.
- This can also be shown using matching conditions for the discontinuity of the delta potential.
- Taking matrix elements of positive energy states allow the computation of the S-matrix.
- Diagonalization of the Hamiltonian leads to the S-matrix but having the S-matrix does not mean that the Hamiltonian is diagonalizable.
- The most general quartic potential which leads to a diagonalizable Hamiltonian was found.
- It may not be hermitian but it is PT symmetric leading to a unitary S-matrix.