Multiloop Gluon Amplitudes

and

AdS/CFT

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Snapshot

In this talk I will be concerned with a particular theory of quantum gravity called $\mathcal{N} = 4$ supersymmetric Yang-Mills theory that has received a large amount of attention over the past decade.

A great deal of the most recent interest stems from evidence that this theory might in fact be solvable (in a certain limited sense that I will make more precise below).

This theory contains all of the interesting physics (including black hole production and decay) that is absent from low-dimensional solvable toy models.

Introduction: Yang-Mills Theory

Of course, other than quantum gravity there are also several other reasons to be interested in YM theory.

- The unique theory of interacting vector bosons
- A great deal of interesting mathematics
- Not to mention, of course, QCD and the ‘Real World’!

The journey towards an analytic solution of this important and rich theory has been long and profitable.

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- And, of course, QCD and the 'Real World'!

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Like in many areas of physics, if we can't solve the theory we're most interested in, we look for a simpler, similar model that we can solve!

This leads us to consider the $\mathcal{N} = 4$ supersymmetric version of the theory, which has even richer mathematical structure.

Roadmap to the Lamppost

$$\begin{array}{c} \text{QCD} \\ \downarrow \\ \mathcal{N}=4 \end{array} \qquad = \qquad \text{strings in } AdS_5 \times S^5$$

Review: $\mathcal{N} = 4$ Yang-Mills

Field content (all in the adjoint reprentation)

- 1 Gauge field A_μ
- 4 Weyl fermions χ^a
- 3 Complex Scalar Fields X, Y, Z

The Lagrangian is essentially unique; it depends only on

- The choice of a gauge group; we choose $SU(N)$
- The coupling constant $g_{\rm YM}$ (a true parameter)

The theory simplifies dramatically in the ‘t Hooft large-N limit

- $\lambda = g_{\rm YM}^2 N$ controls the strength of quantum corrections
- $1/N$ controls the strength of non-planar diagrams

Roadmap to the Lamppost

$$\begin{array}{ccc} \text{QCD} & & \\ \downarrow & & \\ \mathcal{N}=4 & = & \text{strings in } AdS_5 \times S^5 \\ \downarrow & & \\ \text{planar limit } N \to \infty & = & \text{classical strings} \end{array}$$

What does it mean to ‘solve’ planar $\mathcal{N} = 4$ theory?

solve /'sälv/*verb*

To solve planar $\mathcal{N}=4$ Yang-Mills means to determine all correlation functions of gauge invariant operators.

In the planar limit it is sufficient to restrict our attention to single-trace operators. A general gauge-invariant operator is then a trace of the elementary fields and their covariant derivatives

$$\mathcal{O} = \mathrm{Tr}[F^{\mu\nu}\, D_\mu \chi\, X\, \overline{Z}\, F_{\nu\rho}\, D^\sigma\, Y\, D_\sigma \overline{\chi}\, D^\rho Z]$$

and correlation functions are, in general, complicated functions of λ and the positions of the operators

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\cdots\rangle = F_{ij\ldots}(\lambda; x_1, x_2, \cdots)$$

This problem is still too hard in general, so we start by looking just at two-point functions.

The Dilatation Operator

Conformal invariance requires any two-point function take the form

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{c}{|x-y|^{2\Delta}}$$

where the number Δ depends on the operator $\mathcal{O}$ under consideration and is, in general, a complicated function of λ.

The collection of numbers $\Delta(\lambda)$ appearing in $\mathcal{N}=4$ are the eigenvalues of the dilatation operator, which generates scale transformations on $\mathbb{R}^4$.

The dilatation operator is identified, via radial quantization, with the Hamiltonian of the theory on $\mathbb{R}\times S^3$.

Determining all $\Delta(\lambda)$ $\Leftrightarrow$ Finding the spectrum of the theory on S^3

Roadmap to the Lamppost

QCD		
$\downarrow$		
$\mathcal{N} = 4$	$=$	strings in $AdS_5 \times S^5$
$\downarrow$		
planar limit $N \to \infty$	$=$	classical strings
$\downarrow$		
two point functions	$=$	string spectrum
(spectrum on S^3)		(in global time)

Integrability

The current ‘paradigm’ is that all $\Delta(\lambda)$ should be determinable by integrability; a technology familiar from the study of spin chains.

It is instructive to think of a single-trace operator as a spin chain configuration [Minahan & Zarembo]

$$\mathcal{O} = \mathrm{Tr}[F^{\mu\nu}\, D_\mu \chi\, X\, \overline{Z}\, F_{\nu\rho}\, D^\sigma\, Y\, D_\sigma \overline{\chi}\, D^\rho Z]$$

where the different fields (and their derivatives) are the different ‘directions’ in which each ‘spin vector’ can point.

Under radial evolution (generated by the dilatation operator), the spin vectors on different sites can ‘interact’ with each other.

Planar L-loop Feynman diagrams give rise to interactions between spins up to L sites apart from each other.

The S-matrix

An important property of integrable theories is that they can be solved (i.e., the eigenvalues of the Hamiltonian can be found) once we know the 2-particle S-matrix.

In order to set up this scattering problem we choose a ‘ferromagnetic’ ground state

$$\cdots ZZZZZZZZZ \cdots \quad = \quad Z^J$$

This corresponds to a BPS operator whose dimension $\Delta = J$ is protected from quantum corrections. Furthermore we take the infinite volume limit $J \to \infty$ so that we can set up initial and final states of the elementary excitations (‘magnons’) of the spin chain.

[Berenstein, Maldacena, Nastase; Staudacher, Beisert; Hofman, Maldacena, and many others...]

Roadmap to the Lamppost

QCD		
$\downarrow$		
$\mathcal{N}=4$	$=$	strings in $AdS_5 \times S^5$
$\downarrow$		$\downarrow$
planar limit $N \to \infty$	$=$	classical string theory
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operators with infinite J and finite $\Delta - J$	$=$	certain long strings

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Finally, this problem is not only (probably) solvable, but (possibly) has already been solved!

The Worldsheet Guessmatrix of SYM

Beisert showed that the matrix structure of the 2-particle S-matrix is completely fixed by supersymmetry.

The only remaining freedom is an overall phase factor

$$S_0(\lambda; p_1, p_2)$$

which depends on the momenta of the two magnons but not on their species.

A conjecture for this phase factor, which satisfies many desirable properties, has been proposed. [Beisert, Eden, Hernandez, Lopez, Staudacher].

Motivation

The motivation for our work was two-fold

- To unlock previously hidden mathematical richness lurking deep inside multi-loop gluon amplitudes in $\mathcal{N} = 4$ SYM, and
- To exploit that structure to help dramatically simplify the horrendous calculations necessary to check the conjectured 'guessmatrix'.

Target: The Cusp Anomalous Dimension

The cusp anomalous dimension

$$f(\lambda) = \sum_{L=1}^{\infty} f^{(L)}(\lambda) = 4\lambda - 4\zeta(2)\lambda^2 + (4\zeta(2)^2 + 12\zeta(4))\lambda^3 + \mathcal{O}(\lambda^4)$$

governs the behavior of twist-two operators in the limit of very large spin:

$$\Delta\left(\mathrm{Tr}[ZD^S Z]\right) = S + f(\lambda)\log S + \mathcal{O}(S^0), \qquad S \gg 1.$$

This quantity has long played an important role in quantitative checks of AdS/CFT [Gubser, Klebanov, Polyakov].

Target: The Cusp Anomalous Dimension

The BEHLS guessmatrix implies a (complicated!) integral equation for $f(\lambda)$

At strong coupling the yields results consistent with known results from AdS/CFT

$$f(\lambda) = 4\lambda - \frac{3\log 2}{\pi} + \mathcal{O}(1/\lambda)$$

At weak coupling, the perturbative solution leads to the four-loop prediction

$$f^{(4)} = -(4\zeta(2)^3 + 24\zeta(2)\zeta(4) + 50\zeta(6) + 4\zeta(3)^2)$$

The rest of the talk will be devoted to a new method for calculating this quantity from four-particle scattering amplitudes in $\mathcal{N} = 4$ YM.

The Spacetime S-matrix of SYM

For the rest of the talk I will change direction and talk about the 'other' S-matrix of $\mathcal{N} = 4$ SYM—namely, the spacetime S-matrix, which describes the scattering of gluons in four dimensions!

Feynman diagrams are not the most efficient way to calculate scattering amplitudes: too messy, too many terms, hide structure.

Much interest in and progress on the calculation of tree-level amplitudes was stimulated by twistor sring theory. [Witten]

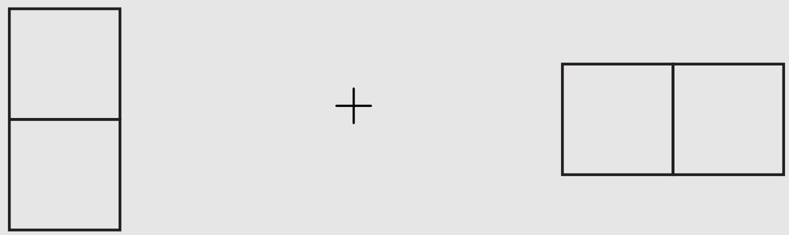
In fact, within a period of less than two years, the problem of calculating closed-form expressions for tree-level scattering amplitudes went from possible only in certain special cases to essentially completely solved.

Many groups, including [Cachazo, Svrcek, Witten] [Britto, Cachazo, Feng, Witten] [Roiban, MS, Volovich] [Brandhuber, Spence, Travaglini] [Dixon, Glover, Khoze] [Bern, Dixon, Kosower] [Badger, Glover, Khoze]

The Spacetime S-matrix of SYM

Progress on loop amplitudes has been steady but less swift.

The first step in calculating an L-loop amplitude is to express it in terms of a (hopefully) small number of relatively simple scalar integrals. For example, the two-loop four-particle amplitude is given by



Unitarity-Based Methods

In general it is very difficult to determine which integrals contribute to any particular amplitude.

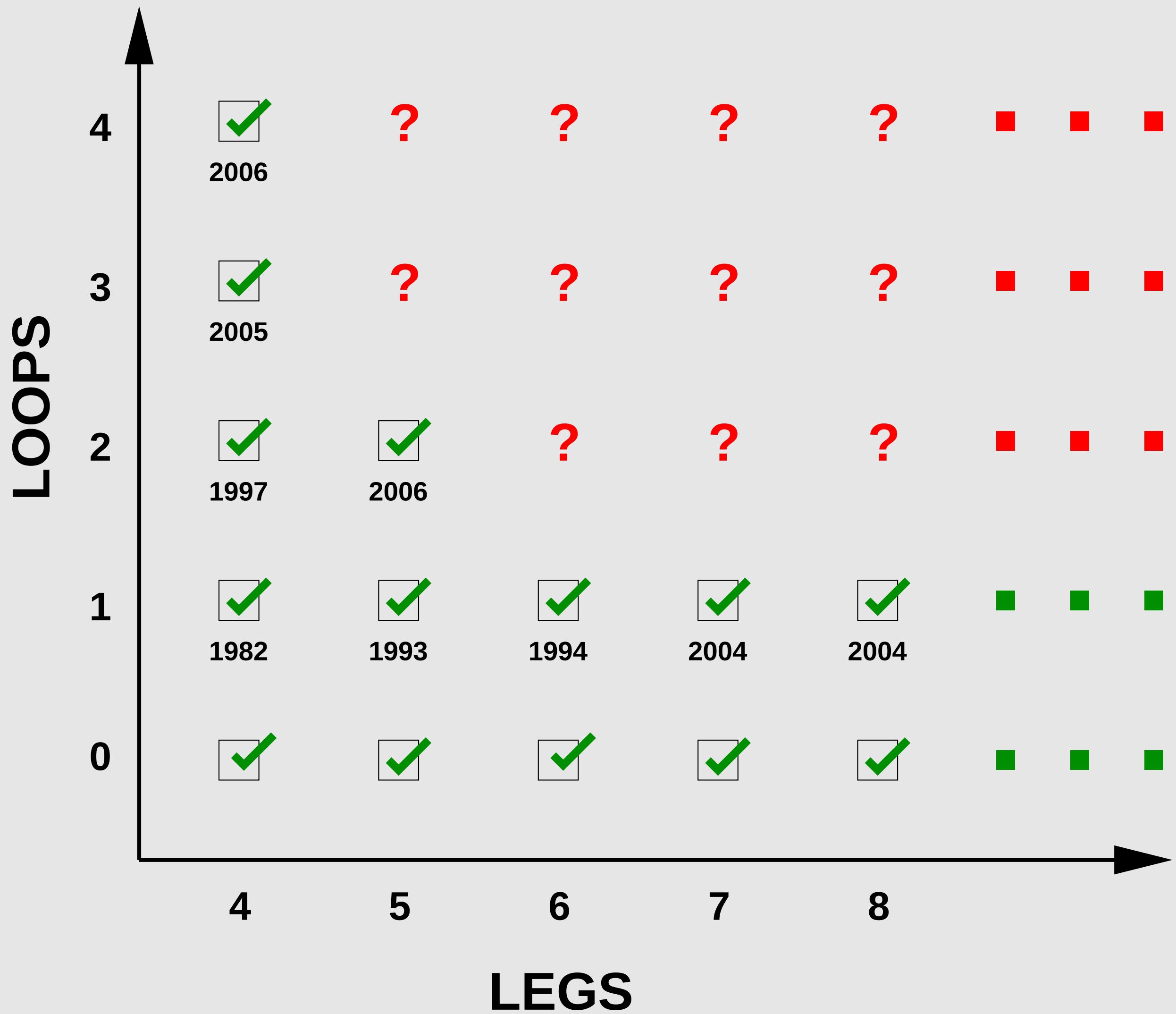
The state of the art involves the use of the unitarity-based methods (or 'generalized unitarity') [Bern, Dixon, Kosower et. al.; Britto, Cachazo, Feng].

Essentially, one can determine an amplitude if one knows all of its cuts (note that in general one needs to evaluate all cuts in D dimensions; knowing them only for $D = 4$ is not sufficient).

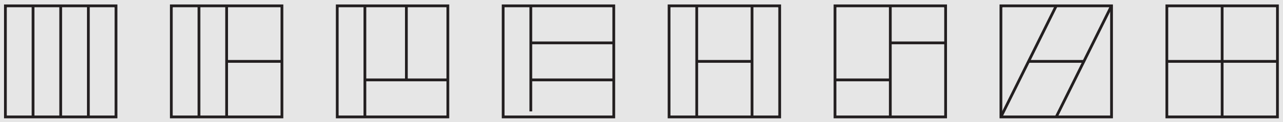
We call this step 'finding the integrand'. For example, the two-loop amplitude on the previous slide is

$$\int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{p^2 q^2 (p-k_1)^2 (p-k_1-k_2)^2 (q+k_4)^2 (q+k_3+k_4)^2 (p-q)^2}.$$

$\mathcal{N} = 4$ Yang-Mills Status Report



The four-loop $\mathcal{A}^{(4)}/\mathcal{A}_{\rm tree}$ amplitude is equal to the sum of 8 integrals:



[Bern, Czakon, Dixon, Kosower, Smirnov]

But unitarity doesn't offer much help with evaluating these nasty integrals, which is the focus of our work...

Universal Infrared Behavior of Loop Amplitudes

Resummation work by Sterman and Tejeda-Yeomans , and infrared singularity work by Catani, shows that in dimensional regularization to $D = 4 - 2\epsilon$, planar n-particle L loop MHV amplitudes satisfy iterative relations of the form

$$M_n^{(L)}(\epsilon) = P^{(L)}(M_n^{(1)}(\epsilon), \ldots, M_n^{(L-1)}(\epsilon)) + (f^{(L)} + \epsilon g^{(L)}) M_n^{(1)}(L\epsilon) + \mathcal{O}(\epsilon^0) \,,$$

where $P^{(L)}$ are some known polynomials.

The quantities $f^{(L)}$ and $g^{(L)}$ are the L-loop terms in the functions

$$f(\lambda) = \sum_{L=1}^{\infty} f^{(L)} \lambda^L, \qquad g(\lambda) = \sum_{L=1}^{\infty} g^{(L)} \lambda^L$$

respectively called the cusp anomalous dimension and collinear anomalous dimension. These two functions capture all independent information about the infrared singularities.

Targets: $f(\lambda)$ and $g(\lambda)$

Much less is known about $g(\lambda)$; I'll mention the AdS/CFT prediction later...

The one-loop four-particle amplitude takes the form

$$M_4^{(1)}(\epsilon) = -\frac{2}{\epsilon^2} + \frac{\log(st)}{\epsilon} - \log s \log t + \frac{2\pi^2}{3} + \mathcal{O}(\epsilon)$$

From the relation

$$M_4^{(L)}(\epsilon) = P^{(L)}(M_4^{(1)}(\epsilon), \ldots, M_4^{(L-1)}(\epsilon)) + (f^{(L)} + \epsilon g^{(L)}) M_4^{(1)}(L\epsilon) + \mathcal{O}(\epsilon^0)$$

we see that we can read off the L loop contribution to $f(\lambda)$ from the $1/\epsilon^2$ and $1/\epsilon$ singularities in the L loop amplitude.

Our interest in exploring the hidden structure in these amplitudes was partly motivated by the desire to develop an efficient algorithm for computing these quantities.

Motive and Opportunity

Generally one only embarks on a calculation of such seriousness if there is very good reason for it.

In other words, what more does one learn by doing one more loop?

In this particular case, there was strong motivation because of some apparent tension in constructing a guess, based on integrability, for $f(\lambda)$ that could interpolate smoothly between weak and strong coupling.

Moreover there were several known opportunities for qualitatively new features that can start only at four loops.

The successful confirmation of the BES conjecture (and a disproof of an earlier conjecture $f^{(4)} = -105.619\ldots$) at loops is significant because it is compatible with all known calculations and provides a smooth, highly nontrivial interpolation between strong and weak coupling.

Brief Technical Details

Our method involves starting with the Mellin transform of an amplitude

$$M(x, \epsilon) = \int_{-i\infty}^{+i\infty} dy\, x^y F(y, \epsilon), \qquad x = t/s,$$

which are **far** easier to compute than the amplitude itself. (One can essentially read off $F(y, \epsilon)$ from a Feynman diagram).

The function $F(y, \epsilon)$ has poles which collide with the integration contour when $\epsilon = 0$

We showed that the cusp anomalous dimension only receives contributions from the leading pole $1/y$.

So, in effect, all one has to do is read off the numerical coefficient sitting on top of the $1/y$ pole. Everything else in $F(y, \epsilon)$ can be thrown away!

Punchline

This method allows us direct access to the cusp anomalous dimension without having to first calculate both sides of the relation

$$M^{(4)} = \frac{1}{4}(M^{(1)})^4 - (M^{(1)})^2 M^{(2)} + M^{(1)} M^{(3)} + \frac{1}{2}(M^{(2)})^2 + \frac{1}{4} f^{(4)} M^{(1)}$$

as (complicated) functions of x, and then relying on delicate cancellations to expose the number $f^{(4)}$ that we are ultimately interested in.

It is sufficient to start with the (relatively far simpler) expressions for the Mellin transform of the amplitude, and then just read off the coefficient of

$$\frac{\delta(y)}{\epsilon^2}$$

since only this particular coefficient contributes

Results

As an application of our method we have obtained the four-loop cusp anomalous dimension

$$f^{(4)} = -117.1789 \pm 0.0002$$

in very good agreement with the BES conjecture

$$f^{(4)} = -\frac{73\pi^6}{630} - 4\zeta(3)^2 = -117.1788285\ldots$$

and an improvement over the [Bern, Czakon, Dixon, Kosower, Smirnov] result

$$f^{(4)} = -117.2 \pm 0.2$$

The numerical improvement is possible because we could identify and eliminate the vast number of terms which we know cannot contribute to $f^{(4)}$

We also found the four-loop collinear anomalous dimension

$$g^{(4)} = -1240.9 \pm 0.3$$

Where is this all going?

I now want to move on to the ‘and AdS/CFT’ part of my talk!

Recall the iterative formula

$$M_n^{(L)}(\epsilon) = P^{(L)}(M_n^{(1)}(\epsilon), \dots, M_n^{(L-1)}(\epsilon)) + (f^{(L)} + \epsilon g^{(L)}) M_n^{(1)}(L\epsilon) + \mathcal{O}(\epsilon^0)$$

The ABDK Relations

Similar relations hold in any gauge theory.

However, it is has been conjectured that in $\mathcal{N} = 4$ Yang-Mills something special happens: the

$$+\mathcal{O}(\epsilon^0)$$

term in the iterative relation is believed to actually be

$$+C^{(L)} + \mathcal{O}(\epsilon^1)$$

where $C^{(L)}$ is a constant (independent of all of the gluon momenta)! [Anastasiou, Bern, Dixon, Kosower], [Bern, Dixon, Smirnov]

This conjecture has only been checked in three cases so far: 4 particles at 2 and 3 loops, and 5 particles at 2 loops (above plus [Cachazo, MS, Volovich], [Bern, Czakon, Kosower, Roiban, Smirnov].) It seems innocent but the consequence is profound...

The BDS Ansatz

If true, then the all-loop, planar, four-particle amplitude sums up to

$$\log(\mathcal{A}/\mathcal{A}_{\text{tree}}) = -\frac{f(\lambda)}{2\epsilon^2} - \frac{g(\lambda)}{\epsilon} + \frac{f(\lambda)}{8}\log^2(t/s) + c(\lambda) + \mathcal{O}(\epsilon^1)$$

where s, t are the usual Mandelstam invariants. [Bern, Dixon, Smirnov]

[A few inconsequential liberties have been taken in writing this equation.]

BDS also formulated an ansatz for the planar, MHV, n-particle amplitude, which very schematically takes the form

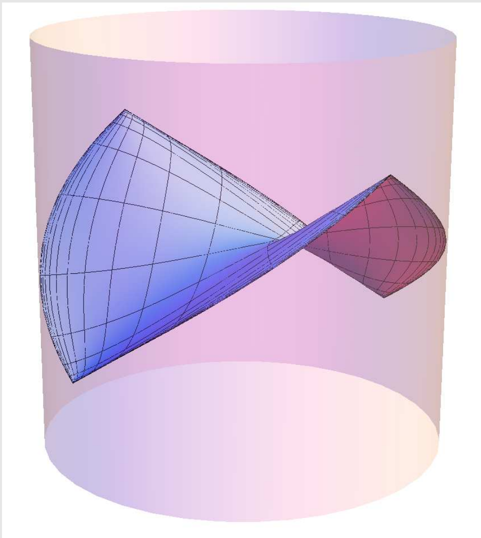
$$\mathcal{A}_n = \mathcal{A}_n^{\text{tree}} \exp[f(\lambda)\mathcal{A}_n^{1-\text{loop}}]$$

This looks just like the kind of formula that begs for an AdS/CFT interpretation.

Alday-Maldacena

Recently, Alday and Maldacena have given a prescription for using AdS/CFT to calculate gluon scattering amplitudes at strong coupling.

Their prescription is computationally equivalent to evaluating a certain Wilson loop composed of null line segments. For four gluons, the relevant classical worldsheet is:



For four particles, an explicit calculation of the (regulated) area of this world-sheet yields precisely the structure

$$\log(\mathcal{A}/\mathcal{A}_{\text{tree}}) = -\frac{f(\lambda)}{2\epsilon^2} - \frac{g(\lambda)}{\epsilon} + \frac{f(\lambda)}{8}\log^2(t/s) + c(\lambda) + \mathcal{O}(\epsilon^1)$$

at strong coupling with numerical coefficients reproducing the known leading behavior

$$f(\lambda) = 4\sqrt{\frac{\lambda}{16\pi^2}} + \mathcal{O}(1)$$

and providing, as a byproduct, the strong coupling prediction

$$g(\lambda) = 2(1 - \log 2)\sqrt{\frac{\lambda}{16\pi^2}} + \mathcal{O}(1)$$

It remains an interesting outstanding problem to find explicit solutions for the classical worldsheet describing more than four gluons scattering (or, alternatively, to find a way to calculate the desired 'regulated area of the worldsheet' without needing to know the explicit solution.

Dual Conformal Symmetry

In fact it has become clear that the functional form of the four and even five particle results are completely fixed by a symmetry that has been called dual conformal symmetry.

What is dual conformal symmetry? [Drummond, Korchemsky, Sokatcehev].

Essentially, conformal invariance in momentum space.

Dual Conformal Invariance at One Loop

The one-loop four-particle amplitude contains the integral

$$\mathcal{I}^{(1)}(k_1, k_2, k_3, k_4) = \int d^4 p_1 \frac{(k_1 + k_2)^2 (k_2 + k_3)^2}{p_1^2 (p_1 - k_1)^2 (p_1 - k_1 - k_2)^2 (p_1 + k_4)^2}.$$

Now, pass to dual coordinates by defining

$$k_1 = x_{12}, \qquad k_2 = x_{23}, \qquad k_3 = x_{34}, \qquad k_4 = x_{41}, \qquad p_1 = x_{15},$$

where $x_{ij} = x_i - x_j$.

In these new variables,

$$\mathcal{I}^{(1)}(x_1, x_2, x_3, x_4) = \int d^4 x_5 \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}.$$

which is easily seen to be invariant under arbitrary conformal transformations on the x_i!

Through four loops, the integrals which contribute to the four-particle amplitude are precisely those integrals which are dual conformally invariant.

Dual conformal invariance has also been used to guide an ansatz for the five-loop amplitude.

[Bern, Czakon, Dixon, Kosower, Smirnov], [Bern, Carrasco, Johansson, Kosower], [Drummond, Korchemsky, Sokatchev]

The origin of this symmetry at weak coupling remains a mystery.

However at strong coupling it has a natural interpretation in the Alday-Maldacena prescription as the isometry of a T-dualized AdS_5.

For six or more particles, dual conformal symmetry is not enough to fix the result (one can have in principle arbitrary functions of conformally invariant cross-ratios). In fact Alday and Maldacena recently claim a contradiction with the BDS ansatz for many particles.

Amplitude/Wilson Loop Equality

The Alday-Maldacena prescription reveals an equality at strong coupling between gluon scattering amplitudes and certain Wilson loops (composed of a sequence of lightlike line segments).

Remarkably this equality has been shown to hold at weak coupling as well in some cases; at one loop for any number of particles [Brandhuber, Heslop, Travaglini] and at two loops for four particles [Drummond, Henn, Korchemsky, Sokatchev].

It could be something really amazing, but there is not yet enough evidence because so much is fixed by dual conformal symmetry.

I mean that if both the amplitude and the Wilson loop have this symmetry, then they can only differ when they are allowed to start differing.

Conclusion

We developed some techniques to aid in direct tests of the conjectured planar $\mathcal{N}=4$ Yang-Mills S-matrix and multiloop iterative relations. As an application, we computed four-loop cusp and collinear anomalous dimensions.

The motivation behind this research is the desire to explore and uncover the rich mathematical structure underlying $\mathcal{N}=4$ Yang-Mills theory.

Discovering such structures also has the pleasant side benefit of making previously difficult calculations much simpler.

It is important to continue unlocking the structure behind gluon scattering amplitudes even if the BDS ansatz is wrong, because historically we have learned a lot from studying alleged 'breakdowns' in AdS/CFT. Of course it has never been AdS/CFT that has broken down, only our naïveté.