Noncommutative spaces and the quantum of time

Juan M. Romero & J. David Vergara
Instituto de Ciencias Nucleares
UNAM
Mexico

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Introduction

- **Noncommutative Quantum Field Theory**
  - Constant Parameter $\theta_{\mu\nu}$
  - Twisted Poincaré invariance

- **Models with nonconstant deformation parameter**
  - κ-Poincaré space-time
  - Snyder space-time
  - Others…

Snyder like Space-Time

Consider a (D+2)-dimensional space:

$$\xi^A = \left(\xi^0, \xi^{0'}, \xi^1, \ldots, \xi^D\right),$$

With quadratic invariant form:

$$\tilde{S}^2 = -\left(\xi^{0'}\right)^2 - \left(\xi^0\right)^2 + \left(\xi^1\right)^2 + \cdots + \left(\xi^D\right)^2.$$

Invariant under the group SO(D,2), with (D+2)(D+1)/2, l_{AB} generators given by

$$l^{\mu 0'} = \xi^{0'} \frac{\partial}{\partial \xi_\mu} - \xi^\mu \frac{\partial}{\partial \xi_{0'}}, \quad l^{\mu\nu} = \xi^\nu \frac{\partial}{\partial \xi_\mu} - \xi^\mu \frac{\partial}{\partial \xi_\nu}.$$

With these generators we can define the following operators:

$$\hat{X}^{\mu} = -ia\left(\xi^{0'}\frac{\partial}{\partial\xi_{\mu}} - \xi^{\mu}\frac{\partial}{\partial\xi_{0'}}\right), \quad \mu = 1,\ldots,D,$$

where a is a constant with units length. Now, using these operators we construct the Hermitian operator invariant under *SO(D,1) Lorentz transformations*,

$$S^2 = X_{\mu}X^{\mu},$$

and from the definition we get,

$$\left[X_{\mu}, X_{\nu}\right] = \frac{ia^2}{\hbar}L_{\mu\nu},$$

Now introducing the momentum operators, defined by

$$P_{\mu} = \frac{-\hbar}{a}\frac{\xi_{\mu}}{\xi^{0'}},$$

We obtain that,

$$L_{\mu\nu} = X_{\mu}P_{\nu} - X_{\nu}P_{\mu},$$

and the following commutation relations,

$$[X_{\mu}, P_{\nu}] = i\hbar\left(\eta_{\mu\nu} + \frac{a^2}{\hbar^2}P_{\mu}P_{\nu}\right), \quad \left[P_{\mu}, P_{\nu}\right] = 0.$$

Quantum of time

Now, it can be shown that $\phi_i = \exp\left[-iL\phi_i\right]$, with $\phi_i = \operatorname{arctanh}\left(\xi^i / \xi^{0'}\right)$, is eigenfunction of X^i,

$$X^i \psi_i = a\tilde{L}\psi_i,$$

In this case $\tilde{L}$ can take any arbitrary value.

However in the case of X_0, the eigenfunction is,

$\psi_0 = \exp\left[iL\phi_0\right]$, with $\phi_0 = \arctan\left(\xi^0 / \xi^{0'}\right)$

As the tangent is 2Π periodic one must constraint the values of L to be integers. Thus, the time is quantized:

$$t_N = \frac{a}{c} N$$

Thus this space-time is discrete in time and consistent with the Lorentz symmetry.

Realization of noncommutative spaces

Starting from the action of a relativistic particle:

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(-m\sqrt{-\eta_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}} \right) \qquad \alpha = 1, \ldots, n$$

In Hamiltonian form we have:

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(p_\alpha \dot{x}^\alpha - \frac{\lambda}{2}\left(p^2 + m^2\right)\right) \qquad \varphi = p^2 + m^2 \approx 0$$

Using the equations of motion to eliminate the momenta:

$$S_* = \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau \left(\frac{\dot{x}^2}{\lambda} + \lambda m^2 \right)$$

We consider now the following generalization**:**

$$S_* = \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau \left(\frac{m_\gamma \dot{x}^2}{\dot{\xi}} + \lambda m \dot{\xi} \right)$$

This action is invariant under reparametrizations and we have the constraint:

$$\phi = P_\xi + \frac{1}{2m_\gamma}\left(P^2 - m^2\right),$$

The Hamiltonian action is,

$$S = \frac{1}{2}\int d\tau \left(P_\xi \dot{\xi} + P_\mu \dot{X}^\mu - \lambda\phi \right).$$

Now using the gauge condition:

$$\chi_1 = A\tau + B\xi + C\xi P_\xi + X^\mu P_\mu, \qquad A, B, C = \text{constants}$$

We obtain a set of second class constraints**:**

$$\{\chi_1, \chi_2\} = B + \frac{P^2}{m_\gamma}\left(1 - \frac{C}{2}\right) + \frac{mC}{2} \neq 0.$$

In reduced phase space the Dirac brackets will be:

$$\left\{X_\mu, X_\nu\right\}^* = -\frac{d}{\hbar^2} L_{\mu\nu}, \qquad L_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu,$$

$$\left\{X_\mu, P_\nu\right\}^* = \eta_{\mu\nu} - \frac{d}{\hbar^2} P_\mu P_\nu, \qquad \left\{P_\mu, P_\nu\right\}^* = 0.$$

Where,

$$d = \frac{\hbar^2}{B m_\gamma + \left(1 - \frac{C}{2}\right)P^2 + \frac{C}{2} m^2}.$$

In the case of:

$$C = 2,\ B = 2m_\gamma \Rightarrow\ d = a^2 = \frac{-\hbar^2}{{m_\gamma}^2 + m^2}$$

The action in the reduced phase space will be**:**

$$S_{rsp} = -\int_{\tau_1}^{\tau_2} d\tau \left(X^{\mu} + \frac{P^{\mu}}{m_{\gamma}} \left(\frac{A + X^{\alpha} P_{\alpha}}{B - Ch} \right) \right) \dot{P}_{\mu},$$

with

$$h = \frac{1}{2m_{\gamma}} \left(P^2 - m^2 \right)$$

Taking A=0, and introducing the metric,

$$g^{\alpha\beta} = \eta^{\alpha\beta} + \frac{P^{\alpha} P^{\beta}}{m_{\gamma} \left(B - Ch \right)}$$

The action takes the form

$$S_{rsp} = -\int_{\tau_1}^{\tau_2} d\tau g^{\alpha\mu} \left(P \right) X_{\alpha} \dot{P}_{\mu},$$

Darboux map

- For an arbitrary Hamiltonian we get

$$S_s = \int_{\tau_1}^{\tau_2} d\tau \left(-g^{\mu\nu} \left(P \right) X_{\mu} \dot{P}_{\nu} - H \left(X, P \right) \right).$$

- Using the local coordinates

$$\tilde{X}^{\mu} = g^{\mu\nu} \left(P \right) X_{\nu}, \qquad \tilde{P}_{\mu} = P_{\mu}.$$

- We obtain the action

$$S_s = \int_{\tau_1}^{\tau_2} d\tau \left(\tilde{X}_{\mu} \dot{P}_{\nu} - H \left(\tilde{X}, P \right) \right).$$

Summary

- **In this work we construct three space-times that are discrete in time and compatibles with the Lorentz symmetry, and we show that these models are noncommutative.**
- **In the case of the Snyder type space we are able to obtain a realization of the model using the action of a reparametrized relativistic particle, and from this result we construct a general action for a particle in this kind of noncommutative spaces.**
- **Furthermore, a general form for the action of a particle in a Snyder-like noncommutative space with arbitrary Hamiltonian is presented.**