A Simple, Direct Finite Differencing of the Einstein Equations

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We investigate a simple variation of the Generalized Harmonic method for evolving the Einstein equations. A flat space wave equation for metric perturbations is separated from the Einstein tensor, with the rest of the Einstein tensor becoming a source for the wave equations. We demonstrate that this allows for the accurate simulation of compact objects, with gravitational field strengths less than or equal to those of neutron stars. This method could thus provide a straightforward path for general relativistic effects to added to astrophysics simulations, such as in core collapse, accretion disks, and extreme mass ratio systems.

I. INTRODUCTION

The Generalized Harmonic (GH) formulation of the Einstein equations was the first version to successfully simulate the inspiral, merger and ringdown of two black holes [1]. It is an offspring of harmonic coordinate condition (given by $\Box x^a = 0$) which place the Einstein equation in a manifestly hyperbolic form. GH coordinates preserve this useful property but also allow any type of coordinates to be used by setting the d'Alembertian of the coordinates to be a general function: $\Box x^a = H^a$.

We report here that a simple rewriting of the harmonic coordinate condition allows for one to numerically evolve the Einstein equations for systems with white dwarf or neutron star strength gravitational fields. This rewriting splits off a flat space wave equation for metric perturbations from the Einstein tensor, and turns the rest of the Einstein tensor into a source for the wave equation. The complexities of the Einstein equations are packaged within this new source term, which we find to be well behaved when the gravitational field strengths are on the order of those for a neutron star or weaker. This could thus form a fairly simple way for general relativistic corrections to be added to astrophysics simulations that have traditionally only used Newtonian gravity. For instance, we could see it being useful in modeling core collapse, the dynamics of accretion disks, and for neutron star or white dwarf orbits around a supermassive black hole.

II. FIELD EQUATIONS

For completeness we begin by reviewing the GH formulation of GR. The metric g_{ab} gives a coordinate map of our spacetime:

$$ds^2 = g_{ab} dx^a dx^b \tag{1}$$

The Einstein equations can be written in trace reversed form:

$$R_{ab} = 4\pi(2T_{ab} - g_{ab}T) \tag{2}$$

with the Ricci tensor R_{ab} given in terms of the connection coefficients Γ^c_{ab}:

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{cb,a} + \Gamma^d_{ab}\Gamma^c_{dc} - \Gamma^d_{cb}\Gamma^c_{da} \tag{3}$$

The lowered indices connection coefficients are given by derivatives of the metric:

$$\Gamma_{abc} = \frac{1}{2}(g_{ab,c} + g_{ac,b} - g_{bc,a}) \tag{4}$$

and we can then raise the first index or get contracted forms using the metric:

$$\Gamma^a_{bc} = g^{ad}\Gamma_{dbc} \tag{5}$$
$$\Gamma^a = g^{bc}\Gamma^a_{bc} \tag{6}$$
$$\Gamma_a = g^{bc}\Gamma_{abc} \tag{7}$$

The contracted connection coefficients can be shown to be equivalent to minus a wave operator acting on the coordinates:

$$g^{bc}\nabla_b\nabla_c x^a = \Box x^a = -\Gamma^a \tag{8}$$

Choquet-Bruhat showed [2] that if we were to pick harmonic coordinates $\Box x^a = 0$, then we could expand the Ricci tensor and transform the the Einstein equations (2) into:

$$\begin{aligned} g^{cd}g_{ab,cd} + 2g^{cd}{}_{(,a}g_{b)d,c} + 2\Gamma^d_{cb}\Gamma^c_{da} \\ = -8\pi(2T_{ab} - g_{ab}T) \end{aligned} \tag{9}$$

A wider range of options available to us however – we have the freedom in general relativity to pick any coordinates we wish. As shown by Friedrich [3] and Garfinkle [4] this freedom can be alternatively expressed by picking source functions $H^a(x,t)$ to drive the waves equations of the coordinates:

$$\Box x^a = H^a(x,t) = -\Gamma^a \tag{10}$$

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Using these source functions we can decompose (2) in the standard GH formulation (see [5] and [6]):

$$g^{cd}g_{ab,cd} + 2g^{cd}{}_{(,a}g_{b)d,c} + 2H_{(a,b)} - 2H_c\Gamma^c_{ab} + 2\Gamma^d_{cb}\Gamma^c_{da} = -8\pi(2T_{ab} - g_{ab}T) \tag{11}$$

We will modify this equation further: one goal of ours is to evolve metric perturbations with a flat space scalar wave equation, and turn the rest of the Einstein tensor into a source for those wave equations (one could then use spherical harmonics to build a split spectral – finite difference code). We will proceed more generally here by considering metric perturbations f_{ab} away from a general background metric $\bar{g}_{ab}$, (such as Minkowski or perhaps Schwarzschild):

$$g_{ab} = \bar{g}_{ab} + f_{ab} \tag{12}$$

We then refer to the raised indices perturbations by h^{ab}:

$$g^{ab} = \bar{g}^{ab} + h^{ab} \tag{13}$$

One can then split the second order piece of (11) into:

$$g^{cd}g_{ab,cd} = (\bar{g}^{cd} + h^{cd})(\bar{g}_{ab,cd} + f_{ab,cd}) \tag{14}$$

We want to separate out a scalar wave equation for the metric perturbations f_{ab} on the background geometry $\bar{g}_{ab}$:

$$\bar{g}^{cd}\bar{\nabla}_c\bar{\nabla}_d f_{ab} = \bar{g}^{cd}f_{ab,cd} - f_{ab,e}\bar{\Gamma}^e \tag{15}$$

We thus need to subtract $f_{ab,e}\bar{\Gamma}^e$ from both sides of (11) in order to get $\bar{\Box}f_{ab}$ (for instance with a flat spherical metric they give the $(2/r)\partial_r f_{ab} + (\cos\theta)/(r^2\sin\theta)\partial_\theta f_{ab}$ terms). We collect the rest of the terms on the LHS of (11) into a source S_{ab} which we will move to the RHS:

$$\begin{aligned} S_{ab} = &-\bar{g}^{cd}\bar{g}_{ab,cd} - h^{cd}(\bar{g}_{ab,cd} + f_{ab,cd}) \\ &-2g^{cd}{}_{(,a}g_{b)d,c} - 2H_{(a,b)} + 2H_c\Gamma^c_{ab} \\ &-2\Gamma^d_{cb}\Gamma^c_{da} - f_{ab,e}\bar{\Gamma}^e \end{aligned} \tag{16}$$

This allows us to rewrite (11) in the simple form:

$$\Box f_{ab} = S_{ab} - 8\pi(2T_{ab} - g_{ab}T) \tag{17}$$

(we have dropped the bar from the wave operator here, as we will switch to a flat background geometry).

A. Constraint Dampening

A crucial element that allowed the breakthrough reported in [1] is the use of constraint dampening. If we define the constraint C^a to be:

$$C^a \equiv H^a - \Box x^a \tag{18}$$

then for initial data that has $C^a = 0$ and $\partial_t C^a = 0$ we would expect that C^a will remain equal to zero throughout an evolution. This is not necessarily true however, as numerical truncation errors can grow rapidly, causing divergence from $C^a = 0$.

This can be prevented by dampening the constraint violations. Following [5] we can add constraint dampening terms to our field equations (17) via:

$$\Box f_{ab} = S_{ab} - 2\kappa(n_{(c}C_{b)} - \frac{1}{2}g_{ab}n^d C_d) - 8\pi(2T_{ab} - g_{ab}T) \tag{19}$$

with a new parameter κ and a time like vector n_a. The constraints themselves are evolved according to:

$$\Box C^a = -R^a{}_b C^b + 2\kappa\nabla_b[n^{(b}C^{a)}] \tag{20}$$

Note that the additional terms are equal to zero for $C^a = 0$, and the dampening action should act to drive any violation of this constraint towards zero: Gundlach at al showed this was true for a related system with perturbations about Minkowski [7].

We note that our code as described later does not in fact need constraint dampening in order to converge to the physical solution. This may be because it only deals with neutron star strength gravitational fields, as opposed to black hole strength fields, where the nonlinear effects in S_{ab} become dominant. It is in general a good idea to include the constraint dampening terms, but in simulations only involving neutron star and white dwarfs it may not prove necessary.

B. Flat space cartesian coordinates

In (16) and (17) we give a general splitting for metric perturbations on a general background geometry $\bar{g}_{ab}$. Here we specialize to the case examined in this paper: we set the background to be flat and use cartesian coordinates $\eta_{ab} = diag(-1, 1, 1, 1)$:

$$g_{ab} = \eta_{ab} + f_{ab} \tag{21}$$

In this case the coordinate source functions H^a are zero and the source S_{ab} simplifies to:

$$S_{ab} = -h^{cd}f_{ab,cd} - 2g^{cd}{}_{(,a}g_{b)d,c} - 2\Gamma^d_{cb}\Gamma^c_{da} \tag{22}$$

(where all of the metric derivatives now stem from the f_{ab} functions).

III. NUMERICAL SIMULATION

A. Matter EOM

In order to test our splitting (16),(17) of the Einstein equations and see if they allow for stable and accurate numerical evolutions, we will build a simplistic model of a binary neutron star inspiral and merger. We will construct and evolve a stress energy tensor T_{ab} by fiat during the simulation, which will source the gravitational fields f_{ab} (which in turn gives rise to the corrective source S_{ab}). In general one would use projections of $T^{ab}{}_{;b} = 0$ to generate the matter equations of motion, but the current version of the code is for only a single processor (with second order accurate derivatives) and thus does not have high enough resolution to accurately resolve an inspiral. For instance, the 2.5 PN radiation reaction force, which is the leading term responsible for the inspiral, is a factor of $v^5 \sim 10^{-5}$ times smaller than the Newtonian forces (and thus the Newtonian forces need to be resolved with relative errors less than $\sim 10^{-5}$ in order to extract the radiation reaction force).

The "neutron stars" are rigid polytropes with a compactness in the range of: $M/R \sim 0.1 - 0.2$. We choose to drive these density profiles on a quasi-circular inspiral path as found by Peters and Mathews: [12],[13]. The inspiral is parameterized by:

$$a(t) = a_0 \left(1 - \frac{t}{t_{decay}}\right)^{1/4} \tag{23}$$

where $a(t)$ is the semimajor axis as a function of time, a_0 is the initial semimajor axis (between the centers of masses of the bodies), and the decay time is given by:

$$t_{decay} = \frac{5}{64}\frac{a_0^4}{M^3} \tag{24}$$

where M is the total mass of the (equal mass) neutron stars. The instantaneous coordinate velocities of the stars are simply given by the Keplerian velocities of a binary with the same mass and separation. Note that eventually the separation $a(t)$ will fall below twice the radius of the stars: here the stress energy tensor is determined by a simple superposition of the individual stars densities, with the separation quickly falling to zero and the velocities also modulated to zero. This is, of course, not a particularly realistic model of a binary neutron star merger, the point is just to demonstrate that the gravitational fields f_{ab} still evolve stably in this strong field and highly dynamical setting.

Having determined the bulk motion of the stars, we need their thermodynamical profiles in order to build the stress energy tensor. We choose the standard perfect fluid form:

$$T^{ab} = (\rho(1+\varepsilon) + p)u^a u^b + p g^{ab} \tag{25}$$

with rest mass density ρ, specific internal energy ε, pressure p, and four velocities u^a. The internal energy and pressure are given in terms of the density through a polytropic equation of state:

$$p = \kappa \rho^\Gamma \tag{26}$$
$$\varepsilon = \frac{\kappa}{\Gamma - 1}\rho^{\Gamma - 1} \tag{27}$$

where we choose $\Gamma = 2$.

The density ρ is in turn derived from a conserved "baryonic" density ρ^* (see e.g. [14]):

$$\rho^* = \rho(-g)^{1/2} u^0 \tag{28}$$

This density has a number of convenient properties, including a flat space conservation law:

$$\partial_t \rho^* = \partial_i (v^i \rho^*) \tag{29}$$

which ensures that the total integrated baryonic mass:

$$M^* = \int \rho^* d^3x \tag{30}$$

is constant throughout the evolution. We thus choose initial radial profiles for ρ^* for each star and hold these constant during the evolution.

B. Field Evolution

We have determined the form of the matter stress energy tensor, and now need to evolve the gravitational fields f_{ab} in response via (17) (along with the simplifications (21),(22)). We go about this in a slightly non-standard way: given previous successes we choose to finite difference scalar wave equation $\Box f_{ab}$ using implicit methods, while we use the standard explicit methods to construct the source S_{ab}.

First consider the implicit finite differencing of $\Box f_{ab}$. We need to use operator splitting to split the 3+1 wave equation into 3 1+1 wave equations (see e.g. [15]). For conciseness replace f_{ab} with ψ and the entire source $S_{ab} - 8\pi(2T_{ab} - g_{ab}T)$ with τ. Operator splitting then divides:

$$-\partial_t^2\psi + \partial_x^2\psi + \partial_y^2\psi + \partial_z^2\psi = \tau \tag{31}$$

into a set of 1+1 problems:

$$-\partial_t^2\psi + \partial_x^2\psi = 1/3\tau \tag{32}$$
$$-\partial_t^2\psi + \partial_y^2\psi = 1/3\tau \tag{33}$$
$$-\partial_t^2\psi + \partial_z^2\psi = 1/3\tau \tag{34}$$

which are sequentially finite differenced implicitly. We use a variation of the Crank-Nicholson method, modified for second time derivatives, to evolve these – for instance (32) becomes:

$$\begin{aligned} \frac{\psi_i^{N+1} - 2\psi_i^N + \psi_i^{N-1}}{\Delta t^2} &= \\ c_{N+1}\frac{\psi_{i+1}^{N+1} - 2\psi_i^{N+1} + \psi_{i-1}^{N+1}}{\Delta x^2} & \\ +c_{N-1}\frac{\psi_{i+1}^{N-1} - 2\psi_i^{N-1} + \psi_{i-1}^{N-1}}{\Delta x^2} & \\ -\frac{1}{3}\tau_i^N & \end{aligned} \tag{35}$$

where the field is discretized at time step N and spatial mesh location i. We have also included generalized coefficients c_{N+1} and c_{N-1} (with $c_{N+1}+c_{N-1}=1$) so that we can transition from Crank-Nicholson $c_{N+1}=c_{N-1}=1/2$ to a fully implicit method $c_{N+1}=1, c_{N-1}=0$, or somewhere in between. More implicit splitting allows us to dissipate high frequency noise if needed. Boundary conditions are added to (35) which is then solved by the Thomas algorithm. We note that in general implicit methods allow one to take as large a time step Δt as desired and still maintain stability, however for reasons of accuracy we take time steps of about the same size that an explicit scheme would take in order to satisfy the Courant stability condition.

This operator-splitting implicit method is combined with a Fixed Mesh Refinement (FMR) grid structure so that a high resolution mesh is available near the origin to resolve the stars, while also extending through successive lower resolution meshes out into the wave zone. Sommerfeld outgoing wave boundary conditions are applied to the largest and coarsest mesh, which is updated first. Successively finer meshes are then updated, with their boundary conditions found via interpolation from the next coarsest mesh. After all the meshes are updated one time step then the coarser mesh points are replaced with the finer mesh's field values wherever there is overlap.

This leaves the evaluation of the metric source S_{ab} given by (22). The spatial derivatives are computed in the standard second order explicit way, but evaluating the time derivatives is more subtle as the updated field values f_{ab}^{N+1} won't be known until $\Box f_{ab}$ is calculated. We solve this iteratively: for the first pass the first and second time derivatives contained in (22) are evaluated using the current and two previous time steps f_{ab}^{N}, f_{ab}^{N-1}, f_{ab}^{N-2} – this allows for a preliminary value of $\hat{f}_{ab}^{N+1}$ to be found. The second iterative pass then uses the preliminary $\hat{f}_{ab}^{N+1}$ to calculate centered time derivatives in (22) and proceed complete the time step.

Finally initial data is needed – we choose very simple initial data: $f_{ab}(t=0)=0$. There is thus a large amount of noise initially as the fields respond to the matter and then settle down to their correct physical solutions, with the transients propagating off the grid. For values of neutron star compactness much larger than $M/R \sim 0.1$ it is useful to also add a large amount of dissipation initially, and then linearly decrease the dissipation to zero over a fraction of an orbit. Otherwise the initial transient spikes can give rise to run away feedback via S_{ab}. Physical initial data would definitely be preferable, but the robustness of the code when given bad initial data counts in its favor.

C. Results

In general we find that splitting off a flat space wave equation for the metric perturbations and turning the rest of the Einstein tensor into a source provides an effective method for simulatiing systems with neutron star strength or weaker gravitational fields. The results for an example evolution are given: we pick two stars, each with baryonic mass $M^*=0.1$ and coordinate radius $R=1$, and with an initial coordinate separation of 4 (which gives a value of $t_{decay}=2500$ via (24)). The finest mesh spacing is 0.25, giving about 8 grid points across the diameter of the stars. At this resolution the Hamiltonian constraint:

$$^3R - K_{ij}K^{ij} + K^2 - 16\pi\rho_H = 0 \tag{36}$$

converges to an error of a couple percent after the initial transients have propagated off the grid. We find that this code is only first order convergent – we believe this is due to taking only one iteration when solving of S_{ab}, and due to the simplistic operator splitting method given in (32-39).

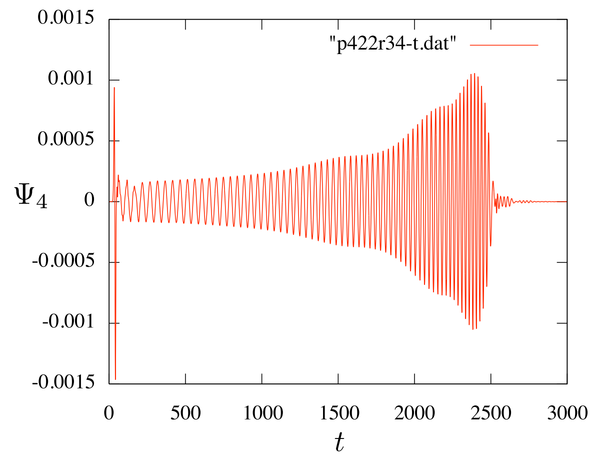

FIG. 1: A plot of the $l=2$, $m=2$ component of Ψ_4 as a function of time.

In figure (1) we see a plot of the $l=2$, $m=2$ component of the gravitational waveform Ψ_4, measured at a radial distance of 34 or $170M^*$. There is a large amount of noise that passes through initially, which then settles down approximately to the standard chirp waveform (there is also some modulation of the amplitude of the waveform, which is an artifact due to the low resolution rather than any eccentricity of the binary). After the merger the waveform dies away particularly quickly, due to the abrupt cessation of motion for the artificially driven sources.

In figure (2) we plot the ADM mass as a function of time. Again there is a large amount of initial noise, which settles down to a mass of $M_{ADM} \sim 0.185 = 0.925M^*$, (modulo oscillations due to the passing waves). After the merger the mass then settles on $M_{ADM} \sim 0.18$.

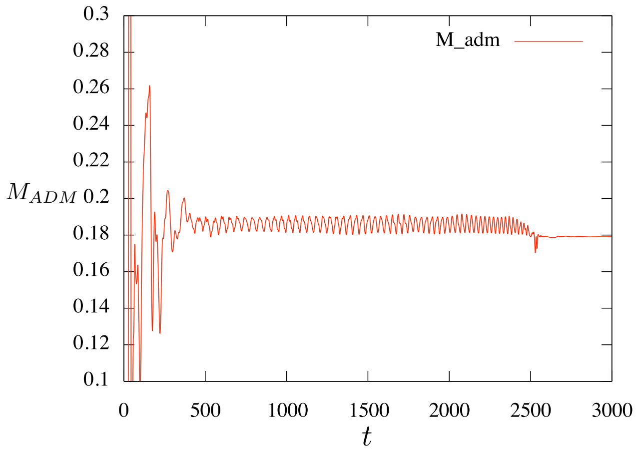

FIG. 2: A plot of ADM mass as a function of time.

IV. CONCLUSIONS

The simple binary inspirals we have evolved show that our direct method (17) successfully solves the Einstein equations, at least in the case of neutron star type systems with a flat Minkowski background (21),(22). It should thus allow for groups in the wider community to add general relativity to their astrophysical simulations in a relatively straightforward manner. We are considering adapting the method to model neutron stars and white dwarfs in orbit around a central supermassive black hole (which would provide the background geometry $\bar{g}_{ab}$).

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