Binary Inspirals in Nordström's Second Theory: Semi-Analytic Calculations

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We find that Gunnar Nordström's second theory of gravitation provides a good testbed for the development of numerical techniques to be used relativistic binary simulations. In this paper we perform analytical calculations to determine the nature of binary orbits in Nordström's theory. We use the analytic solutions to check the accuracy of our numerical simulations in a companion paper. First we show that to first PN order the orbits are Keplerian with small corrections - i.e. there is no Nordtvedt effect. We then calculate the energy and angular momentum flux carried by the waves produced by a Keplerian binary, which gives rise to a secular inspiral similar to the one found in general relativity.

I. INTRODUCTION

One of the first predictions of general relativity was found by Einstein only a year after he developed the main theory: the existence of gravitational waves in spacetime. By splitting the metric into a small perturbation $h_{\mu\nu}$ and a Minkowski background spacetime $\eta_{\mu\nu}$ one finds that the perturbation follows a wave equation: $\Box \bar{h}_{\mu\nu} = 8\pi T_{\mu\nu}$ (with $\bar{h}_{\mu\nu} = h_{\mu\nu} - (1/2)h$). However, the question of whether gravity waves actually carry energy was contentious for many decades after Einstein's original prediction – Einstein himself changed his mind on the issue (see e.g. [1]).

Over time the issue was resolved in favor of the waves being physical, and furthermore capable of being generated by bodies freely falling along geodesics. The resolution was helped the concept of general covariance: the waves do carry energy, but the energy can not be defined at individual points in spacetime (since coordinates can always be constructed such that spacetime is flat at those points), and instead needs to be averaged over an entire wavelength. By 1964 Peters and Mathews had calculated ([2],[3]) the energy and angular momentum carried by the waves emitted by a pair of stars in Keplerian orbit, which in turn results in the secular decay of the semi-major axis a and eccentricity e. Chandrasekhar found the same result by developing the post-Newtonian framework out to $2\frac{1}{2}$ order for extended fluid bodies [4]. This was dramatically confirmed by Hulse and Taylor's discovery [5] of the binary pulsar PSR B1913+16, whose orbit decays at the rate predicted by Peters and Mathews.

In our research we have determined that Nordström's second theory (see e.g. [6]) provides a good test bed for the development of numerical techniques to be used in relativistic binary simulations. We need analytical calculations to determine the rate at which a binary system's orbital parameters evolve in order to compare with the results from a numerical evolution of the binary. We thus perform a calculation in this paper similar to the one done by Peters and Mathews in order to find the rate at which semi-major axis a and eccentricity e change due to the emission of gravity waves for Nordström's theory. The method we develop follows the general procedure given by Will in chapter 10.3 of [7] for determining the gravitational radiation produced by alternative theories of gravitation. First we need show that two stars in Nordström's theory do in fact move in Keplerian orbit to lowest order, which is not true in all gravitational theories. We then calculate the energy radiated due to the gravity waves on a large two-sphere out in the radiation zone. This allows us to determine the rate at which a binary orbit decays.

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II. NORDSTRÖM'S SECOND THEORY

Following Einstein and Fokker's geometric description of the Nordström's theory we find that instead of the Einstein tensor being equated to the stress energy tensor we have instead the Ricci scalar being generated by the trace of the stress energy tensor:

$$R = 24\pi T \tag{1}$$

with the additional criteria that the theory is conformally flat:

$$C_{\alpha\beta\gamma\delta} = 0. \tag{2}$$

If we pick the metric to be of the form:

$$g_{\mu\nu} = (1+\varphi)^2 \eta_{\mu\nu} \tag{3}$$

then we can expand (1) into:

$$\Box\varphi = 4\pi(1+\varphi)^3(\rho + \rho\varepsilon - 3p) \tag{4}$$

where we have used the standard perfect fluid stress energy tensor:

$$T^{\mu\nu} = [\rho(1+\varepsilon) + p]u^\mu u^\nu + g^{\mu\nu} p \tag{5}$$

We can also find a conserved energy-momentum complex, $t^{\mu\nu}$, which includes the energy contained in the field:

$$t^{\mu\nu} = \frac{1}{4\pi}\left[\eta^{\mu\alpha}\eta^{\nu\beta}\varphi_{,\alpha}\varphi_{,\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}\right] + (1+\varphi)^6 T^{\mu\nu} \quad (6)$$

This stress energy tensor follows the flat space conservation law ${t^{\mu\nu}}_{,\nu} = 0$, which is made possible by the flat background metric $\eta_{\mu\nu}$. We will use this to calculate the energy flux in a later section.

A first post-Newtonian calculation performed in the next section reveals that Nordström's theory is fully conservative, with post-Newtonian parameters $\beta = 1/2$ and $\gamma = -1$. There is thus no lowest order Nordtvedt effect [8] in this theory (although the results given in the companion paper suggest that Nordtröm's theory violates the Strong Equivalence Principle at 2PN, which would give rise to higher order Nordtvedt effects). This allows us to utilize Keplerian orbits to calculate the quadrupole moment tensors, which are used to find the energy flux radiated by the system. The fact that the theory is fully conservative also allows for the usage of "Hydro-without-Hydro" approximation [9] since there are no "star-crushing" effects (see e.g. [10]). Therefore in an eccentric orbit the stars will not undergo radial pulsations, which in general can give rise to monopole radiation for scalar fields. This allows us to approximate the stars as point bodies in Keplerian orbit when finding the energy flux.

III. BINARY ORBITS TO 1ST PN ORDER

We will give a quick outline of the post-Newtonian calculations here. Following Will [7], we first determine the metric to 1 PN order (following chapter 4 of TEGP), and then calculate orbital motion (chapter 6 of TEGP).

A. The Metric to 1st PN Order

To make contact with Newtonian physics we only need to know the g_{00} component of the metric to v^2 order. To 1st post-Newtonian order we need to expand g_{00} to v^4, g_{i0} to v^3 and g_{ii} to v^2. We split φ into pieces that scale as the second and fourth powers of the velocity $\varphi = \varphi^{(2)} + \varphi^{(4)}$ (higher order corrections are not needed for 1PN). This enables us to split (4) into:

$$\nabla^2\varphi^{(2)} = 4\pi\rho \quad (7)$$

and

$$-\partial_t^2\varphi^{(2)} + \nabla^2\varphi^{(4)} = 4\pi(3\varphi^{(2)}\rho + \rho\varepsilon - 3P). \quad (8)$$

Equation (7) has the solution

$$\varphi^{(2)} = -\int_V \frac{\rho(\mathbf{x}', t)}{|\mathbf{x}-\mathbf{x}'|} d^3x' = -U \quad (9)$$

where we use Will's definition of the Newtonian potential U.

To solve equation (8) first introduce the superpotential χ:

$$\chi = -\int_V \rho(\mathbf{x}', t)|\mathbf{x}-\mathbf{x}'| d^3x' \quad (10)$$

the Laplacian of which is related to U:

$$\nabla^2\chi = -2U \quad (11)$$

We thus find a solution for $\varphi^{(4)}$:

$$\varphi^{(4)} = \frac{1}{2}\partial_t^2\chi + 3\Phi_2 - \Phi_3 + 3\Phi_4 \quad (12)$$

where

$$\Phi_2 = \int \frac{\rho' U'}{|\mathbf{x}-\mathbf{x}'|} d^3x' \quad (13)$$

$$\Phi_3 = \int \frac{\rho' \Pi'}{|\mathbf{x}-\mathbf{x}'|} d^3x' \quad (14)$$

$$\Phi_4 = \int \frac{p'}{|\mathbf{x}-\mathbf{x}'|} d^3x' \quad (15)$$

We can also expand out $\partial_t^2\chi$:

$$\partial_t^2\chi = A + B - \Phi_1 \quad (16)$$

where

$$A = \int \frac{\rho'}{|\mathbf{x}-\mathbf{x}'|^3}(\mathbf{v}'\cdot(\mathbf{x}-\mathbf{x}'))^2 d^3x' \quad (17)$$

$$B = \int \frac{\rho'}{|\mathbf{x}-\mathbf{x}'|}(\mathbf{x}-\mathbf{x}')\cdot\frac{d\mathbf{v}'}{dt} d^3x' \quad (18)$$

$$\Phi_1 = \int \frac{\rho' v'^2}{|\mathbf{x}-\mathbf{x}'|} d^3x' \quad (19)$$

With $g_{\mu\nu} = (1 + \varphi^{(2)} + \varphi^{(4)})^2\eta_{\mu\nu}$ we thus find:

$$g_{00} = -1 + 2U - U^2 - A - B + \Phi_1 - 6\Phi_2 + 2\Phi_3 - 6\Phi_4 \quad (20)$$

$$g_{0i} = 0 \quad (21)$$

$$g_{ij} = \delta_{ij}(1-2U) \quad (22)$$

If we follow Will and transform into the Standard Post Newtonian gauge (with $\lambda_1 = -1/2$ and $\lambda_2 = 0$) then we find:

$$g_{00} = -1 + 2U - U^2 - 6\Phi_2 + 2\Phi_3 - 6\Phi_4 \quad (23)$$

$$g_{0i} = g_{i0} = 1/2(V_i - W_i) \quad (24)$$

$$g_{ij} = \delta_{ij}(1-2U) \quad (25)$$

where V_i and W_i are defined to be:

$$V_i = \int \frac{\rho' v_i'}{|\mathbf{x}-\mathbf{x}'|} d^3x' \quad (26)$$

$$W_i = \int \frac{\rho'\mathbf{v}'\cdot(\mathbf{x}-\mathbf{x}')(x-x')_i}{|\mathbf{x}-\mathbf{x}'|^3} d^3x' \quad (27)$$

By comparing to the PPN metric in Will we can see that $\gamma = -1, \beta = 1/2$ and all the other parameters are zero: $\xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$.

Transforming to the Standard Post Newtonian gauge does not affect the equations of motion for circular orbits. This is because in transforming from our original coordinates to the Standard Post Newtonian gauge we subtract $-\chi_{,00}$ from g_{00} and add $\chi_{,0j}$ to g_{0j}. When we compute the connection coefficients we get need $g_{00,j}$ and $g_{0j,0}$, thus both gauges end up contributing $\chi_{,00j}$. In addition the coordinates are unchanged by the gauge transformation in circular orbits, where the velocity is orthogonal to the separation vector: $\mathbf{v} \cdot \mathbf{x}_{ab} = 0$.

B. Orbits to 1st PN Order

To calculate the equations of motion for the bodies we first need to introduce a useful variable: the conserved density ρ^* (which will also be useful later when we do a multipole expansion of φ). We define ρ^*:

$$\rho^* = \rho\sqrt{-g}u^0 = \frac{(1+\varphi)^3\rho}{\sqrt{1-v^2}} \tag{28}$$

which follows an "Eulerian" conservation rule:

$$\partial_t \rho^* = -\partial_i(\rho^* v^i) \tag{29}$$

This allows for the general rule:

$$(d/dt)\int_V \rho^* f d^3x = \int_V \rho^*(df/dt)d^3x \tag{30}$$

and allows us to define the conserved (baryon) mass:

$$m = \int_V \rho^* d^3x, \qquad \frac{dm}{dt} = 0 \tag{31}$$

We then define the center of mass for the a'th body to be:

$$x_a^j = \frac{1}{m_a}\int_a \rho^* x^j d^3x \tag{32}$$

The acceleration of the center of mass in the x^j direction is then:

$$\ddot{x}_a^j = \frac{1}{m_a}\int_a \rho^* \frac{dv^j}{dt} d^3x \tag{33}$$

We thus need an expression for $\rho^* dv^j/dt$ to use in (33). We act on the divergence of the stress energy tensor $T^{\mu\nu}{}_{;\nu} = 0$ with the projection operator $Q^\alpha_\mu = u^\alpha u_\mu + \delta^\alpha_\mu$ to this end:

$$Q^j_\mu T^{\mu\nu}{}_{;\nu} = \rho h u^\nu u^j{}_{;\nu} + Q^{j\nu}p_{,\nu} = 0 \tag{34}$$

with $h = 1 + \varepsilon + p/\rho$ (the relativistic specific enthalpy). Expanding this out we find:

$$\rho^*\frac{dv^j}{dt} = -\rho^*\Gamma^j_{\alpha\beta}v^\alpha v^\beta - (u^0)^{-1}\rho^* v^j \frac{du^0}{dt} - (u^0)^{-2}\frac{\rho^*}{\rho h}Q^{j\nu}p_{,\nu} \tag{35}$$

Expanding further and keeping only post-Newtonian terms we find:

$$\begin{aligned}\rho^*\frac{dv^j}{dt} &= \rho^* U_{,j}[1 - v^2 + U] \\ &+ p_{,j}(-1 + 3U + \frac{1}{2}v^2 + \Pi + p/\rho^*) \\ &- \frac{1}{2}\rho^*(V^j - W^j)_{,0} + v^j(\rho^* U_{,0} - p_{,0}) \\ &+ \rho^*[-3\Phi_{2,j} + \Phi_{3,j} - 3\Phi_{4,j}]\end{aligned} \tag{36}$$

After plugging (36) into (33) we find that most of the terms cancel. We find that the acceleration scales as:

$$\ddot{x}_a^j = -\frac{M_b x_{ab}^j}{r_{ab}^3}\left(1 + \frac{M_b}{r_{ab}} - v_a^2 - \frac{3}{2}\left(\frac{\mathbf{v}_b\cdot\mathbf{x}_{ab}}{r_{ab}}\right)^2\right) + \frac{M_b(v_a^j - v_b^j)(\mathbf{v}_b\cdot\mathbf{x}_{ab})}{r_{ab}^3} \tag{37}$$

where M_b is the gravitational mass of star b:

$$M_b = \int_b \rho^*(1 + (1/2)\bar{v}^2 - (1/2)\bar{U} + \varepsilon)d^3x. \tag{38}$$

We thus find that to lowest order the binary is in Keplerian orbit, with corrections that scale as M/r_{ab} coming in at first post-Newtonian order. Note that this is a non-trivial result. Individual terms in (36) would give rise to deviations from Keplerian orbits at lowest order. For instance, the $V^j{}_{,0}$ term evaluates to:

$$-\frac{1}{2m_a}\int_a \rho^* V^j{}_{,0}d^3x = -\frac{1}{2m_a}\int_a \rho^* U_a U_{b,j}d^3x \sim \frac{1}{2}\frac{M_b x_{ab}^j}{r_{ab}^3}U_a \tag{39}$$

which by itself would cause a constant deviation (proportional to U_a) from the Keplerian value for the acceleration, even for arbitrarily large separations. However, since this theory is fully conservative, the other terms in (36) precisely cancel out this one, and there is thus no Nordtvedt effect at 1PN order.

In the following section we calculate the lowest order contribution to energy loss due to radiation, which kicks in at quadrupole order. We will thus only use the lowest order contribution to the acceleration: $\ddot{x}_a^j = -M_b x_{ab}^j/r_{ab}^3$ (which gives rise to Keplerian orbits). However, if we were to calculate the energy loss to the next level of accuracy (which would include octupole terms) we would need to include the corrections in (37).

IV. CALCULATION OF ORBITAL EVOLUTION

A. Energy Loss at Outer Boundary

In the first step of the calculation we find the energy radiated on a 2-sphere S^2 far out in the radiation zone. The key is provided by the energy-momentum complex $t^{\mu\nu}$ which follows the standard flat-space conservation law:

$$t^{\mu\nu}{}_{,\nu} = 0. \tag{40}$$

We can thus integrate $t^{00}{}_{,0} = -t^{0i}{}_{,i}$ inside the volume of the S^2:

$$\int t^{00}{}_{,0} d^3x = -\int t^{0i}{}_{,i} d^3x \tag{41}$$

or, using Gauss's law:

$$\partial_t E = -\int t^{0i} n_i dS. \tag{42}$$

We now need an expression for t^{0i}, which we get from equation (6):

$$t^{0i} = -\frac{1}{4\pi}\partial_t\varphi\partial_i\varphi. \tag{43}$$

$\partial_i\varphi$ can be transformed into $-n^i\partial_t\varphi$ since at large radius φ is approximately a spherical wave $\varphi \sim \sin(t-r)/r$. We thus find:

$$\partial_t E = -\frac{1}{4\pi}\int (\partial_t\varphi)^2 r^2 d\Omega \tag{44}$$

for the energy loss.

We now need an expression for $\partial_t\varphi$. We rewrite the equation of motion for the field (4) with the conserved mass density ρ^*:

$$\Box\varphi = 4\pi\rho^*(1-v^2)^{1/2}(1+\varepsilon - 3p/\rho). \tag{45}$$

This can be solved via a Green's function:

$$\varphi = -\int \frac{[\rho^{*\prime}(1-(1/2)v'^2+\varepsilon' - 3p'/\rho' + \mathcal{O}(v^4))]_{ret}}{|\mathbf{x}-\mathbf{x}'|} d^3x' \tag{46}$$

which is evaluated at the retarded time $t' = t - |\mathbf{x}-\mathbf{x}'|$. The $1/|\mathbf{x}-\mathbf{x}'|$ term can be expanded into:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{x^j x^{j\prime}}{r^3} + \ldots \tag{47}$$

We will only use the first term since we are working at large radius. Likewise the expansion:

$$r - |\mathbf{x}-\mathbf{x}'| = \frac{x^j x^{j\prime}}{r} + \frac{x^j x^k}{2r}\frac{(x^{j\prime}x^{k\prime} - r'^2\delta^j_k)}{r^2} + \ldots \tag{48}$$

will prove useful (again we just use the first term: $r - |\mathbf{x}-\mathbf{x}'| = x^j x^{j\prime}/r$). We now Taylor expand the numerator in (46) about $t-r$. Let $f(t',x') = f(t-|\mathbf{x}-\mathbf{x}'|, x') = [\rho^{*\prime}(1-(1/2)v'^2+\varepsilon'-3p'/\rho'+\mathcal{O}(v^4))]_{ret}$, then:

$$f(t',x') = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\partial^n}{\partial t'^n}f(t',x')\right]_{t'=t-r}\left(\frac{x^j x^{j\prime}}{r}\right)^n \tag{49}$$

Equation (46) can thus be expanded out in multipole moments:

$$\varphi = -\frac{1}{r}\left[M + n_i\partial_t D^i + \frac{1}{2}n_i n_j \partial_t^2 Q^{ij} + \ldots\right] \tag{50}$$

with

$$M = \int \rho^{*\prime}(1-(1/2)v'^2+\varepsilon'-3p'/\rho'+\mathcal{O}(v^4))d^3x' \tag{51}$$

$$D^i = \int \rho^{*\prime}(1-(1/2)v'^2+\varepsilon'-3p'/\rho'+\mathcal{O}(v^4))x^{i\prime}d^3x' \tag{52}$$

$$Q^{ij} = \int \rho^{*\prime}(1-(1/2)v'^2+\varepsilon'-3p'/\rho'+\mathcal{O}(v^4))x^{i\prime}x^{j\prime}d^3x' \tag{53}$$

We need $\partial_t\varphi$ for equation (44), and thus need to calculate the time derivatives of the three multipole moments. Looking ahead we find that the quadrupole contribution $\partial_t^3 Q^{ij}$ scales as v^5, so we will drop all terms that are smaller than this. First we calculate the time derivative of the monopole:

$$\partial_t M = \int \rho^{*\prime}\frac{d}{dt}(1-(1/2)v'^2+\varepsilon'-3p'/\rho')d^3x' \tag{54}$$

where we have used (30) to pass the time derivative through ρ^*. Note that if the two stars are on a quasi-circular orbit then the total time derivatives in the integrand are on the radiation reaction timescale, and thus contribute radiation at a far lower scale than v^5. They may contribute on elliptical orbits however. First we evaluate the $d\varepsilon/dt$ term, and to simplify we will pick $\Gamma = 2$, although the result holds in general. We find $\varepsilon = \kappa\rho = \kappa\rho^*(1-(1/2)v^2 - 3\varphi + \mathcal{O}(v^4))$. We only keep $\varepsilon = \kappa\rho^*$ since this is already a higher order term. The term reduces to $d\varepsilon/dt = -\kappa\rho^*\partial_i v^i$. Thus any monopole radiation of order v^5 from the $d\varepsilon/dt$ term stems from the "breathing" motion as the star expands and contracts during the elliptical orbit. However, Nordström's second theory is fully conservative, and thus the stellar matter undergoes no "breathing" motion: the central density is constant to 1PN order. In fact, this justifies our use of the "Hydro-without-Hydro" assumption where we hold the stars to be rigid bodies throughout the evolution. Thus the entire $d\varepsilon/dt$ term does not contribute at v^5 order. The same holds for the $d(3p/\rho)/dt$ term. We thus find the monopole contribution to the radiation:

$$\partial_t M = -\frac{1}{2}\int \rho^{*\prime}\partial_t(v'^2)d^3x' + \mathcal{O}(v^7) \tag{55}$$

which enters in at v^5 order if the orbit is eccentric and is much smaller otherwise (note also that the total time derivative has been switched to a partial derivative since the velocity is now essentially constant throughout the star). It is also convenient that radial pulsations do not contribute at the order we are considering since this allows us to treat the stars as point bodies in later calculations.

We now calculate the contribution from the dipole via $\partial_t^2 D^j$. After applying two time derivatives we find:

$$\partial_t^2 D^j = \int \rho^{*\prime}[(1 - v^{\prime 2} + \varepsilon' - 3p'/\rho')\frac{dv^{j\prime}}{dt} - \frac{d}{dt}\left(\frac{dv^{i\prime}}{dt}v^{i\prime}\right)x^{j\prime} - 2\frac{dv^{i\prime}}{dt}v^{i\prime}v^{j\prime}]d^3x' \tag{56}$$

At first glance the $\int \rho^{*\prime}(dv^{j\prime}/dt)d^3x'$ term appears to contribute at v^4 order, but in fact the term is zero due to Newton's third law. Any corrections would enter at v^6 order at the soonest, and all the other terms in equation (56) also scale as v^6. Thus the dipole does not radiate to the order we are considering:

$$\partial_t^2 D^j = 0 + \mathcal{O}(v^6) \tag{57}$$

The final multipole moment we consider is the quadrupole. After similar calculations it reduces to:

$$\partial_t^3 Q^{ij} = \partial_t^3 \int \rho^{*\prime} x^{i\prime} x^{j\prime} d^3x' + \mathcal{O}(v^7) \tag{58}$$

Putting the terms together we find:

$$\partial_t E = -\frac{1}{4\pi}\int (\partial_t M + (1/2)n_i n_j \partial_t^3 Q^{ij})^2 d\Omega \tag{59}$$

for the rate of energy loss. With the integrals:

$$\int n_i n_j d\Omega = \frac{4\pi}{3}\delta^{ij} \tag{60}$$

$$\int n_i n_j n_k n_l d\Omega = \frac{4\pi}{15}[\delta^i_j \delta^k_l + \delta^i_k \delta^j_l + \delta^i_l \delta^j_k]$$

we finally find:

$$\partial_t E = -(\partial_t M)^2 - \frac{1}{3}\partial_t M \partial_t^3 Q^{ii} - \frac{1}{60}(\partial_t^3 Q^{ii})^2 - \frac{1}{30}(\partial_t^3 Q^{ij})^2. \tag{61}$$

This reduces to:

$$\partial_t E = -\frac{1}{30}(\partial_t^3 Q^{ij})^2. \tag{62}$$

for zero eccentricity, which is six times smaller than the value found in GR.

B. Angular Momentum Loss at Outer Boundary

We perform a similar calculation to find the rate at which the angular momentum is radiated. With equation (40) and:

$$L_i = \epsilon_{ijk}\int x^j t^{0k} d^3x \tag{63}$$

we find that

$$\frac{dL_i}{dt} = -\epsilon_{ijk}\int x^j t^{kl}{}_{,l} d^3x = -\epsilon_{ijk}\int x^j t^{kl} n^l dS \tag{64}$$

The stress energy tensor components are:

$$t^{kl} = \frac{1}{4\pi}\partial_k\varphi\partial_l\varphi - \frac{1}{8\pi}\delta^k_l \eta^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi \tag{65}$$

Thus

$$\frac{dL_i}{dt} = -\frac{1}{4\pi}\epsilon_{ijk}\int x^j \partial_k\varphi\partial_l\varphi n^l dS. \tag{66}$$

The $\eta^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi$ term is zero due to the anti-symmetry of ϵ_{ijk}.

It is tempting at first to transform both of the spatial partial derivatives into time derivatives via $\partial_i\varphi = -n^i\partial_t\varphi$, but the resulting expression is zero due to the anti-symmetry of ϵ_{ijk}. We will thus only switch one of them into a time derivative:

$$\frac{dL_i}{dt} = -\frac{1}{4\pi}\epsilon_{ijk}\int x^j \partial_k\varphi \frac{1}{r}(\partial_t M + (1/2)n_a n_b \partial_t^3 Q^{ab})dS. \tag{67}$$

while the other spatial derivative needs to be calculated out to higher order to find the first non-vanishing contribution (in fact, ϵ_{ijk} determines that it is the $\partial_k\varphi$ term that needs to be expanded to higher order, while the $\partial_l\varphi$ is approximated with the time derivative). When we apply ∂_k to (50) one of the terms we get is:

$$\partial_k\left(-\frac{1}{r}\right)M = \frac{x^k}{r^3}M. \tag{68}$$

However when we insert this term into (67) we get zero, again due to the anti-symmetry of ϵ_{ijk}, and in fact all the terms stemming from derivatives of powers of r are zero for the same reason. This leaves:

$$\partial_k\varphi = -\frac{1}{r^2}\partial_t D^k - \frac{x^c}{r^3}\partial_t^2 Q^{ck} \tag{69}$$

Examination of $\partial_t D^k/r^2$ shows that in general it is of order v^3/r. However, we work in the center of mass coordinate system so the lowest order contribution to this term is in fact zero, and lowest order that any corrections can appear is at v^5/r. In turn, the quadrupole term scales as v^4/r, therefore we will keep only it.

Equation (67) now becomes:

$$\frac{dL_i}{dt} = \frac{1}{4\pi}\epsilon_{ijk}\int n^j n^c \partial_t^2 Q^{ck}(\partial_t M + (1/2)n_a n_b \partial_t^3 Q^{ab})d\Omega. \tag{70}$$

Using the integrals (60) again we finally find:

$$\frac{dL_i}{dt} = \frac{1}{15}\epsilon_{ijk}\partial_t^3 Q^{jc}\partial_t^2 Q^{ck} \tag{71}$$

where the $\partial_t M$ term has dropped out, again due to ϵ_{ijk}. This expression is precisely one sixth of the value given by general relativity.

C. Application to Keplerian Orbits

We now apply the equations for the rates of energy (61) and angular momentum loss (71) to a binary star system in Keplerian orbit. The stars have gravitational masses m_1 and m_2, a semi-major axis a and eccentricity e. The separation d between the two stars is determined by the phase ϕ:

$$d = \frac{a(1-e^2)}{1+e\cos(\phi)} \tag{72}$$

which thus also gives the distances of the stars from the center of mass:

$$d_1 = \left(\frac{m_2}{m_1+m_2}\right) d, \qquad d_2 = \left(\frac{m_1}{m_1+m_2}\right) d. \tag{73}$$

The non-zero quadrupole moment components are:

$$\begin{aligned} Q_{xx} &= \mu d^2 \cos^2\phi \\ Q_{yy} &= \mu d^2 \sin^2\phi \\ Q_{xy} = Q_{yx} &= \mu d^2 \sin\phi\cos\phi \end{aligned} \tag{74}$$

with $\mu = m_1 m_2/(m_1+m_2)$. Note also that for consistency we are expressing the Q_{ij} in terms of the gravitational mass instead of the rest mass $m^* = \int \rho^* d^3x$ as used in the previous section. Switching between the two involves corrections of order v^2 which do not affect our lowest order calculation. We need their second and third time derivatives for use in (61) and (71). Making use of the angular velocity:

$$\omega = \frac{[(m_1+m_2)a(1-e^2)]^{1/2}}{d^2} \tag{75}$$

we find the second derivatives to be:

$$\begin{aligned} \frac{d^2Q_{xx}}{dt^2} &= -\gamma(4\cos(2\phi) + e(3\cos(\phi)+\cos(3\phi))) \\ \frac{d^2Q_{yy}}{dt^2} &= \gamma(4\cos(2\phi) + e(4e + 7\cos(\phi)+\cos(3\phi))) \\ \frac{d^2Q_{xy}}{dt^2} = \frac{d^2Q_{yx}}{dt^2} &= -2\gamma\sin(\phi)(4\cos(\phi) + e(3+\cos(2\phi))) \end{aligned} \tag{76}$$

where γ defined as:

$$\gamma = \frac{m_1 m_2}{2a(1-e^2)} \tag{77}$$

and third derivatives are:

$$\begin{aligned} \frac{d^3Q_{xx}}{dt^3} &= \beta(1+e\cos\phi)^2(2\sin 2\phi + 3e\sin\phi\cos^2\phi) \\ \frac{d^3Q_{yy}}{dt^3} &= -\beta(1+e\cos\phi)^2(2\sin 2\phi + e\sin\phi(1+3\cos^2\phi)) \\ \frac{d^3Q_{xy}}{dt^3} = \frac{d^3Q_{yx}}{dt^3} &= \\ -\beta(1+e\cos\phi)^2&(2\cos 2\phi - e\cos\phi(1-3\cos^2\phi)) \end{aligned} \tag{78}$$

with β defined as:

$$\beta^2 = \frac{4m_1^2 m_2^2(m_1+m_2)}{a^5(1-e^2)^5}. \tag{79}$$

Finally we need $\partial_t M$. We have:

$$\partial_t M = -\frac{1}{2}(m_1\partial_t v_1^2 + m_2\partial_t v_2^2) = \frac{1}{2}\beta e\sin\phi(1+e\cos\phi)^2 \tag{80}$$

Putting the parts together we find:

$$\partial_t E = -\frac{1}{15}\beta^2(1+e\cos\phi)^4(4+2e^2+8e\cos\phi+2e^2\cos^2\phi) \tag{81}$$

and

$$\partial_t L_z = -\frac{2}{15}\beta\gamma(1+e\cos\phi)^3(8-2e^2+12e\cos\phi+6e^2\cos^2\phi) \tag{82}$$

In general relativity the energy can't be localized at individual points in space, and therefore the expressions for the dE/dt and dL_z/dt need to be averaged over an orbit. While the energy can be localized in Nordström's theory, we will also average (81) and (82) in order to compare with general relativity. This is valid since the orbital parameters change little over the span of an orbit (for moderately large separations). For the energy we average:

$$\langle \dot{E} \rangle = \frac{\int_0^T \dot{E}dt}{T} = \frac{\int_0^{2\pi} \dot{E}(dt/d\phi)d\phi}{\int_0^{2\pi}(dt/d\phi)d\phi} \tag{83}$$

and find:

$$\langle \dot{E} \rangle = -\frac{16}{15}\frac{m_1^2 m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left[1+\frac{13}{4}e^2+\frac{7}{16}e^4\right] \tag{84}$$

which compares to the value given by Peters [3] for general relativity:

$$\langle \dot{E} \rangle = -\frac{32}{5}\frac{m_1^2 m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left[1+\frac{73}{24}e^2+\frac{37}{96}e^4\right] \tag{85}$$

Likewise for the angular momentum we average to find:

$$\langle \dot{L_z} \rangle = -\frac{16}{15}\frac{m_1^2 m_2^2(m_1+m_2)^{1/2}}{a^{7/2}(1-e^2)^2}\left[1+\frac{7}{8}e^2\right] \tag{86}$$

which again is one sixth the value found in general relativity.

Finally, by using

$$a = -m_1 m_2/2E, \tag{87}$$

$$L^2 = m_1^2 m_2^2 a(1-e^2)/(m_1+m_2) \tag{88}$$

we can convert $\langle \dot{E} \rangle$ $\langle \dot{L_z} \rangle$ into $\langle \dot{a} \rangle$ and $\langle \dot{e} \rangle$:

$$\left\langle \frac{da}{dt} \right\rangle = -\frac{32}{15}\frac{m_1 m_2(m_1+m_2)}{a^3(1-e^2)^{7/2}}\left[1+\frac{13}{4}e^2+\frac{7}{16}e^4\right] \tag{89}$$

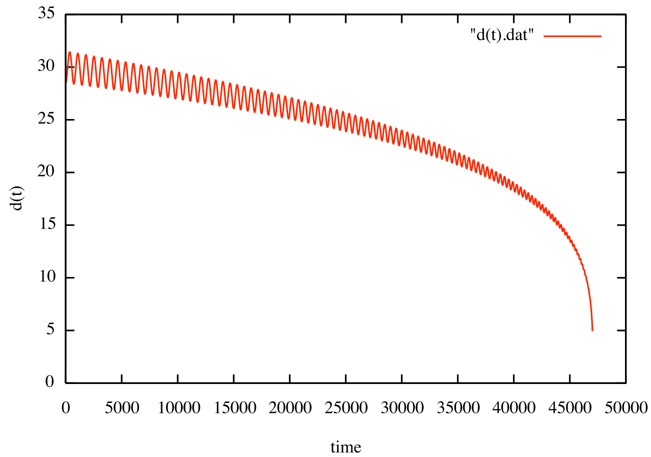

FIG. 1: Binary separation as a function of time.

and

$$\left\langle \frac{de}{dt} \right\rangle = -\frac{18}{5}\frac{m_1 m_2(m_1+m_2)}{a^4(1-e^2)^{5/2}}e\left[1+\frac{7}{18}e^2\right] \tag{90}$$

We can solve for $a(t)$ exactly for the special case $e = 0$. We find:

$$a(t) = \left(a^4(0) - \frac{128}{15}m_1 m_2(m_1+m_2)t\right)^{1/4} \tag{91}$$

An example orbital evolution based on equations (90) and (89) is shown in figure (1). The system is given an initial separation of $30M$ and an eccentricity of 0.05 and then evolved forward until merger. The overall profile of the inspiral is quite close to the expression given in (91) due to the low eccentricity. We also note that the orbit circularizes over time, as it does in general relativity. This inspiral can be compared to the numerical inspirals we describe in the companion paper. A plot of e as a function of a for a numerical inspiral as compared to the theoretical profile is given in figure (2).

V. CONCLUSIONS

Nordström's second theory is a weak emitter of gravitational waves compared to general relativity, which is itself a weakly radiating theory. In the case of quasi-circular orbits, Nordström's theory radiates energy and angular momentum six times more slowly than in GR. This is somewhat surprising at first, since in general scalar theories can emit monopole radiation. This low radiative power is a reflection of the conservation laws that hold at lowest order, similar to those in GR, so that quadrupole level terms are the first to appear. As Will points out in [7], most alternative theories contain dipole radiation since they violate the Strong Equivalence Principle (SEP) and their gravitational and inertial masses differ. Nordström's theory likely also violates the SEP, but not at 1PN, so any Nordtvedt effects occur at high order.

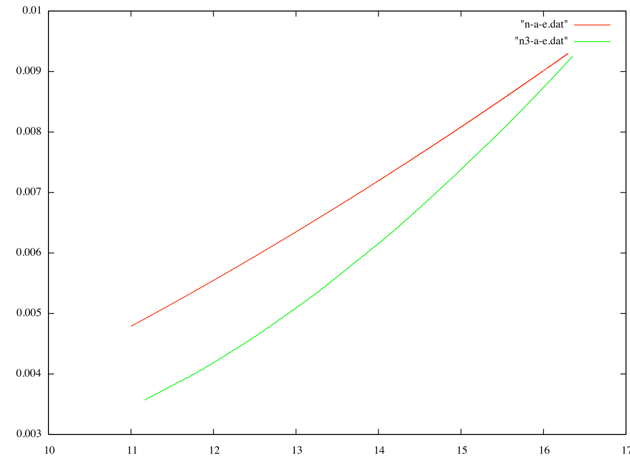

FIG. 2: Eccentricity e as a function of semimajor axis a for a numerical inspiral and a theoretical inspiral with the same Keplerian parameters.

As noted before, binary orbits will circularize over time in Nordström's theory. In our numerical simulations we thus place the stars in quasi-circular orbit and then evolve the system forward in time. The numerical techniques we have developed allow for long stable evolutions composed of many orbits. We compare the inspirals produced by the code to the predicted profiles given in (91) and find that they match to high accuracy. Thus the numerical and analytical methods mutually confirm each other. This success demonstrates that Nordström's theory is quite useful for developing numerical techniques to be used in numerical relativity. The next step is to apply the techniques developed and insight derived from Nordström's theory to the problem of binary inspirals in general relativity.

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