Probing Space-like Singularities With Quantum Fields

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Some mathematical structures taken from earlier work
with Corichi and Kesavan, Phys. Rev. D 102, 023512 (2020).

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Preamble

• Space-like singularities are taken to be the absolute beginning or end of space-time in GR. Geodesics of test particles end there. Tidal forces between them become infinite. But what if one uses quantum probes? It has been long argued that singularities may be tamer for physically more realistic probes. Examples:

Horowitz and Marolf (1995): In certain static space-times with time-like singularities: Dynamics of test quantum particles well-defined in some cases.)

Ishibashi & Hoyasa (1999); Stalker& Tahvildar-Zadeh (2004):Dynamics of classical fields well-defined across certain time-like singularities.

Hofmann and Schneider (2015): The Schwarzschild space-like singularity probed with test quantum fields. Found to be tame. But the arguments are formal; infinite number of degrees of freedom did not receive due care.

• Goal: Revisit the issue for the physically most important dynamical singularities with precision required to handle the infinite number of DOF of QFT carefully. Due to time limitation, this talk focuses on the Big Bang/Big Crunch singularities in the FLRW models. Very similar results on the Schwarzschild singularity.

• Main Question: Do test quantum fields $\hat{\phi}(x)$ and observables constructed from them such as $\langle\hat{\phi}(x)\,\hat{\phi}(x\,')\rangle$, $\langle\hat{\phi}^2(x)\rangle_{\rm ren}$, $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$ remain regular in the sense of QFT across the big bang?

• Apparent Problem: In the Friedmann-Lemaître-Robertson-Walker (FLRW) space-times, $\hat{\phi}(x)$ is given by

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[A(\vec{k})e(k,\eta) + A^\dagger(-\vec{k})e^\star(k,\eta)\right]e^{i\vec{k}\cdot\vec{x}}$$

but the mode functions $e(k,\eta)$ diverge at the Big Bang!

• Recall, however, that already in Minkowski space, $\hat{\phi}(x)$ is not an operator but an operator-valued (tempered) distribution (OVD): $\int_{\mathring{M}} \hat{\phi}^\circ(x)\, f(x)\, \mathrm{d}^4\mathring{V}$ is an operator of the Fock space on all test functions $f(x) \in \mathcal{S}$, the Schwartz space. (Since $f \in C_0^\infty \Rightarrow f \in \mathcal{S}$, a tempered distribution is in particular an (ordinary) distribution.)

• Main results: In all cosmological FLRW models, $\hat{\phi}(x)$ remains well-defined across the big bang as OVDs. I will focus on the k=0, radiation and dust filled universes and summarize results in more general cases. In particular, the expectation values $\langle\hat{\phi}(x)\,\hat{\phi}(x\,')\rangle$ are well-defined bi-distributions in the extended space-time. Interestingly, correlations between fields evaluated at spatially and temporally separated points exhibit an asymmetry that is reminiscent of the Belinskii, Khalatnikov, Lifshitz behavior. The renormalized products of fields $\langle\hat{\phi}^2(x)\rangle_{\rm ren}$ and $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$ also remain well-defined as distributions. Conformal coupling is not necessary for these considerations to hold. Thus, when probed with observables associated with quantum fields, the big bang (and the big crunch) singularities are quite harmless.

• LQG perspective: First reaction: Why care? We know that non-perturbative quantum geometry effects resolve space-like singularities. True, BUT:

(i) QFT in CST has a vast Damian of applicability. Usually one just says it fails in the Planck regime. How exactly does LQG cure semi-classical gravity? To answer this, we need to know: What exactly fails in QFT?

(ii) In some LQG scenarios of BH evaporation, there is a large portion of the 'transition surface' in which the geometry is 'adiabatic, slowly varying'. What happens to the stress-energy tensor of the modes that fell into the DH as one approaches Planck curvature? What is their back-reaction likely to do to the geometry?

Organization

1. Nature of QFs in Minkowski space-time
2. FLRW space-times: Radiation filled universes
3. FLRW space-times: Dust filled universes
4. Summary, Generalizations and Broader Perspective

1. Nature of Quantum Fields: Minkowski Space

- In Minkowski space $(\mathring{M}, \mathring{g}_{ab})$ is an (OVD) :

$$\hat{\phi}^\circ(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left[\hat{A}(\vec{k})\frac{e^{-i\omega\eta}}{\sqrt{2\omega}} + \hat{A}^\dagger(-\vec{k})\frac{e^{i\omega\eta}}{\sqrt{2\omega}}\right] e^{i\vec{k}\cdot\vec{x}} \quad \text{with } x \equiv (\vec{x}, \eta)$$

satisfying $(\mathring{\Box} - m^2)\hat{\phi}^\circ(x) = 0$. That is, $\int_{\mathring{M}} \mathrm{d}^4\mathring{V}\, \hat{\phi}^\circ(x)(\Box - m^2)f(x) = 0$, for all test functions $f(x) \in \mathcal{S}$, the Schwartz space.

- The distributional character is not a mere technicality but is conceptually important. For example in

$$[\hat{\phi}^\circ(x),\, \hat{\phi}^\circ(x')] = i\hbar\,(G_{\rm ad} - G_{\rm ret})(x,\, x')\,\hat{I} \quad \text{and}$$

$$\langle\hat{\phi}^\circ(x)\,\hat{\phi}^\circ(x')\rangle_\circ = \frac{\hbar}{4\pi^2}\,\frac{1}{|\vec{x}-\vec{x}\,'|^2-((\eta-\eta\,')-i\epsilon)^2}$$

the right sides are genuine distributions; not functions. Meaning of $i\epsilon$: first integrate $\langle\hat{\phi}^\circ(x)\,\hat{\phi}^\circ(x')\rangle_\circ$ smeared with a test function *and then* take the limit $\epsilon \to 0$. More importantly, products $\hat{\phi}^2(x)$ have to be regularized precisely because $\hat{\phi}(x)$ is an OVD. The textbook terminology of 'field operators' and '2-point functions' (and Dirac 'δ function') can be very misleading if taken literally.

- So the question is: Do quantum fields $\hat{\phi}(x)$ continue to be well-defined across the big-bang as OVDs ? For example, in 3-d, $1/r$ is singular as a function but C^∞ as a tempered distribution (satisfying $\vec{\nabla}^2(1/r) = 4\pi\delta^3(\vec{r})$).

2. FLRW space-times

• Extending space-time beyond the Big Bang: Recall that every FLRW space-time (M, g_{ab}) is conformally flat: If K=0:

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b \;=\; a^2(\eta)\,\mathring{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b \;\equiv\; a^2(\eta)\,(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2) \quad \text{with } a(\eta) = a_\beta\,\eta^\beta;\ \beta \geq 0.$$

$\eta > 0$ on M, and the big bang corresponds to $\eta = 0$. We can extend $a^2(\eta)$ and hence g_{ab} to the full Minkowski manifold $\mathring{M}$ with $\eta \in (-\infty, \infty)$ as a continuous tensor field (degenerate at $\eta = 0$). Similarly for Schwarzschild (D'Ambrosio & Rovelli).

• Systematic Rationale: The traditional (ADM) Hamiltonian framework for the initial value problem of full GR based on 3-metrics and their momenta breaks down if the 3-metric becomes degenerate. But equations satisfied by ('connection' variables) do not. The two are equivalent when the 3-metric is non-degenerate but connection variables allow for degenerate metrics. (AA, Henderson & Sloan). In FLRW (as well as Bianchi models and the Schwarzschild solution) this procedure enables one to evolve across the singularity unambiguously (Koslowski, Mercati & Sloan; Mercati; AA & Valdes).

In FLRW models, the extension yields just the simple prescription given above. As a tensor field, g_{ab} is smooth if $\beta \in \mathbb{Z}$ (but not invertible at $\eta = 0$) .

QFT in FLRW Space-times

• For definiteness, consider the massless scalar field: $\Box\phi = 0$. Since $a(\eta) = a_\beta\,\eta^\beta$, on M, $\phi^\circ = a(\eta)\,\phi(x)$ satisfies a simple equation with respect to the Minkowski metric $\mathring{g}_{ab}$ in presence of a 'universal' time dependent potential:

$$(\mathring{\Box} - V(\eta))\phi^\circ(x) = 0 \qquad \text{with} \qquad V(\eta) = \beta(\beta-1)/\eta^2.$$

The operator algebra can be constructed unambiguously but we need new input to build a 'Fock' representation since the potential is time dependent.

Rigorous Result: Because of the form of the potential, one can introduce a canonical $\pm$ frequency decomposition (i.e. a canonical Kähler structure) on the space of classical solutions and write the general solution as

$$\phi^\circ(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[z(\vec{k})\mathring{e}(k,\eta) + z^\star(-\vec{k})\mathring{e}^\star(k,\eta)\right]e^{i\vec{k}\cdot\vec{x}}$$

where $\mathring{e}(k,\eta)$ are the positive-frequency modes and $z(\vec{k})$ are regular coefficients (in $\mathcal{S}$). Then the putative OVD on FLRW space-time is given by:

$$\hat{\phi}(x) = \frac{1}{a(\eta)}\hat{\phi}^\circ(x) = \frac{1}{a(\eta)}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\hat{A}(\vec{k})\mathring{e}(k,\eta) + \hat{A}^\dagger(-\vec{k})\mathring{e}^\star(k,\eta)\right]e^{i\vec{k}\cdot\vec{x}}.$$

• The mode functions $\mathring{e}(k,\eta)$ are explicitly known. Generically they diverge at $\eta = 0$. For example, for dust ($\beta = 2$), they are $(e^{ik\eta}/\sqrt{2k})(1 - i/k\eta)$. As functions, they diverge at the big bang. And there is another $1/a(\eta)$ overall factor in $\hat{\phi}(x)$. Is $\hat{\phi}(x)$ nonetheless well-defined as an OVD across the big bang on full $\mathring{M}$?

$K = 0$ Radiation-filled Universe

- Is $\hat{\phi}(f) = \int_{\mathring{M}} \mathrm{d}^4V\, \hat{\phi}(x) f(x)$ well-defined on the extended space-time, and satisfy $\int_{\mathring{M}} \mathrm{d}^4V \hat{\phi}(x)(\Box f) = 0\ \forall f \in \mathcal{S}$ i.e. for all $\eta \in (-\infty, \infty)$?

- Radiation-filled universe: We have $a(\eta) = a_1\, \eta$ (i.e. $\beta = 1$) $\Rightarrow V(\eta) = 0$, whence, $\hat{\phi}^\circ(x) = a(\eta)\, \hat{\phi}(x)$ now satisfies $\mathring{\Box}\hat{\phi}^\circ = 0$ in Minkowski space! Hence mode functions $\mathring{e}(k, \eta)$ same as in Minkowski space; trivially regular for $\eta \in (-\infty, \infty)$. But the physical field on FLRW space-time is $\hat{\phi}(x) = a^{-1}(\eta)\, \hat{\phi}^\circ(x)$ and $a(\eta) = 0$ at $\eta = 0$! How could it then be regular on Minkowski Fock space?

Answer: The physical volume element is $\mathrm{d}^4V = a^4\, \mathrm{d}^4x$. Hence:

$$\hat{\phi}(f) = \int_{\mathring{M}} \mathrm{d}^4V\, \hat{\phi}(x) f(x) = \int_{\mathring{M}} \mathrm{d}^4x\, \hat{\phi}^\circ(x)\, (a^3(\eta) f(x))$$

and $a^3(\eta) f(x) \in \mathcal{S}$ if $f \in S$. Hence $\hat{\phi}(f)$ is in fact a well-defined operator on the Minkowski Fock space $\Rightarrow$ $\hat{\phi}(x)$ well defined OVD for all $\eta \in (-\infty, \infty)$!

Next, the expectation value of the product of fields

$$\langle \hat{\phi}(x)\, \hat{\phi}(x')\rangle = \frac{1}{a_1^2 \eta\eta'} \langle \hat{\phi^\circ}(x)\, \hat{\phi^\circ}(x')\rangle_\circ = \frac{1}{a_1^2 \eta\eta'}\ \frac{\hbar}{4\pi^2}\ \frac{1}{r^2 - (t - i\epsilon)^2}$$

(where $r = |\vec{x} - \vec{x}\,'|$ and $t = \eta - \eta'$) is also a well-defined bi-distribution because $\mathrm{d}^4V = (a_1^4 \eta^4)\ \mathrm{d}^4x$ and d^4x is well defined on all of $\mathring{M}$.

Renormalized operator products

• In the radiation filled case, $\langle\hat{\phi}^2(x)\rangle_{\rm ren} = 0$ (could have been expected on dimensional grounds since $R = 0$ in this case). But $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$ is non-trivial:

$$\langle\hat{T}_{ab}(x)\rangle_{\rm ren} = \frac{\hbar}{720\pi^2 a_1^2\,\eta^6}\,\nabla_a\eta\,\nabla_b\eta + \frac{\hbar}{576\pi^2\,a_1^2\,\eta^6}\,\mathring{g}_{ab}.$$

Since $\mathrm{d}^4V = (a_1^4\,\eta^4)\,\mathrm{d}^4x$ the product $a^4(\eta)\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$ diverges at the big-bang as η^{-2}. Is it nonetheless a well-defined distribution on the extended space-time?

• Two basic properties: (i) Every locally integrable function is a (Schwartz) distribution; and (ii) (Schwartz) Distributions are infinitely differentiable. Now, $\ln|\eta|$ is a locally integrable function (since $\int \ln\eta\,\mathrm{d}\eta = \eta\,\ln\eta - \eta$). Hence all its derivatives are tempered distributions! Therefore, as a distribution,

$$\underline{\eta}^{-m} := \frac{(-1)^{m-1}}{(m-1)!}\;\frac{\mathrm{d}^m\;\ln|\eta|}{\mathrm{d}\eta^m} \qquad \text{i.e.,} \qquad \underline{\eta}^{-m} : f \;\to\; -\frac{1}{(m-1)!}\int \mathrm{d}\eta\;\ln|\eta|\,\frac{\mathrm{d}^m\,f}{\mathrm{d}\eta^m}\,.$$

is well-defined on $(\mathbb{R}, \mathrm{d}\eta)$. They satisfy intuitively expected properties, including

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\,\underline{\eta}^{-m} = -m\,\underline{\eta}^{-m-1} \qquad \text{and,} \qquad \eta\,\underline{\eta}^{-m} = \underline{\eta}^{-m+1}\;.$$

• This is why even when $a^4(\eta)\,\langle\hat{\phi}^2(x)\rangle_{\rm ren}$ & $a^4(\eta)\,\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$ diverge as functions (as they do in general FLRW models), they can be well-defined tempered distributions on the extended space-time. Recall: Even in Minkowski space-time, observables of quantum fields are tempered distributions, not functions. Cannot ask them to be better at singularities!!

3. $K = 0$ Dust-filled FLRW Universe

• More interesting case: Dust-filled universe where $a(\eta) = a_2\,\eta^2$. Hence $(1/a(\eta))$ diverges faster at $\eta = 0$ and mode functions $\mathring{e}_k(\eta) = (e^{ik\eta}/\sqrt{2k})(1 - i/k\eta)$ also diverge at the big bang (unlike in the radiation-filled case).

• Now, the 1-particle Hilbert space is built out of solutions:

$$\phi(x) \;=\; \frac{1}{a(\eta)} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left[z(\vec{k})\mathring{e}(k,\eta) + z^\star(-\vec{k})\mathring{e}^\star(k,\eta)\right] e^{i\vec{k}\cdot\vec{x}} \text{ with } z(\vec{k}) \in \tilde{\mathcal{S}}$$

and they all diverge at $\eta = 0$. So how can there be a well-defined Fock space? There is, because the 1-particle norm $||\phi(x)||$ is perfectly finite (and non-zero) at $\eta = 0$ because the divergence in the value of $\phi(x)$ is precisely compensated by the vanishing of the 3-volume element there!

$$||\phi(x)||^2 \;=\; \frac{1}{\hbar}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\, |z^\star(\vec{k})|^2.$$

This is analogous to the QM fact that while $\Psi(\vec{x}) := (1/r)e^{-\alpha r}$ is divergent as a function, it represents a well-defined state in $\mathcal{H} := L^2(\mathbb{R}^3)$, because the volume element d^3x goes as r^2.

• Since $\mathrm{d}^4V = a_2^4\eta^8\,\mathrm{d}^4x$, $\hat{\phi}(x)$ is a well-defined OVD. (However, there is an infrared subtlety (Ford and Parker (1977)). Already for $\eta > 0$, the action of $\hat{\phi}(x)$ is well-defined on a co-dimension 1 subspace $\mathcal{S}_1$ of $\mathcal{S}$ and there is a 1-parameter freedom in extending its action on full $\mathcal{S}$, representing an infrared cutoff ℓ. But this cutoff has nothing to do with the big bang. Once $\hat{\phi}(x)$ is defined for $\eta > 0$ with an IR regulator, it continues to be well-defined for $\eta \le 0$).

Dust filled Universe: 'BKL Behavior'

- The expectation value of the product of fields is given by

$$\langle \hat{\phi}(x)\,\hat{\phi}(x')\rangle = \frac{\hbar}{4\pi^2}\,\frac{1}{a_2^2\eta^2\eta'^2}\Big[\frac{1}{(r^2-(t-i\epsilon)^2)} + \frac{1}{2\eta\eta'}[2(1-\gamma)+\ln\frac{r^2-(t-i\epsilon)^2}{\ell^2}]$$

(where $r = |\vec{x}-\vec{x}'|$; $t = |\eta-\eta'|$ and γ the Euler-Mascheroni constant.) It is a well-defined bi-distribution, i.e., $\int_{\mathring{M}} \mathrm{d}^4V\, \mathrm{d}^4V'\, \langle \hat{\phi}(x)\,\hat{\phi}(x')\rangle\, f_1(x) f_2(x)$ is well-defined because $\mathrm{d}^4V = a_2^4\eta^8\mathrm{d}^4x$, and d^4x is well-defined on all of $\mathring{M}$.

Now, for space-like and time-like separated points, one interprets $\langle \hat{\phi}(x)\,\hat{\phi}(x')\rangle$ as a 'correlation function'. In Minkowski space, correlations decay as $1/\mathrm{Dist}^2$ for both space-like and time-like separations.

- Now, there is an interesting space vs time asymmetry as one approaches the singularity: Consider points that are space-like or time-like separated by a fixed proper (geodesic) distance D. As one approaches the big bang, space-like correlations dominate over time-like ones: $\lim_{\eta_\circ\to 0} \frac{\langle\hat{\phi}(\vec{x},\eta_o)\,\hat{\phi}(\vec{x}',\eta_\circ)\rangle}{\hat{\phi}(\vec{x}_\circ,\eta_o)\hat{\phi}(\vec{x}_\circ,\eta)} = \infty$ (as $2D/a_2\eta^3$). Strong correlations $\sim$ smaller variations $\Rightarrow$ smaller derivatives. Therefore, "time derivatives dominate over space-derivatives" as in the well-known BKL behavior of GR. But one has to keep in mind that conceptually these are quite different statements: this behavior refers to test quantum fields on a given FLRW background while the BKL behavior refers to the gravitational field itself.

Dust filled Universe: Operator-Products

- $\hat{\phi}(x)$ is a 'dimension 1' OVD, while $\langle\hat{\phi}^2(x)\rangle_{\rm ren}$ is a 'dimension 2' OVD and $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$ a 'dimension 4' OVD. So, a priori the fact that $\hat{\phi}(x)$ is well-behaved across the big bang does not mean that these operator-products would be well-defined. Are they?

- Older works (Bunch, Davies, ...) imply $\langle\hat{\phi}^2(x)\rangle_{\rm ren} = \frac{\hbar\mathcal{R}}{288\pi^2}\,(5 - 2\ln\frac{2\mathcal{R}}{3a_2\ell^3\mu^3})$. At the big bang, $\mathcal{R}\sim 1/\eta^6$ is divergent as a function (but a C^∞ tempered distribution). Since $\mathrm{d}^4V = a_2^4\eta^8\mathrm{d}^4x$, $\eta^8\langle\hat{\phi}^2(x)\rangle_{\rm ren}(x)$ is in fact a C^2 function! Unlike in the radiation-filled case, it does not vanish because $R\neq 0$.

- Older works also provide the expression of $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$. Being a 'dimension 4' OVD, it involves products and second derivatives of curvature tensors. The explicit expression is long but has the simple form $\langle\hat{T}_{ab}(x)\rangle_{\rm ren} = T_1(\eta)\nabla_a\eta\nabla_b\eta + T_2(\eta)\mathring{g}_{ab}$, where the most divergent term in T_1 and T_2 go as $(\eta^{-8}\ln|\eta|)$. Now, $\mathrm{d}^4V\sim\eta^8$ and $\eta^8T_1\sim\eta^8T_2\sim\ln|\eta|$, which is a locally integrable function and hence in particular, a C^∞ tempered distribution!

- Summary: Dynamics of $\hat{\phi}$ is much more non-trivial in the dust-filled case: It represents the generic case where the scalar curvature does not vanish. Still, $\hat{\phi}$ is a well-defined OVD in every sense one asks in QFT in CST!

Summary and Generalizations

• Summary: There is a long history of probing classical GR singularities with classical fields and quantum particles. But most analyses were for (conformally) static space-times with time-like singularities.

• Here we considered time-dependent space-times with space-like singularities, which are also physically far more interesting. But time-dependence forces one to consider quantum fields as probes. Somewhat surprisingly, the big bang and big crunch singularities are remarkably tamer when probed with observables associated with quantum fields, when one keeps in mind that these are OVDs.

Classical fields $\phi(x)$ that define 1-particle states do diverge at the big bang singularity. But their norm in the 1-particle Hilbert space is finite because the shrinking of the volume element exactly compensates for this divergence. Recall from QM: the wave function $\Psi(\vec{r}) = (1/r)e^{-r/r_o}$ diverges at the origin but is a well-defined element of the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$. Similarly, the mode functions that enter the expansion of $\hat{\phi}(x)$ diverge but it is a well-defined OVD: smeared operators $\hat{\phi}(f)$ are well defined. $\langle \hat{\phi}(x)\, \hat{\phi}(x')\rangle$ –and even $\langle \hat{\phi}^2(x)\rangle_{\rm ren}$ and $\langle \hat{T}_{ab}(x)\rangle_{\rm ren}$– are well defined tempered distributions, just as they are in Minkowski space.

• Generalization: The main results on tame behavior of linear, test quantum fields extend to other FLRW models with $\beta > 0$. I used Radiation and dust filled cases because the mode functions are sufficiently simple to display explicit results.

Generalizations: $K = \pm 1$ FLRW models

• Open and closed FLRW models have also been investigated in detail. Again, the big-bang/crunch is tame. However, some differences arise in the analysis:

1. The FLRW metric is again conformally flat. But if written as $g_{ab} = a^2(\eta)\bar{g}_{ab}$, the metric $\bar{g}_{ab}$ is non-flat (but ultra-static). Therefore, while the $K = 0$ procedure goes through, the spatial dependence of basis functions is no longer $\exp(i\vec{x} \cdot \vec{k})$; it is much more complicated.

2. One has to use C_0^∞ test functions in place of $\mathcal{S}$, and hence ordinary distributions rather than tempered.

3. Infrared divergences don't arise even in the dust-filled case; K provides an effective IR cutoff. So the closed and open FLRW models are simpler that the spatially flat one!

4. The bi-distribution $\langle \hat{\phi}(x)\, \hat{\phi}(x')\rangle$ continues to have the Hadamard form not only away from the $\eta = 0$ surface, but also when x, x' lie on the two sides of the $\eta = 0$ surface! (If either lies on the $\eta = 0$ surface, the notion breaks down.)

5. Expressions of $\langle \hat{\phi}^2(x)\rangle_{\rm ren}$ and $\langle \hat{T}_{ab}(x)\rangle_{\rm ren}$ are more complicated. $a^4(\eta)\, \langle \hat{\phi}^2(x)\rangle_{\rm ren}$ continues to be a regular function on the full extended space-time. But, regarded as a function, $a^4(\eta)\, \langle \hat{T}_{ab}(x)\rangle_{\rm ren}$ diverges as in the $k = 0$ case. Hence it is again well-defined as a distribution. Non-trivial check: It is conserved, as in the $k = 0$ case.

Generalizations: Contd

• Higher spins: Since FLRW space-times are conformally flat, quantum (as well as classical) Maxwell fields are trivially regular across the big bang and big crunch. Results on the massless scalar field imply that tameness persists also for spin 2 (i.e. linearized gravitational) fields.

• Bianchi models: allowing anisotropies, but retaining homogeneity. Ae expect similar results. Nice open problem for those who have already studied quantum fields in these space-times.

• What about black hole singularities? (Work in Progress with del Rio and Schneider)

The Schwarzschild singularity: One can focus on the 'interior' region inside the horizon (Kontowski-Sachs metric). We have analytic expressions of mode functions as infinite convergent series. Work is in progress to identify the states in the 'interior region' that correspond to the Unruh (and Hartle-Hawking) vacuum of the (right) asymptotic region, and compute $\langle\hat{\phi}(x)\,\hat{\phi}(x\,')\rangle$, $\langle\hat{\phi}^2(x)\rangle_{\rm ren}$, $\langle\hat{T}_{ab}(x)\rangle_{\rm ren}$ in these states at the singularity. Should be completed in the near future.

However: Generic BH singularities are expected to be null (Cauchy horizon instability) rather than space-like. This case seems much more difficult technically. But there is no obvious obstacle 'of principle'. Nice problem for researchers familiar with scalar fields on Kerr space-time.

Broader Perspective

- Extension of QFT in CST: Since $(\mathring{M}, g_{ab})$ includes the big-bang, it is not globally hyperbolic. Yet we could meaningfully extend the QFT to it. The procedure suggests an avenue to go beyond global-hyperbolicity that has been a bed-rock of QFT in CST.

- Semi-classical gravity: g_{ab} classical; $\hat{\phi}(x)$ an OVD subject to eqns:

$$\Box\hat{\phi}(x) = 0 \quad \text{and} \quad G_{ab} = 8\pi G_N \, \langle \hat{T}_{ab}(x)\rangle_{\rm ren}.$$

But $\langle \hat{T}_{ab}(x)\rangle_{\rm ren}$ is no longer smooth. It is a genuine distribution, forcing us to seek distributional semi-classical solutions g_{ab}. Existence is not obvious because of non-linearity of Einstein's equations. But quite possible; examples are known. Possibility opened up because the investigation pin-pointed what exactly is different in QFT on CST at the big-bang: Einstein's equations have to be solved in a distributional sense, whence we have to allow the semi-classical g_{ab} to be a genuine distribution.

- Full QG: In the Planck regime, excitations of quantum geometry have support in 2 (space-time) dimensions in many approaches (see, e.g., Carlip's 2009 short review). A concrete example is provided by the distributional nature of quantum geometry in loop quantum gravity and spinfoams. Interesting Challenge: Can we systematically show that the semi-classical distributional geometry is an approximation to the LQG distributional geometry? Advances along this direction would provide a concrete bridge between LQG and the mathematical QFT community.