Quantum cuboids:

Numerical investigations of the EPRL-FK state sum

Benjamin Bahr
II. Institute for Theoretical Physics
University of Hamburg
Luruper Chaussee 149
22761 Hamburg

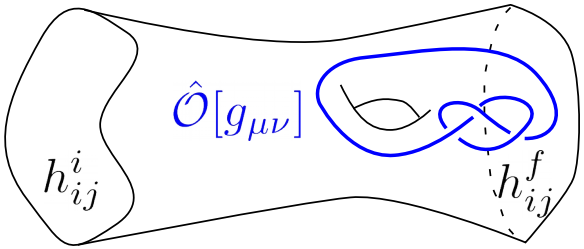
ILQGS seminar, 1st September 2015

joint work with Dr Sebastian Steinhaus
arXiv:1508.07961

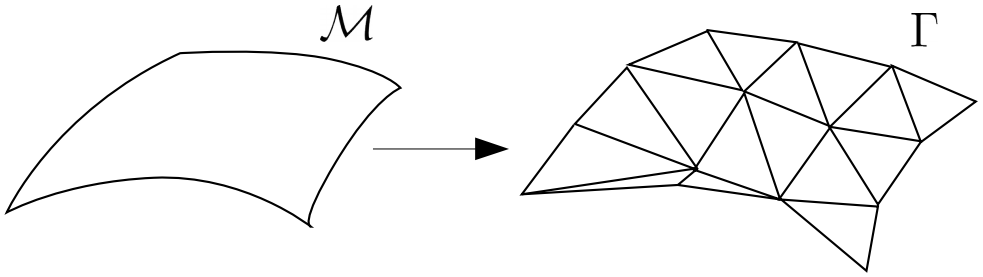
Motivation

Spin Foam Models: Attempt to give rigorous meaning to path integral for quantum gravity.

$$\langle \psi^i | \hat{\mathcal{O}} | \psi^f \rangle_{\text{phys}} = \int_{g_{\mu\nu}|_{\Sigma^{i,f}} = h_{ij}^{i,f}} \mathcal{D}g_{\mu\nu} \; \exp^{\frac{i}{\hbar} S_{\text{EH}}[g_{\mu\nu}]} \, \hat{\mathcal{O}}[g_{\mu\nu}]$$



Discretization of space-time: triangulation (2-complex)



$$Z_\Gamma = \sum_{j_f, \iota_e} \prod_f \mathcal{A}_f \prod_e \mathcal{A}_e \prod_v \mathcal{A}_v$$

[Reisenberger '94, Barrett, Crane '99, Livine, Speziale '07, Engle, Pereira, Rovelli, Livine '07, Freidel, Krasnov '07, Kamiński, Kisielowski, Lewandowski '09, Han, Thiemann '10, Oriti Baratin '11, ...]

Motivation

EPRL-FK spin foam model: many desirable properties

- Discrete gravity in semiclassical limit [Barrett et al '08, Freidel, Conrady '08]
- Connection to loop quantum gravity [Alexandrov, Livine '03, Rovelli '11, Thiemann, Zipfel '13]
- Connection to cosmology [Bianchi, Rovelli, Vidotto '10, Gielen, Oriti, Sindoni '13]

Also many open questions: more than one building block?

- Continuum limit?
- RG flow? e.g. [Rovelli, Smerlak '12, Dittrich '14]
- Renormalizability? [Dittrich '08, Bahr, Dittrich '09, Dittrich '12]
- Diffeomorphism invariance? [...]
- … make precise physical predictions?

To tackle many of them, numerical methods and approximations have to be used!

Motivation

Steps in that direction:

- Computation of self-energy term in EPRL model [Perini, Rovelli, Speziale '09, Riello '13]
- Numerical methods for spin net models (Migdal-Kadanoff-type approximations) [Dittrich, Eckert, Martin-Benito '12, Bahr, Dittrich, Hellmann, Kaminski '13]
- Tensor network renormalization methods to investigate phase structure of analogue spin foam models [Dittrich, Martin-Benito, Schnetter '13, Dittrich, Mizera, Steinhaus '14]
- Several topological fixed points of the RG flow, in particular for finite and quantum groups! [Dittrich, Matrin-Benito, Steinhaus '14 Steinhaus '15]

Motivation

In this article:

- Use restriction of state sum to derive interesting properties of the EPRL-FK model for many building blocks

General strategy (also employed in lattice gauge theories):

- Do not sum over all states in the path integral
- Sum only over those, which are relevant for the situation at hand (physical input!)

Specific strategy in this article:

- Use <u>4d hypercubic lattice</u>
- Only use <u>*quantum cuboids*</u> as intertwiners

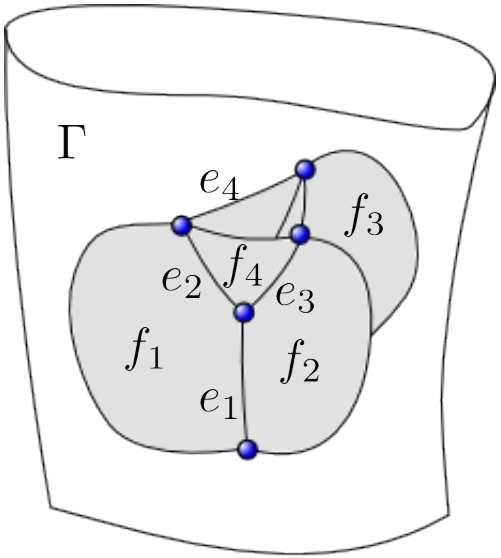
Table of contents

Setup: EPRL-FK model

"States": assignments of spins & intertwiners
to faces and edges of 2-complex Γ :

$$j_f \in \frac{1}{2}\mathbb{N} \qquad \iota_e \in \mathrm{Inv}_{SU(2)}\left(\bigotimes_{f\supset e} V_{j_f}\right)$$

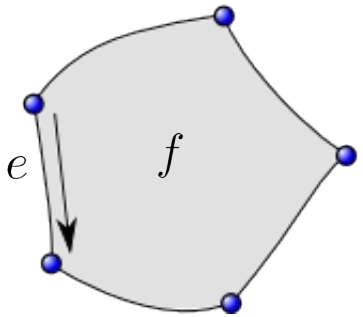
$$j_f^{\pm} := \frac{|1\pm\gamma|}{2} j_f$$



Vertex amplitude: $\mathcal{A}_v = \mathrm{tr}\left(\bigotimes_{e\to v} \Phi\iota_e \bigotimes_{e\leftarrow v} \Phi\iota_e^\dagger\right)$

Edge amplitude: $\mathcal{A}_e = \|\iota_e\|^{-2}$

Face amplitude: $\mathcal{A}_f = 2j_f+1,\ (2j_f^+ + 1)(2j_f^- + 1), \ldots$



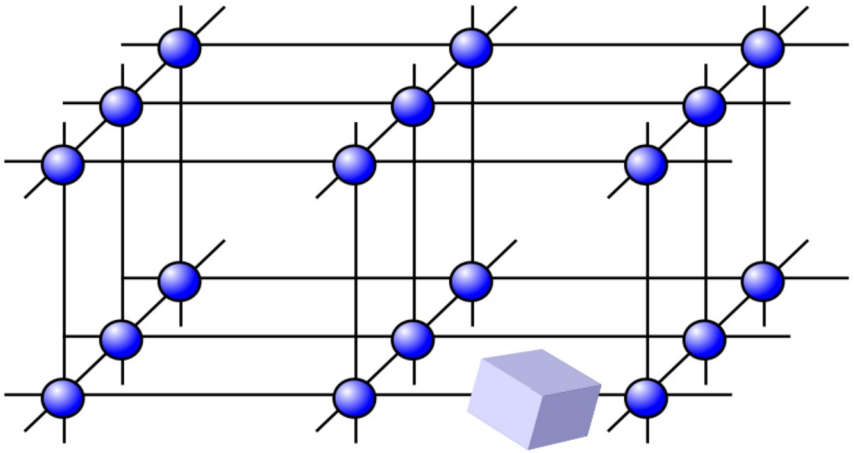
$$Z_\Gamma = \sum_{j_f,\iota_e} \prod_f \mathcal{A}_f \prod_e \mathcal{A}_e \prod_v \mathcal{A}_v$$

[Review: Perez '12]

Symmetry restriction: quantum cuboids

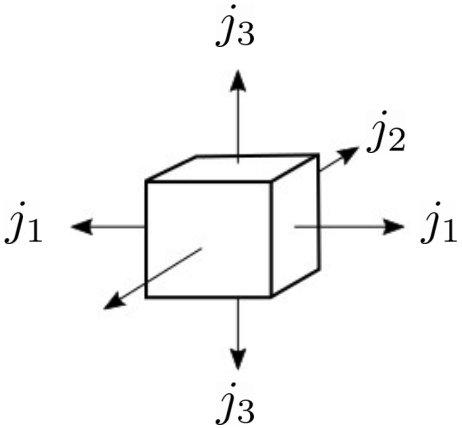
Approximation to the path integral:

2-complex: regular 4d hypercubic lattice



restrict intertwiners: only <u>quantum cuboids</u>

$$|\iota_{j_1,j_2,j_3}\rangle = \int_{SU(2)} dg\; g \triangleright \left(\prod_{i=1}^{3} |j_i\, e_i\rangle |j_i\, -e_i\rangle \right)$$

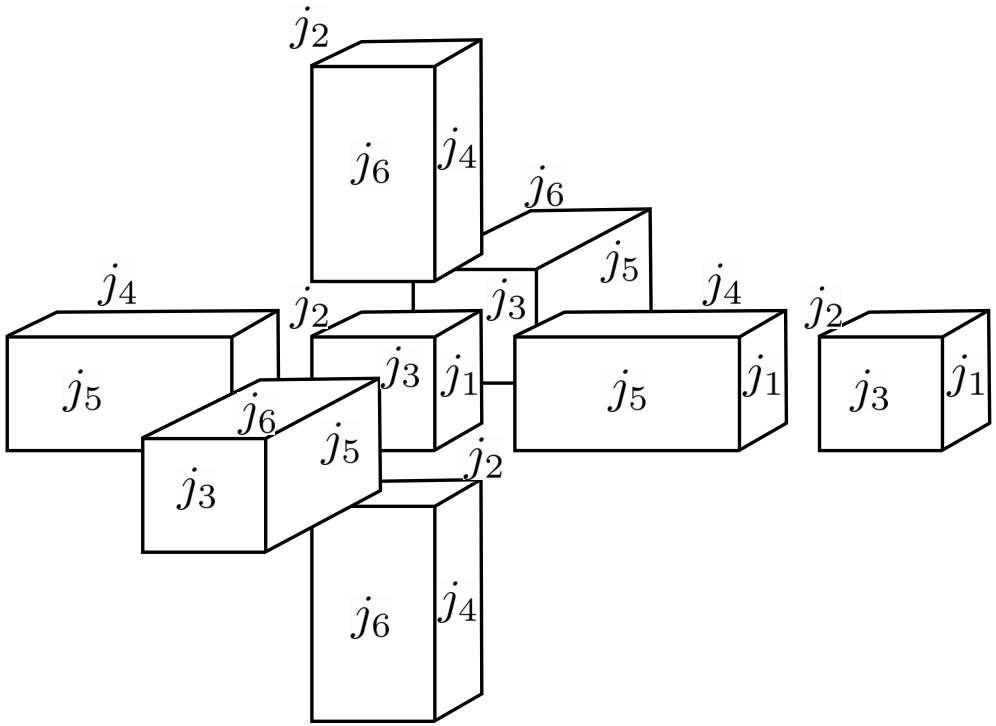


(unique for three given spins: omit sum over intertwiners in notation)

[Livine, Speziale '07, Bainchi, Doná, Speziale '11, Alesci, Cianfrani '15]

Symmetry restriction: hypercuboids

Results in vertices all being <u>hypercuboids</u>



Six spins per hypercuboid: can be freely chosen

Symmetry restriction: hypercuboids

Remarks about symmetry:

- Right angles everywhere: no deficit angles ϵ_t all configurations flat!
- With quantum cuboids, there are still local degrees of freedom: non-geometricity
- An infinite-dimensional (Abelian) subgroup of diffeomorphisms acts on states!
  $\Rightarrow$ test-bed for diffeom symmetry in spin foam models
- Stark simplification: don't expect reliable physical predictions.
  But: can easily be generalized to include more states
  (e.g. quantum prisms, to capture some curvature degrees of freedom)
- Method of restricting states which are summed over:

  useful approximation for certain situations: physical input!
  (e.g. exclude high curvature states when only interested in gravitational waves)

II. Semiclassical Regime

Semiclassical regime: non-geometricity

Large j -asymptotics: $$a_f \;=\; j_f \ell_P^2$$

Distributions of areas $a_{\mu\nu}^{\vec{n}}$ along faces of hypercubic lattice $\vec{n} \in \mathbb{Z}^4$

$$a_{\mu\nu}^{\vec{n}} \;=\; a_{\mu\nu}^{\vec{n}+ke_\sigma+le_\rho} \qquad k,l \in \mathbb{Z}, \qquad \text{all } \mu,\nu,\sigma,\rho \text{ distinct}$$

Six areas per hypercuboid: two non-geometric degrees of freedom

geometricity constraints: $$a_{xy}a_{zt} \;\stackrel{!}{=}\; a_{xz}a_{yt} \;\stackrel{!}{=}\; a_{xt}a_{yz}$$

(generalized) 4-volume: $$V^{\vec{n}} \;:=\; \left(a_{xy}a_{zt}a_{xz}a_{yt}\; a_{xt}a_{yz}\right)^{\frac{1}{3}}$$

non-geometricity: $$\xi^{\vec{n}} \;:=\; \frac{1}{V^{\vec{n}}} \begin{pmatrix} a_{xy}a_{zt} - a_{xz}a_{yt} \\ a_{xz}a_{yt} - a_{xt}a_{yz} \\ a_{xy}a_{zt} - a_{xt}a_{yz} \end{pmatrix}$$

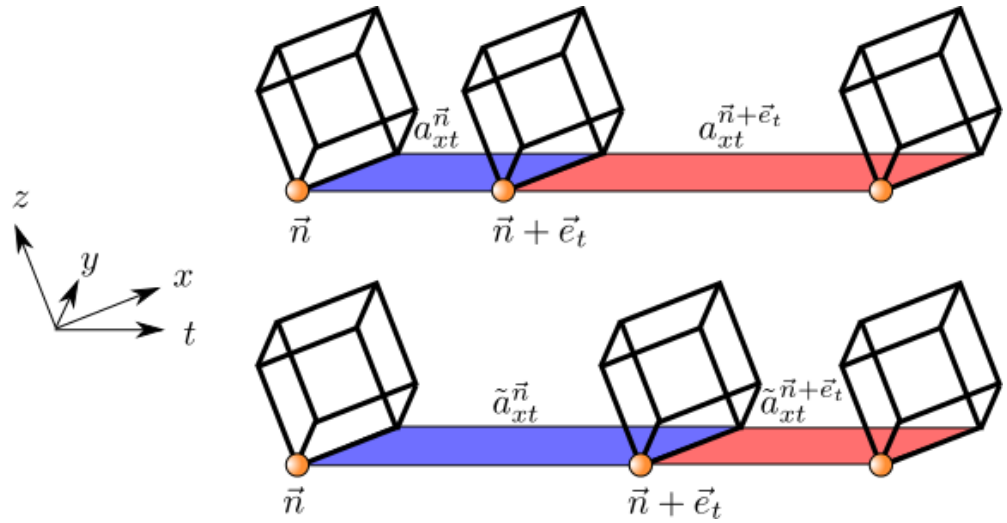
[Freidel, Speziale '10, Freidel, Ziprick '14]

Semiclassical regime: vertex translations

change of areas which should leave geometry invariant

transformation: $i, j = x, y, z$

$$
\begin{aligned}
a_{ij}^{\vec{n}} &\longrightarrow a_{ij}^{\vec{n}} \\
a_{it}^{\vec{n}} &\longrightarrow a_{it}^{\vec{n}}(1+\Delta) \\
a_{it}^{\vec{n}+e_t} &\longrightarrow a_{it}^{\vec{n}+e_t}(1-\Delta')
\end{aligned}
$$



such that: $V^{\vec{n}}\Delta = V^{\vec{n}+e_t}\Delta'$

- this is the discrete version of an Abelian subgroup of 4-diffeos in the continuum
- non-geometricity ξ and total 4-volume are invariant
- these are not the diffeos from LQG! In spirit much more like vertex-displacement symmetry in Regge calculus

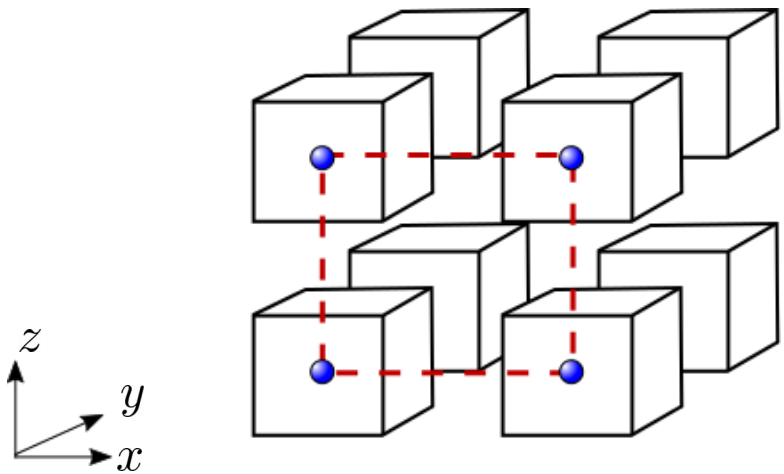
Semiclassical regime: torsion on the boundary

[Freidel, Speziale '10, Freidel, Ziprick '14]

Torsion is understood as the non-closing of a geodesic square.

continuum:

$$T^{\rho}_{\mu\nu} = e^{\rho}_{i}(\nabla_{\mu} e^{i}_{\nu} - \nabla_{\nu} e^{i}_{\mu} - (e_{\mu} \wedge e_{\nu})^{i})$$

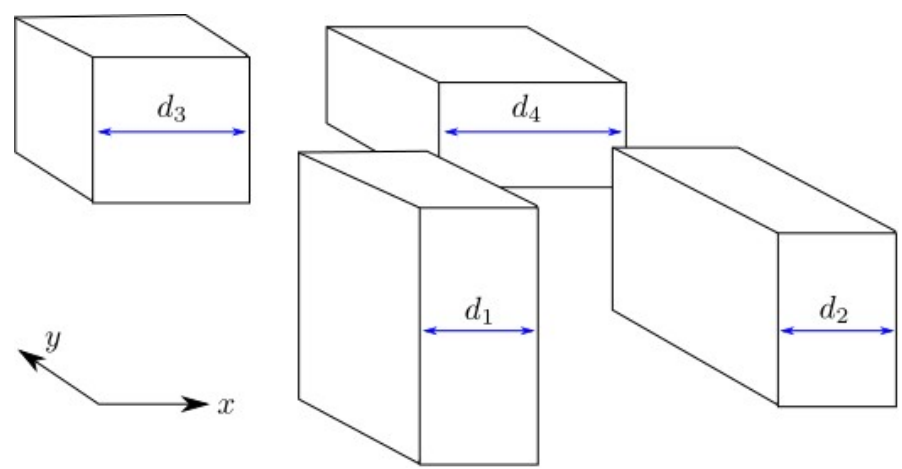


discrete:

use the diameters of cuboids

e.g.: $$d_x = \left(\frac{a_{xy}a_{xz}}{a_{yz}}\right)^{\frac{1}{2}}$$

$$T^{x}_{xz} := \frac{1}{2}\left(d_x^{\vec{n}} + d_x^{\vec{n}+e_x} - d_x^{\vec{n}+e_z} - d_x^{\vec{n}+e_x+e_z}\right)$$



III. Vertex Asymptotics

Large j-asymptotics of the hypercuboid:

Choice: EPRL-FK vertex with $0 < \gamma < 1$

This leads to factorization in + and - sector:

$$|\iota_{j_1,j_2,j_3}\rangle \;=\; \int_{SU(2)} dg\; g \triangleright \Big(\prod_{i=1}^{3} |j_i\, e_i\rangle |j_i\; - e_i\rangle \Big)$$

“boosted” intertwiner:

$$\Phi_\gamma |\iota_{j_1,j_2,j_3}\rangle \;=\; |\iota_{\frac{1+\gamma}{2}j_1,\frac{1+\gamma}{2}j_2,\frac{1+\gamma}{2}j_3}\rangle \otimes |\iota_{\frac{1-\gamma}{2}j_1,\frac{1-\gamma}{2}j_2,\frac{1-\gamma}{2}j_3}\rangle$$

Amplitude is contraction of boosted intertwiners, so:

$$\mathcal{A}_v(j) \;=\; \mathcal{A}_v^+ \mathcal{A}_v^-$$

Asymptotics factorizes as well:

$$\mathcal{A}_v(j)^{(0)} \;=\; \mathcal{A}_v^{(0),+} \mathcal{A}_v^{(0),-}$$

Large j-asymptotics of the hypercuboid:

Integral over: $SU(2)^8$ (for both + and - part) details: similar to Barrett et al '08, Freidel, Conrady '08

But gauge symmetry: only $SU(2)^7$ remains

2^7 -fold discrete symmetry of the integrand: $g_i \to -g_i$

Two distinct critical stationary points:

$$\mathcal{A}_v^{(0)\pm} \sim \frac{1}{\sqrt{-\det H(\vec{j})}} + \frac{1}{\sqrt{-\det \overline{H(\vec{j})}}}$$

determinant of the Hessian matrix (21x21):

No Einstein-Hilbert-part! (not surprising, because of flatness)

Two „coupling constants“: α, γ

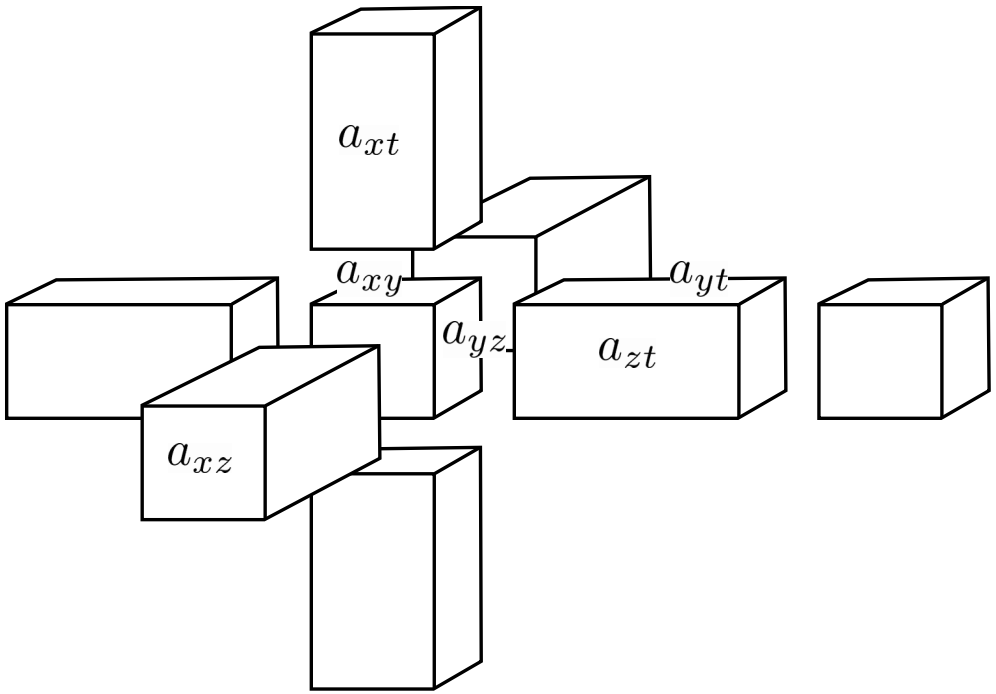
$$Z^{\alpha,\gamma} = \sum_{j_f} \prod_e (d_f^+ d_f^-)^\alpha \prod_e \frac{1}{\|\Phi_\gamma \iota_{j_1,j_2,j_3}\|^2} \prod_{v,\pm} \mathcal{A}_v^{(0),\pm}\left(\frac{1\pm\gamma}{2} j_f\right)$$

$$= \left(\frac{1-\gamma^2}{4}\right)^{(6\alpha-9/2)V} \sum_{j_f} \prod_v \hat{\mathcal{A}}^\alpha{}_v(\vec{j})$$

III. Results

Non-geometricity in the path integral

perturb asymptotic amplitude around regular hypercube (all areas equal):



Fix a_{xy}, a_{yz}, a_{xz} and total 4-volume

$\Rightarrow$ Four d.o.f.

the two remaining ones correspond to non-geometricity:

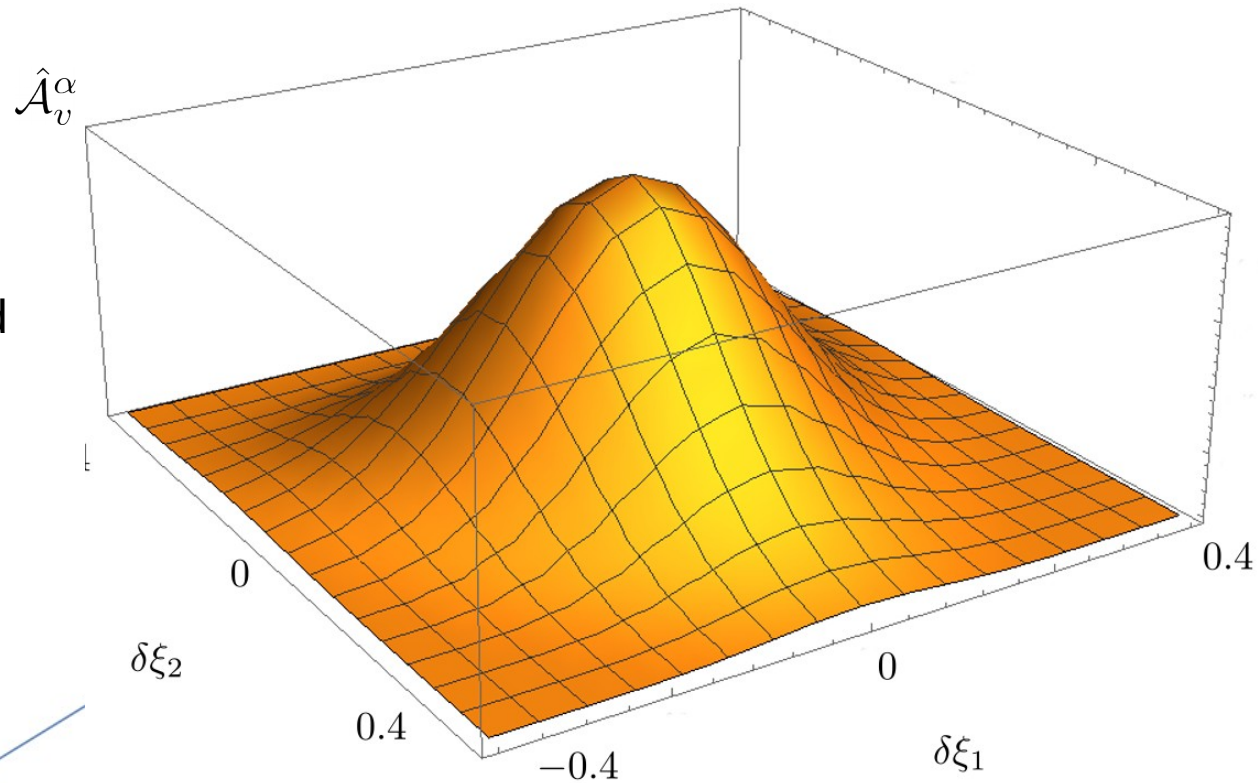
$$\delta\xi_1 = \frac{1}{\sqrt{2}a}(\delta a_{xt} - \delta a_{yt})$$

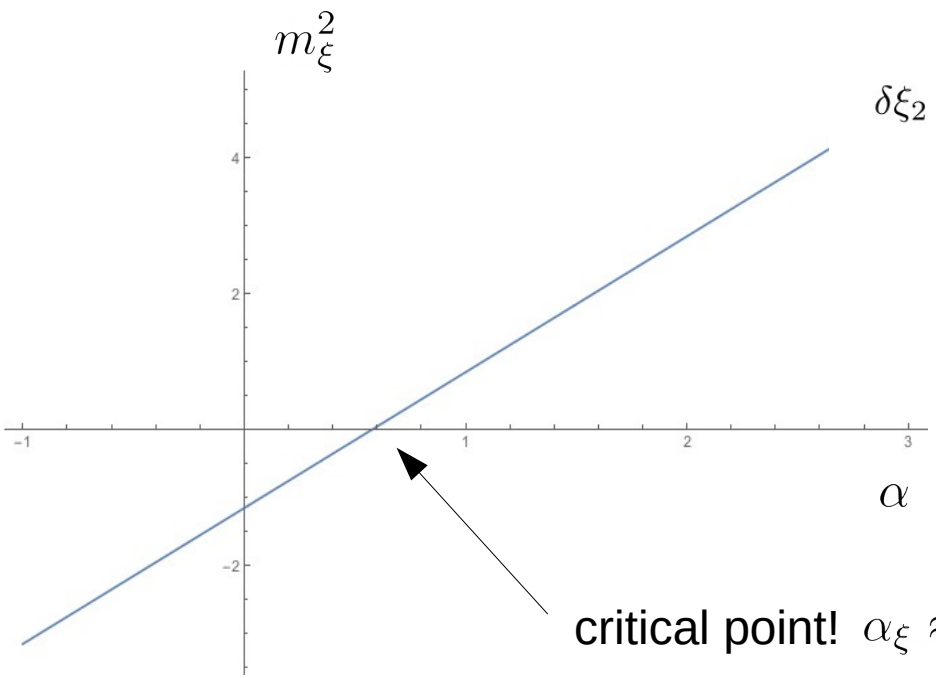
$$\delta\xi_2 = \frac{1}{\sqrt{6}a}(\delta a_{xt} + \delta a_{yt} - 2\delta a_{zt})$$

Non-geometricity in the path integral

Amplitude:

non-geometricity fluctuates
around regular hypercuboid





critical point! $\alpha_\xi \approx 0.58$

Width of that bell curve:
“inverse mass squared”

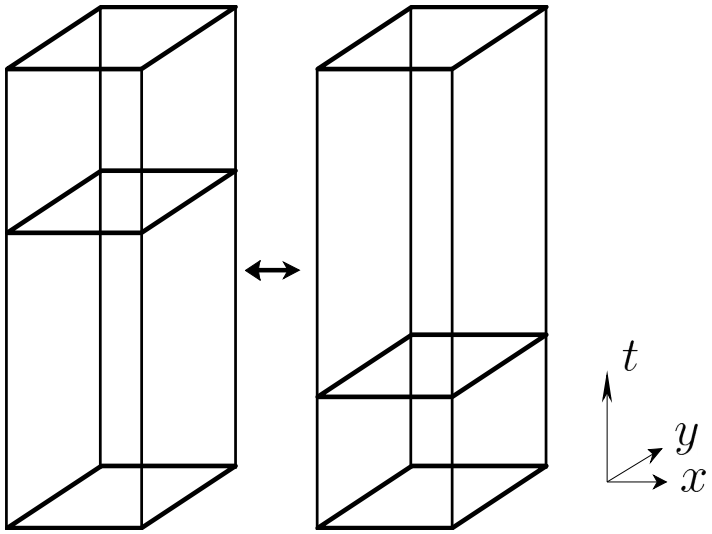
$$\hat{\mathcal{A}}_v^\alpha \sim e^{-V m_\xi^2 \xi^2}$$

Vertex-translation symmetry

Consider two hypercuboids, and the product of their amplitudes

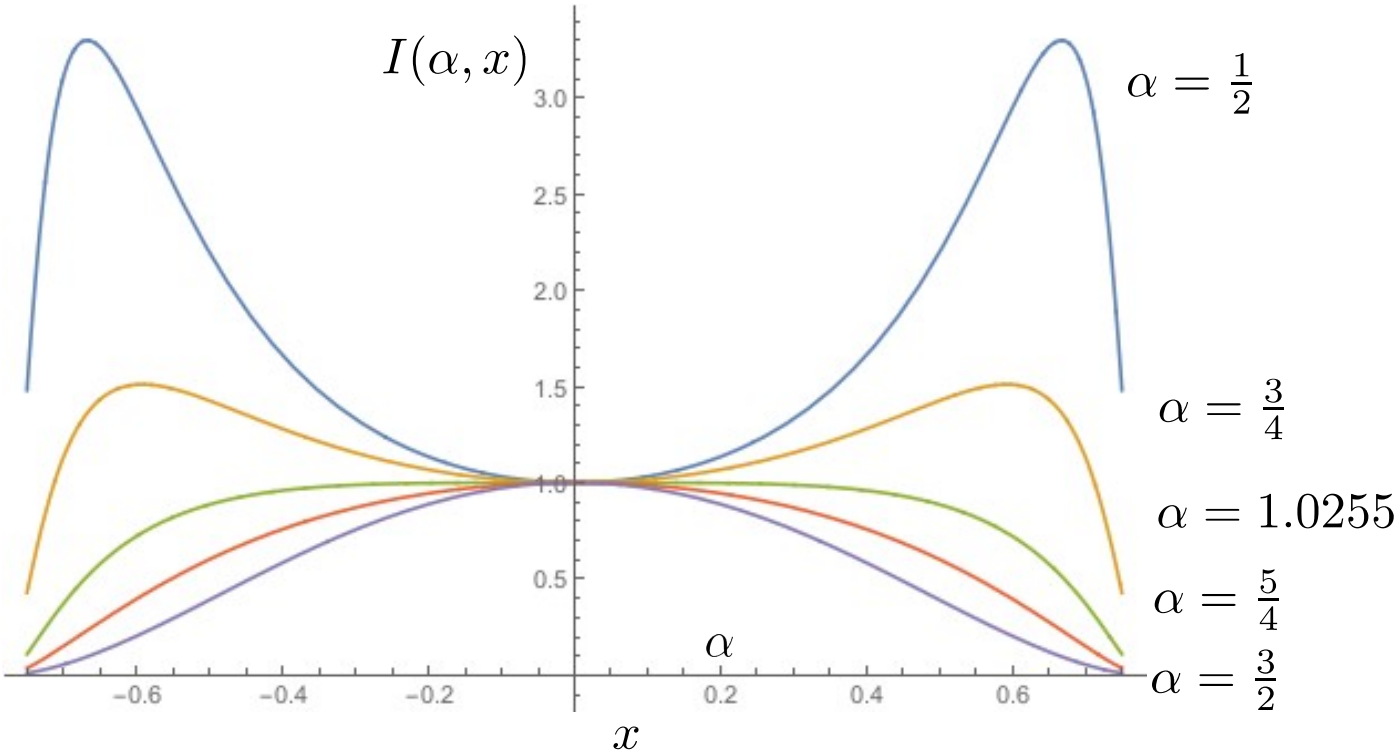
$$\hat{\mathcal{A}}_v^\alpha(\vec{j},\vec{k}) := \hat{\mathcal{A}}_v^\alpha\Big(j_{xy}, j_{xz}, j_{yz}, k_{xt}, k_{yt}, k_{zt}\Big)$$

$$I(\alpha,\lambda) := \hat{\mathcal{A}}_v^\alpha(\vec{j},(1+\lambda)\vec{j})\hat{\mathcal{A}}_v^\alpha(\vec{j},(1-\lambda)\vec{j})$$



Parameter λ labels orbit of vertex translation

Vertex-translation symmetry



$$I''(\alpha, 0) \approx -0.6137\alpha + 0.6294$$

For small values of α, maximum is where one hypercube is large, the other is small.

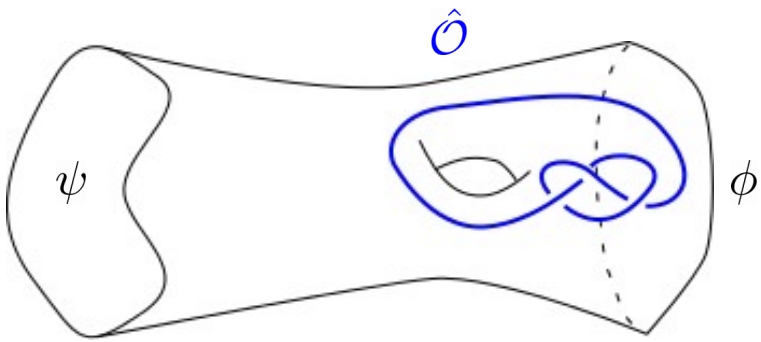
For large values of α, maximum is when both hypercubes are of same size.

Note: this means that in quantum theory, diffeomorphism symmetry is broken even in the flat case!

Physical norm of boundary states:

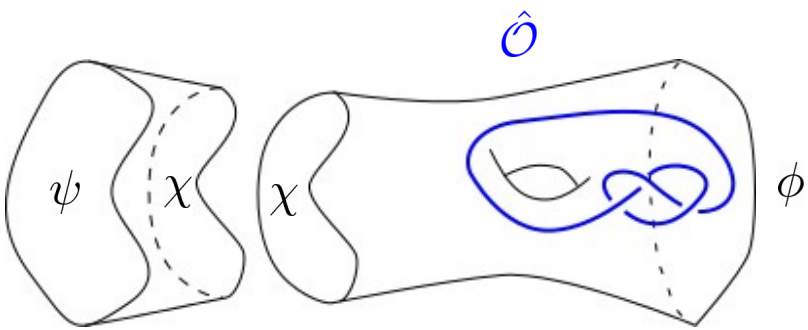
The path integral functions as a way to compute <u>physical</u> inner product.

Formally (or, if kinematical Hilbert space is finite-dimensional):



$$\langle\psi|\hat{\mathcal{O}}|\phi\rangle_{\text{phys}} = \sum_{\chi} \underbrace{\langle\psi|\chi\rangle_{\text{phys}}}_{\|\cdot\|^2 \leq \|\psi\|^2_{\text{phys}}\|\chi\|^2_{\text{phys}}} \langle\chi|\hat{\mathcal{O}}|\phi\rangle_{\text{phys}}$$

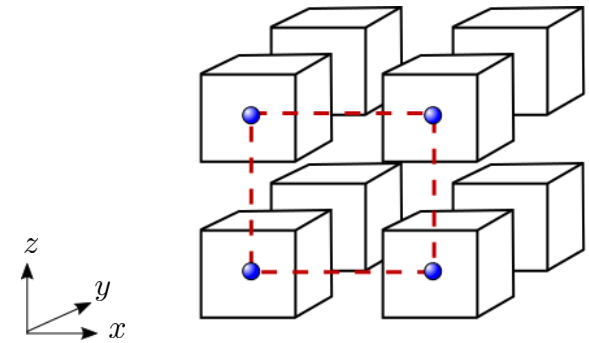
$= \sum_{\chi}$



So, states with small physical norm should contribute little to the path integral.

Torsion in the physical inner product

Compute the physical norm of kinematical (boundary) states, depending in their torsion.



Lattice: $2 \times 2 \times 2 \times N$

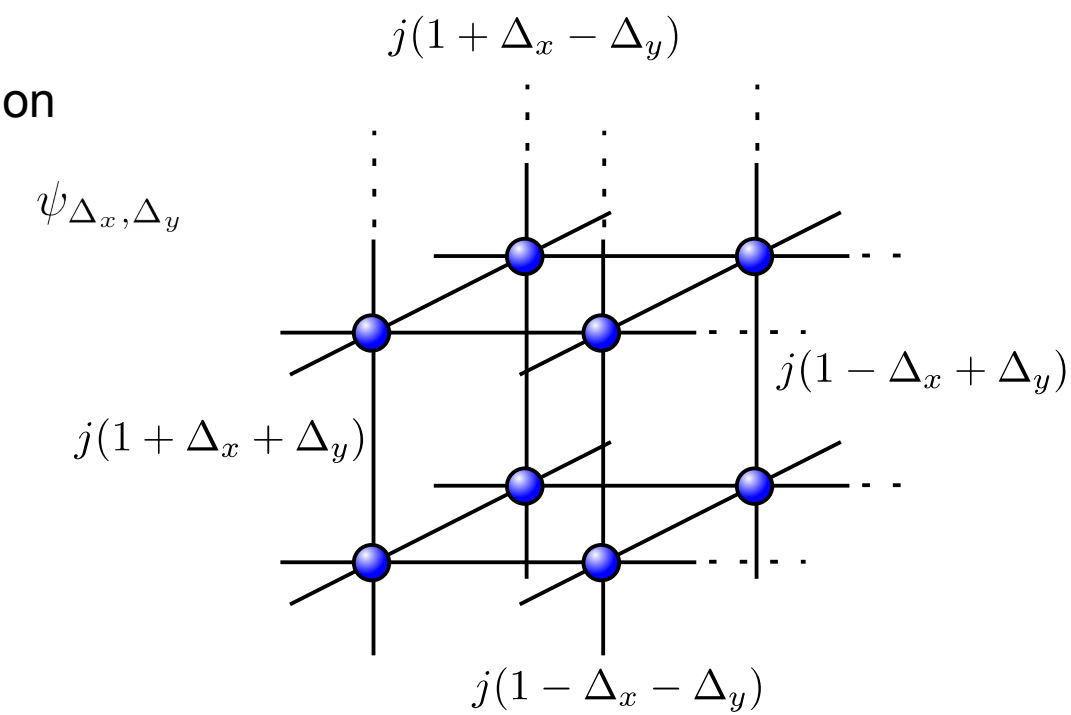
Boundary spin network: $\psi_{\Delta_x, \Delta_y}$

All spins equal to some (large) spin j on edges in x- and y-direction

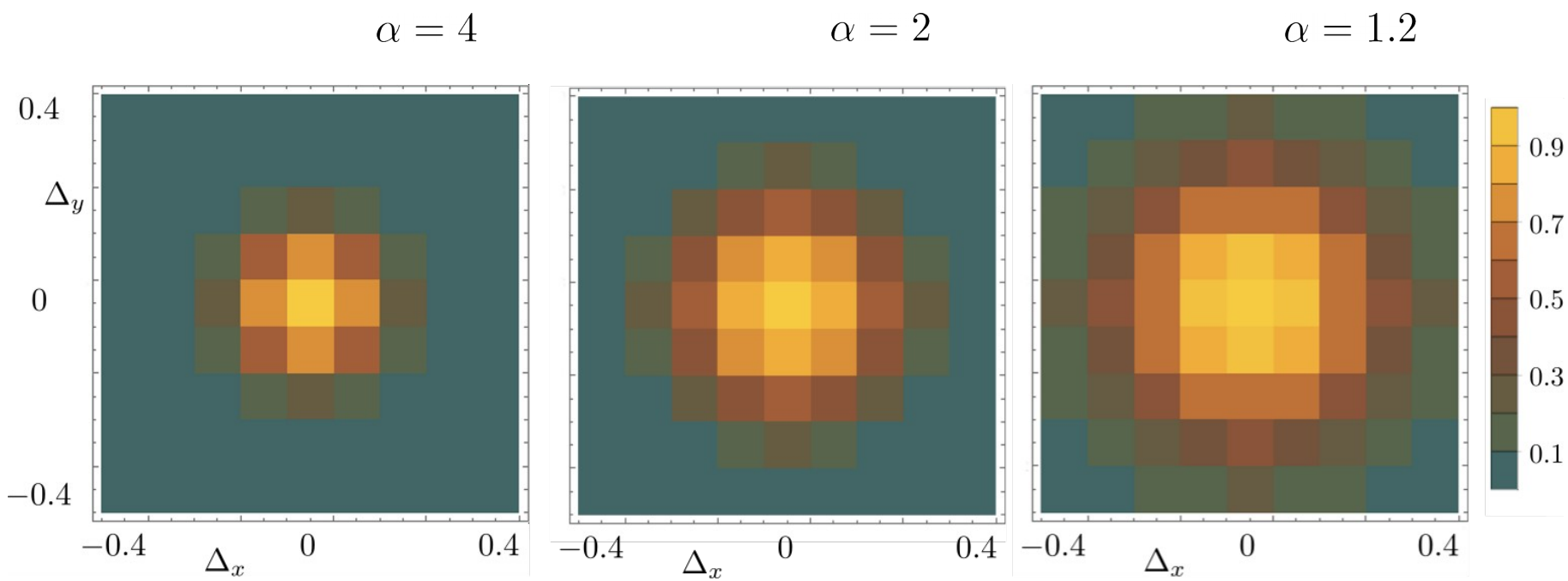
spins on edges in z-direction have value: $j(1 \pm \Delta_x \pm \Delta_y)$

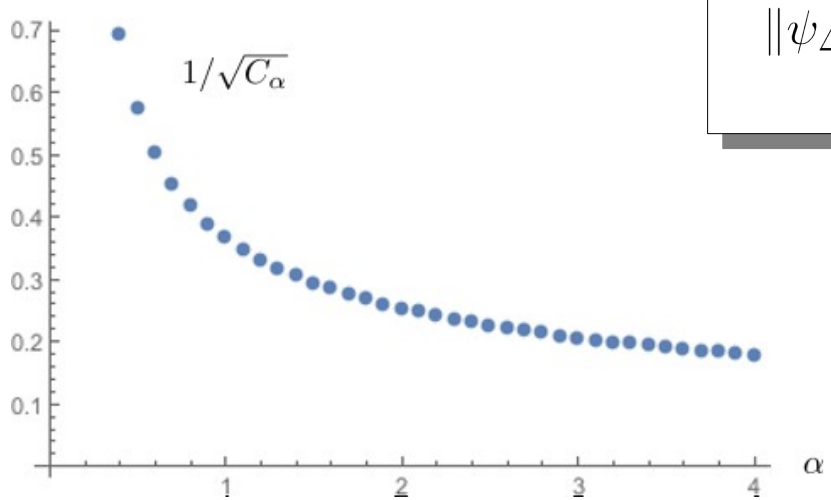
Components of the torsion tensor:

$$T^x_{xy} = \frac{1}{2}\Delta_x \qquad T^y_{xy} = \frac{1}{2}\Delta_y$$



Torsion in the physical inner product (results)





$$\|\psi_{\Delta_x,\Delta_y}\|_{\text{phys}} \sim \exp\left(-\frac{1}{2}C_\alpha N(\Delta_x^2+\Delta_y^2)\right)$$

Excellent fit: $C_\alpha \approx 8.4319\alpha$

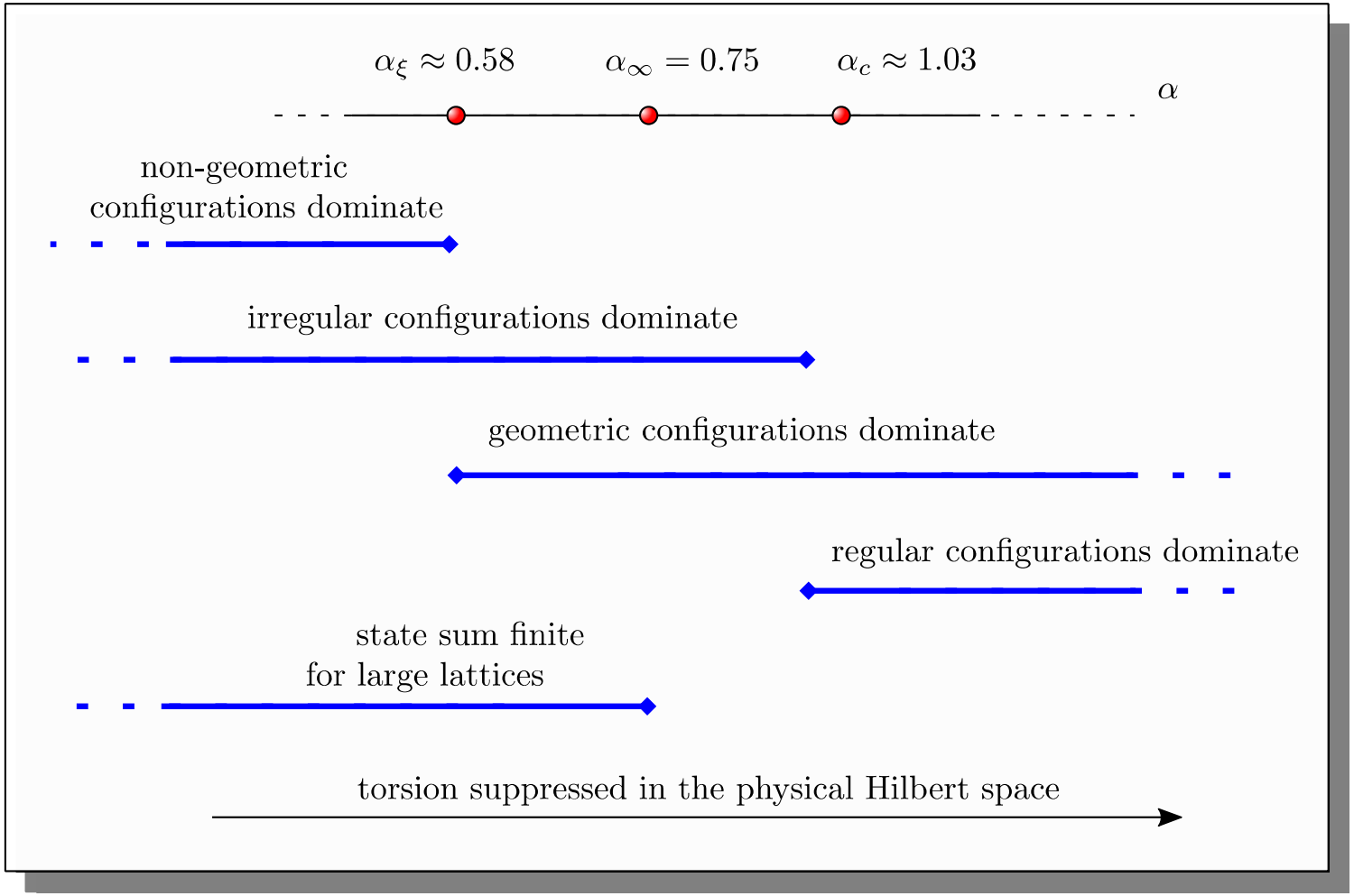
Boundary states with high torsion are suppressed!

IV. Summary & Conclusion

Summary

- Spin foam models need approximations to evaluate state sum (to compute observables, physical inner products, RG flow, …)
- Method: state restriction! Only sum over those states which are important for specific question at hand (physical input!)
- First step: Take regular hypercubic lattice and only sum over *quantum cuboids* (coherent polyhedra: cuboids in semiclassical limit)
- Second step: consider regime of large spins (determined by boundary data)
- Despite drastic simplifications, surprising amount of information can be gathered about the path integral!
- Path integral behaviour crucially depends on value of coupling constant α

Summary


$\alpha_\xi \approx 0.58$
$\alpha_\infty = 0.75$
$\alpha_c \approx 1.03$
α
non-geometric configurations dominate
irregular configurations dominate
geometric configurations dominate
regular configurations dominate
state sum finite for large lattices
torsion suppressed in the physical Hilbert space

Summary

For $\alpha > \alpha_c$ the state sum behaves very nice:

- There exists a vacuum state Ψ_j which resembles Euclidean space!
- Non-geometric degrees of freedom arise as excitations around flat background Ψ_j
- State with torsion can be omitted in the path integral (non-local condition on spins)
- At the point $\alpha = \alpha_c$ the state sum appears to be invariant under vertex translation symmetry. Chance for second order phase transition?

Artefacts of simplification or genuine properties of the path integral?

Outlook

To Do, in order to improve the analysis:

- Use more states (include local curvature!)
- Go beyond large spin asymptotics (probably only viable numerically)
- Repeat analysis for Lorentzian signature
- Replace bare amplitude by renormalized one
  (equivalently, include sum over finer 2-complexes: GFT!)
- Include more coupling constants!
- Couple matter: make contact to Lattice Gauge Theory!

Thank you for your attention!

Large j-asymptotics of the hypercuboid: validity

$$\mathcal{A}_v(j) \;=\; \mathcal{A}_v^{(0)}(j)\,\left(1+\mathcal{A}_v^{(1)}(j)+\ldots\right)$$

with $\mathcal{A}_v^{(1)}(\lambda j)/\mathcal{A}_v^{(0)}(\lambda j) = o(\lambda)$

Assume that approximation is good for $\lambda > J$, i.e. $\mathcal{A}_v^{(1)}/\mathcal{A}_v^{(0)} \ll \lambda$

Large j-asymptotics of more than one building block:

$$\prod_v \mathcal{A}_v(j_f) \;= \prod_v \mathcal{A}_v^{(0)}(j)\left(1+\mathcal{A}_v^{(1)}+\ldots\right)$$

$$= \left(\prod_v \mathcal{A}_v^{(0)}(j)\right) + \textstyle\sum_v \left(\mathcal{A}_v^{(1)} \prod_{v'\neq v} \mathcal{A}_{v'}^{(0)}\right) + \ldots$$

vertices

first term can be expected to be a good approximation if $\mathcal{A}_v^{(1)}/\mathcal{A}_v^{(0)} \ll N\lambda$

so $j_f > NJ$

Further restricts the validity of approximation!

still: assuming $J \approx 100$ allows for femtometer precision
with $10^9 \times 10^9 \times 10^9 \times 10^9$ lattice *if small spins can be ignored*