# (SOME COMMENTS ON)

# CANONICAL GAUGE THEORIES

# WITH BOUNDARIES

*Alejandro Corichi*

Center for Mathematical Sciences, UNAM, Morelia, México

**ILQGS, LSU**
**September, 2019**

*Collaboration with: T. Vukasinac.*

# PLAN

1. Questions
2. Canonical Hamiltonian Formalism
3. Boundaries
4. Examples: Maxwell + Pontryagin
5. Comments

# QUESTIONS

- **Why bother about boundaries?**
- **What about boundary degrees of freedom?**
- **When do we have a boundary contribution to the Symplectic Structure?**
- **Do we have to modify the Dirac-Regge-Teitelboim prescription?**
- **Is there a relation between boundary conditions and constraints?**

# CANONICAL HAMILTONIAN FORMALISM.

Let us recall how one constructs the canonical Hamiltonian formalism. The starting point is a configurations space $\mathcal{C}$. Fundamental object is the momentum function: $P : V^a \in T_q\mathcal{C} \mapsto \mathbb{R}$. Usually $P[V] := P_a V^a$. One then defines the phase space as $\Gamma := T_*\mathcal{C}$, and defines a 1-form $\theta$ on $\Gamma$ such that

$$\theta[\dot{X}] = P[\dot{Q}] \tag{1}$$

where $\dot{X}$ is the velocity on $\Gamma$ and $\dot{Q}$ is the velocity on $\mathcal{C}$. The 1-form $\theta$ is called the symplectic potential and can be defined as:

$$\theta := P_a \, \mathrm{d}Q^a \tag{2}$$

The symplectic two-form on $\Gamma$ is then defined as $\Omega := \mathrm{d}\theta$.

**A fundamental equation is given by.**

$$\mathrm{d}F = \Omega(X_F, \cdot) \tag{3}$$

**where $X_F$ is the Hamiltonian vector field associated to the function $F$. If we now contract with an arbitrary (Hamiltonian) vector field $Y$, we have,**

$$\pounds_Y F = \mathrm{d}F(Y) = \Omega(X_F, Y)\,. \tag{4}$$

**If $Y$ itself is the Hamiltonian vector field of $G$, then we can define the Poisson Bracket between $F$ and $G$:**

$$\{F, G\} := \Omega(X_F, Y_G) \tag{5}$$

**Hamilton's equations generated by the Hamiltonian $H$ read,**

$$\dot{F} = \pounds_{X_H} F = \{F, H\} := \Omega(X_F, X_H) \tag{6}$$

**with $X_H = \dot{X}$ the velocity on $\Gamma$.**

**How about field theories? The geometric ideas are the same, just that the dimensionality of the phase space is infinite. One has to be careful but the formalism is the same.**
**For example for a field theory with phase space $\Gamma = (\phi, P)$, the momentum mapping looks like**

$$P[V] = \int_\Sigma \mathrm{d}^3x \tilde{P}\, V \tag{7}$$

**This means that $\theta$ is of the form,**

$$\theta = \int_\Sigma \mathrm{d}^3x\, \tilde{P}\, \mathbb{d}\phi \tag{8}$$

**and the symplectic structure has the form,**

$$\Omega = \mathbb{d}\theta = \int_\Sigma \mathrm{d}^3x\, \mathbb{d}\tilde{P} \wedge \mathbb{d}\phi \tag{9}$$

## BOUNDARIES

**Why are boundaries relevant? Because in the standard Dirac analysis of gauge field theories, one normally disregards boundaries, and all the boundary terms that appear as one integrates by parts are discarded. One can no longer do that.**
**In the standard Regge-Teitelboim analysis of field theories with boundaries, the main theme is to make all functio(al)s differentiable. This means that, when one computes the gradient $\mathbb{d}F$ of the function $F$, there should be no contributions from the boundaries. This is equivalent to saying that all boundary terms that appear when taking the "variation" of the function, should vanish. Standard lore is that this approach to field theories with boundaries is sufficient to deal with all cases. Method is generic. But, Is it?**

## Boundary Contributions to the Symplectic Structure?

**How could a boundary contribution to the symplectic structure arise?**

**Recall the momentum map! Suppose that we have a map that looks like,**

$$P[V] = \int_{\Sigma} \mathrm{d}^3x \, \tilde{P}^a V_a + \int_{\partial\Sigma} \mathrm{d}^2x \, \tilde{P}^a_{\partial} v_a \,, \tag{10}$$

**namely, it as a contribution from the boundary $\partial\Sigma$. This means that**

$$\theta = \int_{\Sigma} \mathrm{d}^3x \; \tilde{P}^a \, \mathbb{d} A_a + \int_{\partial\Sigma} \mathrm{d}^2x \; \tilde{P}^a_{\partial} \, \mathbb{d} A^{\partial}_a \,. \tag{11}$$

**From which the symplectic structure,**

$$\Omega = \mathbb{d}\theta = \int_{\Sigma} \mathrm{d}^3x \, \mathbb{d}\tilde{P}^a \wedge \mathbb{d}A_a + \int_{\partial\Sigma} \mathrm{d}^2x \, \mathbb{d}\tilde{P}^a_{\partial} \wedge \mathbb{d}A^{\partial}_a \,, \tag{12}$$

**acquires a boundary term. $(A^{\partial}, P_{\partial})$ are the boundary DOF.**

**What are the practical implications of such a boundary contribution?**

**Recall the basic Hamiltonian equation,**

$$\mathrm{d}F(Y) = \Omega(X_F, Y)\,. \tag{13}$$

**If there are no boundary terms in the RHS, then there should be no boundary terms in the LHS, that is, in the gradient. This is precisely the standard Regge-Teitelboim case, where we require the boundary terms in the gradient to vanish.**

**But now if we have boundary terms in the RHS due to the existence of boundary contributions to $\Omega$, then we must have boundary terms in the LHS, namely, in the gradient.**

**Regge-Teitelboim needs to be revisited/extended!**

**But, how does it work then ?**

# Maxwell-Pontryagin vs Maxwell-Chern-Simons

**Let us consider these theories that are known to be equivalent. One on the bulk, M-P and one with a bulk term and a boundary term, M-CS.**
**Let us start by considering the bulk theory. It is given by a covariant action:**

$$S_{MP}[A] = S_M[A]+S_P[A] = -\frac{1}{4}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{|g|}g^{ac}g^{bd}F_{cd}F_{ab}-\frac{\theta}{4}\int_{\mathcal{M}} \mathrm{d}^4x\,\tilde{\varepsilon}^{abcd}F_{ab}F_{cd}\,, \tag{14}$$

**In the Dirac analysis, the momenta is of the form,**

$$\tilde{\Pi}^a := \frac{\delta \mathcal{L}_{MP}}{\delta(\pounds_t A_a)} = \frac{\sqrt{h}}{N}\,h^{ac}(\pounds_t A_c - \nabla_c\phi + N^d F_{cd}) - \theta\,\tilde{\varepsilon}^{abc}F_{bc}\,. \tag{15}$$

**It has no boundary contribution, which means the symplectic structure, as expected, has only a bulk term.**

There is a primary constraint

$$\tilde{\Pi}_\phi := \hat{t}_a \tilde{\Pi}^a = 0 \quad \text{or} \quad \tilde{\Pi}_\phi[\lambda] := \int \mathrm{d}^3x \, \lambda(x) n_a \tilde{\Pi}^a(x) \approx 0 \,, \tag{16}$$

The canonical Hamiltonian is then given by,

$$H_C(A, \tilde{\Pi}) = \int_\Sigma \mathrm{d}^3x \, \left[ \frac{\sqrt{h}}{4} N h^{ac} h^{bd} F_{ab} F_{cd} + (\nabla_a \phi - N^b F_{ab}) \tilde{\Pi}^a \right. \tag{17}$$

$$\left. + \frac{N}{2\sqrt{h}} h_{ab} (\tilde{\Pi}^a + \theta \, \tilde{\varepsilon}^{acd} F_{cd})(\tilde{\Pi}^b + \theta \, \tilde{\varepsilon}^{bkn} F_{kn}) \right] \,. \tag{18}$$

In order to proceed, we need to ensure that both the primary constraint and the canonical Hamiltonian are differentiable, the last of which yields some conditions to be satisfied at the boundary.

**The term at the boundary reads:**

$$V_{\rm bound} = \int_{\partial\Sigma} {\rm d}^2 y \left[ (\sqrt{h} r_a F^{ab} + \theta \tilde{\varepsilon}^{ba} \nabla_a \phi) \delta A_b - (r_a \tilde{P}^a - \theta \tilde{\varepsilon}^{ab} F_{ab}) \delta\phi \right] , \quad (19)$$

*Boundary Conditions.* **An important issue when dealing with theories with boundaries are the boundary conditions that we impose. There are basically two viewpoints:**

**i) Impose physically motivated boundary conditions throughout the evolution. (i.e. isolated horizons).**
**ii) Do not impose conditions a priory, but along the way to make the Hamiltonian formalism well defined.**

**Here we shall not focus on this issue very much, just to mention that there are indeed BC that make both terms in (19) vanish.**

**Once we have ensured that both primary constraint and Hamiltonian are differentiable, we impose the condition that the constraint be preserved under evolution:**

$$\dot{\tilde{\Pi}}_\phi := \{\tilde{\Pi}_\phi, H_T\} \approx 0\,. \tag{20}$$

**This consistency condition leads to a new constraint**

$$\chi := \nabla_a \tilde{\Pi}^a \approx 0 \qquad \Rightarrow \qquad \chi[w] := \int_\Sigma \mathrm{d}^3x\, w\, \nabla_a \tilde{\Pi}^a \approx 0\,. \tag{21}$$

**which is Gauss' law. This smeared constraint is in turn differentiable and its consistency condition**

$$\{\chi[w], H_T\} = -\int_{\partial\Sigma} \mathrm{d}^2y\, N\sqrt{h}(\nabla_a w) r_b F^{ba} = 0\,, \tag{22}$$

**is satisfied under the same conditions needed for the differentiability of $\chi[w]$. So, there are no tertiary constraints, and the form and the algebra of the constraints are the same as in Maxwell theory. They are not affected by the Pontryagin term.**

**Since $H_T$ is differentiable, the corresponding equations of motion can be calculated through Poisson brackets as:**

$$\pounds_t \phi = \{\phi, H_T\} = u \,, \tag{23}$$

$$\pounds_t \tilde{P}_\phi = \{\tilde{P}_\phi, H_T\} = \nabla_a \tilde{P}^a \approx 0 \,, \tag{24}$$

$$\pounds_t A_a = \{A_a, H_T\} = \nabla_a \phi - N^b F_{ab} + \frac{1}{\sqrt{h}} N h_{ab} \tilde{P}^a \,, \tag{25}$$

$$\pounds_t \tilde{P}^a = \{\tilde{P}^a, H_T\} = \nabla_c (\sqrt{h} N F^{ca} - \tilde{P}^c N^a + \tilde{P}^a N^c) \tag{26}$$

**These are the same as Maxwell equations. There are no boundary contributions to equations of motion.**
**The only effect of the Pontryagin term is through boundary conditions needed to make the formalism consistent.**
**Regge-Teitelboim is well defined!**

# Maxwell-Chern-Simons

**Now we have the "same" theory but with a bulk term and a boundary term,**

$$S_{MCS}[A] = S_M[A]+S_{CS}[A] = -\frac{1}{4}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{|g|}g^{ac}g^{bd}F_{cd}F_{ab}-\frac{\theta}{2}\int_{\partial\mathcal{M}} \mathrm{d}^3x\,\tilde{\varepsilon}^{abc}A_aF_{bc}\,. \tag{27}$$

**The Chern-Simons term can be rewritten when the boundary is decomposed,**

$$S_{CS}[A] = \theta\int_I \mathrm{d}t\int_{\partial\Sigma} \mathrm{d}^2y\,\tilde{\varepsilon}^{abc}r_c[(\pounds_t A_a)A_b + F_{ab}\,\phi]\,. \tag{28}$$

**Together with the standard Maxwell term in the bulk, we can now compute the momentum maps:**

$$\mathbf{P}_\phi[f] = \int_\Sigma \mathrm{d}^3x\, \tilde{P}_\phi\, f\,, \tag{29}$$

$$\mathbf{P}[v] = \int_\Sigma \mathrm{d}^3x\, \tilde{P}^a\, V_a + \int_{\partial\Sigma} \mathrm{d}^2x\, \tilde{P}^a_\partial\, v_a\,, \tag{30}$$

**where in the surface integral $v_a$ is a pullback of the form $V_a$ to $\partial\Sigma$.**

**We have acquired a boundary term since the Chern-Simons term has a time derivative of the connection. We will have a boundary term in the symplectic potential and the symplectic structure.**

**We are beyond Regge-Teitelboim!**

**The theory has primary constraints**

$$\tilde{C} := \tilde{P}_\phi = 0\,, \tag{31}$$

$$\tilde{C}^a_\partial := \tilde{P}^a_\partial + \theta\,\tilde{\varepsilon}^{ab} A_b = 0\,. \tag{32}$$

**The canonical Hamiltonian is given by**

$$H = \mathbf{P}[\pounds_t A] + \mathbf{P}_\phi[\pounds_t \phi] - L = H_M + H_{CS}\,, \tag{33}$$

**where** $H_M$ **and** $H_{CS}$ **are the canonical Hamiltonians of the Maxwell theory and the Chern-Simons theory (defined on the boundary** $\partial\Sigma$**), respectively, with**

$$H_{CS} = -\theta \int_{\partial\Sigma} \mathrm{d}^2x\,\tilde{\varepsilon}^{ab} F_{ab}\,\phi\,. \tag{34}$$

**In order to proceed with the consistency condition we need to take into account the boundary contributions to the symplectic structure, and use the equation,**

$$\Omega(X_T, Y) = \mathbb{d} H_C(Y) + \mathbb{d} C(Y) + \mathbb{d} C_\partial(Y)\,, \tag{35}$$

**where the two last terms correspond to the smeared constraints, and $X_T$ is the Hamiltonian vector field of the total Hamiltonian $H_T$.**

**We can now use Eq.(35) to find the Hamiltonian vector field on the bulk and the boundary. The bulk components give the same equations of motion as in Maxwell-Pontryagin. We have now contributions to the boundary,**

$$X_{Aa} = \mu_a\,,$$

$$X^{\partial a}_{\tilde{P}} = r_c(\sqrt{h} N F^{ca} - \tilde{P}^c N^a + \tilde{P}^a N^c) + \theta \tilde{\varepsilon}^{ca}(-2\nabla_c \phi + \mu_c)\,, \tag{36}$$

and an additional constraint on the boundary

$$\tilde{B}^1_\partial := r_a \tilde{P}^a - \theta \tilde{\varepsilon}^{ab} F_{ab} = 0 \,|_{\partial\Sigma} \,. \tag{37}$$

The consistency conditions of the primary constraints are

$$\pounds_t \tilde{C} = \iota_X \mathbb{d} \tilde{P}_\phi = X_{\tilde{P}_\phi} = \nabla_a \tilde{P}^a = 0 \,,$$

$$\pounds_t \tilde{C}^a_\partial = \iota_X \mathbb{d} \tilde{C}^a_\partial = X^{\partial a}_{\tilde{P}} + \theta \tilde{\varepsilon}^{ab} X_{Ab} = r_c(\sqrt{h} N F^{ca} - \tilde{P}^c N^a + \tilde{P}^a N^c) - 2\theta \tilde{\varepsilon}^{ca} \nabla_c \phi = 0 \,,$$

leading to the Gauss constraint

$$\tilde{G} := \nabla_a \tilde{P}^a = 0 \,, \tag{38}$$

and a new constraint on the boundary,

$$\tilde{B}^{2a}_\partial := r_c(\sqrt{h} N F^{ca} - \tilde{P}^c N^a + \tilde{P}^a N^c) - 2\theta \tilde{\varepsilon}^{ca} \nabla_c \phi = 0 \,|_{\partial\Sigma} \,. \tag{39}$$

The Gauss constraint is preserved in time since,

$$\pounds_t \tilde{G} = \iota_X \mathbb{d}\, \tilde{G} = \nabla_a X^a_{\tilde{P}} = 0 \,. \tag{40}$$

**The continuity of $X_{Aa}$ implies that on the boundary**

$$\mu_a = \nabla_a \phi - N^b F_{ab} + \frac{1}{\sqrt{h}} N h_{ab} \tilde{P}^b \,, \tag{41}$$

**As a result, the boundary primary constraints (32) are second class.**

**Due to (39) and (41), the components of the boundary Hamiltonian vector field (36) are of the form**

$$X_{Aa} = \nabla_a \phi - N^b F_{ab} + \frac{1}{\sqrt{h}} N h_{ab} \tilde{P}^a \,,$$

$$X^{\partial a}_{\tilde{P}} = -\theta \tilde{\varepsilon}^{ac} (\nabla_c \phi - N^b F_{cb} + \frac{1}{\sqrt{h}} N h_{cb} \tilde{P}^b) \,. \tag{42}$$

**Note that**

$$X^{\partial a}_{\tilde{P}} = -\theta \tilde{\varepsilon}^{ac} X_{Ac} \,, \tag{43}$$

**consistent with the primary constraint (32).**

**We have a different constraint structure. In the MP system we only have first class constraints, while in the MCS case we have additional constraints on the boundary that turn out to be second class.**

**We have the choice to treat boundary conditions as that or as constraints in the traditional Dirac sense, subject to the consistency conditions. In some cases, this leads to a tower of conditions to be satisfied on the boundary.**

**Within this extended Dirac-Regge-Teitelboim formalism, there is no need to introduce new degrees of freedom. Depending on the particular choices of boundary conditions, we might have remaining degrees of freedom that are not cancelled by gauge at the boundary, or we might have even reduced degrees of freedom. As usual, we let the theory guide us and tell us what is gauge and what is not. We should not impose any preconceived notions of what gauge should be.**

## SUMMARY

- **We have a consistent procedure for addressing gauge theories with boundaries.**

- **The structure of the theory tells us whether we have a boundary $\Omega_b$.**

- **If there is no boundary $\Omega_b$, standard differentiability conditions yield a consistent theory**

- **If $\Omega_b \neq 0$ then we need to extend the DRT procedure. There are boundary contributions to the symplectic structure and to the gradient.**

- **In Maxwell+Topological term, full control over all the issues.**

- **Isolated horizons present another interesting example (another seminar ...).**

# REFERENCES

**Caninical analysis of Topological theory:**
AC, T. Vukasinac **arXiv:1809.09248 (IJMPD, 2019).**

**Canonical analysis of Maxwell+Pontryagin:**
AC, T. Vukasinac, **stay tuned, 2019).**

**Canonical analysis of Isolated Horizons:**
AC, JD Reyes, T. Vukasinac, **stay tuned, 2019).**