Toward LQG effective dynamics

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2. cosmology from full LQG?
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Effective dynamics: some important developments

2006 Ashtekar, Pawlowski and Singh find that LQC quantum dynamics of semiclassical states can be approximated by effective dynamics of

$$H_{LQC}^{eff} := -\frac{3}{8\pi G\gamma^2\Delta} V \sin^2(\sqrt{\Delta} b)$$

2008 Taveras shows that H_{LQC}^{eff} can be obtained as an expectation value of LQC quantum Hamiltonian on Gaussian states:

$$\langle\Psi_{(V,b)}|\hat{H}_{LQC}|\Psi_{(V,b)}\rangle = -\frac{3}{8\pi G\gamma^2\Delta} V\left[\sin^2(\sqrt{\Delta} b) + \mathcal{O}(\epsilon)\right]$$

where $\Psi_{(V,b)}(v) \sim e^{-\frac{\epsilon^2}{2}(V-v)^2 - i\frac{\sqrt{\Delta}}{2}b(V-v)}$.

2013 Alesci and Cianfrani apply Taveras's idea to QRLG, finding the same result for certain Livine-Speziale coherent states.

2017 AD and Liegener apply the same idea to LQG (on a fixed cubic graph), finding for certain complexifier coherent states $\Psi^{\mu,t}_{(c,p)}$

$$\langle\Psi^{\mu,t}_{(c,p)}|\hat{H}_{LQG}|\Psi^{\mu,t}_{(c,p)}\rangle = -\frac{3}{8\pi G\gamma^2\mu^2}\sqrt{p}\left[\sin^2(\mu c) - (1+\gamma^2)\sin^4(\mu c) + \mathcal{O}(t)\right]$$

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Brief reminder on complexifier coherent states [Thiemann, Winkler 2000]

Consider a graph Γ with N edges. Given N $SL(2,\mathbb{C})$ elements $h := (h_1, .., h_N)$,

$$\Psi_h(g_1, .., g_N) = \psi_{h_1}(g_1)..\psi_{h_N}(g_N), \quad \psi_{h_e}(g_e) = \frac{1}{N}\sum_{j=0}^{\infty}(2j+1)e^{-j(j+1)t/2}\chi^j(g_e h_e^\dagger)$$

Properties:

1. Overlap: $\langle\Psi_h|\Psi_{h'}\rangle \sim e^{-X(h,h')/t}$ where $X(h,h') = 0$ iff $h = h'$
2. Expectation values: writing $h_e = e^{-i\tau_I p_e^I} u_e$ with $u_e \in SU(2)$, we have

   $$\langle\Psi_h|\hat{U}_e|\Psi_h\rangle = u_e[1+\mathcal{O}(t)], \qquad \langle\Psi_h|\hat{E}_e^I|\Psi_h\rangle = \frac{1}{\alpha}p_e^I[1+\mathcal{O}(t)]$$

   with $\alpha := t/(16\pi\ell_P^2\gamma)$.
3. Peakedness: $\frac{\Delta U}{\langle\hat{U}\rangle}, \frac{\Delta E}{\langle\hat{E}\rangle} \sim \mathcal{O}(t)$, so t is called "semiclassicality parameter"
4. Matrix elements: $\langle\Psi_h|\hat{A}|\Psi_{h'}\rangle = \langle\Psi_h|\Psi_{h'}\rangle\langle\Psi_h|\hat{A}|\Psi_h\rangle[1+\mathcal{O}(t)]$

Ψ_h is not gauge-invariant. Consider the group-averaging of state Ψ_h:

$$\Psi_h^G(g) := \int dg' \prod_{e\in\Gamma} D(g'_{s_e})D(g'_{t_e})\psi_{h_e}(g) = \int dg' \prod_{e\in\Gamma} \psi_{h_e}(g'^{\dagger}_{t_e} g g'_{s_e})$$
$$= \int dg' \prod_{e\in\Gamma} \psi_{g'_{t_e} h_e g'^{\dagger}_{s_e}}(g)$$

where $D(g_{s_e})$ and $D(g_{t_e})$ are the gauge-transformations at start and target of e.

Now, if $\hat{A}$ is gauge-invariant, we have

$$\langle \Psi_h^G|\hat{A}|\Psi_h^G\rangle = \int dg \overline{\Psi_h^G(g)}(\hat{A}\Psi_h^G)(g) = \int dg dg' dg'' \prod_{e\in\Gamma} \overline{\psi_{g'_{t_e} h_e g'^{\dagger}_{s_e}}(g)}(\hat{A}\psi_{g''_{t_e} h_e g''^{\dagger}_{s_e}})(g)$$
$$= \int dg dg' \prod_{e\in\Gamma} \overline{\psi_{g'_{t_e} h_e g'^{\dagger}_{s_e}}(g)}(\hat{A}\psi_{g'_{t_e} h_e g'^{\dagger}_{s_e}})(g)[1+\mathcal{O}(t)]$$
$$= \int dg dg' \prod_{e\in\Gamma} \overline{\psi_{h_e}(g)}(D(g'_{t_e})^{\dagger} D(g'_{s_e})^{\dagger}\hat{A}D(g'_{s_e})D(g'_{t_e})\psi_{h_e})(g)[1+\mathcal{O}(t)]$$
$$= \int dg \prod_{e\in\Gamma} \overline{\psi_{h_e}(g)}(\hat{A}\psi_{h_e})(g)[1+\mathcal{O}(t)]$$
$$= \langle \Psi_h|\hat{A}|\Psi_h\rangle[1+\mathcal{O}(t)]$$

Complexifier coherent states for cosmology

- RW metric: in adapted coordinates, $ds^2 = -dt^2 + p(t)[dx^2 + dy^2 + dz^2]$
- Ashtekar-Barbero variables: $A_a^I = c\delta_a^I$ and $E_I^a = p\delta_I^a$
- fix the graph: cubic lattice embedded in space along the coordinate axes
- read off the classical holonomy and flux on each edge:

$$u_e = e^{-c\mu\tau_e}, \qquad p_e^I = \delta_e^I \alpha\mu^2 p$$

  where μ is the *coordinate length* of each edge

- construct the $SL(2, \mathbb{C})$ elements $h_e = e^{-i\tau_I p_e^I} u_e$, and use them as label:

$$\Psi_{(c,p)}^{\mu,t} := \Psi_h$$

By construction, $\Psi_{(c,p)}^{\mu,t}$ is peaked on classical RW data along the edges of Γ:

$$\langle\Psi_h|\hat{U}_e|\Psi_h\rangle = e^{-c\mu\tau_e} + \mathcal{O}(t), \qquad \langle\Psi_h|\hat{E}_e^I|\Psi_h\rangle = \delta_e^I\mu^2 p + \mathcal{O}(t)$$

Evaluation of LQG Hamiltonian

Consider non-graph-changing Thiemann Hamiltonian [Giesel, Thiemann 2006], $\hat{H}_{LQG}$.

As already announced, the expectation value on $\Psi^{\mu,t}_{(c,p)}$ is

$$\langle\Psi^{\mu,t}_{(c,p)}|\hat{H}_{LQG}|\Psi^{\mu,t}_{(c,p)}\rangle = -\frac{3}{8\pi G\gamma^2\mu^2}\sqrt{p}\left[\sin^2(\mu c) - (1+\gamma^2)\sin^4(\mu c) + \mathcal{O}(t)\right]$$
$$= H^{eff}_{LQC}\left[1-(1+\gamma^2)\sin^2(\mu c)\right] + \mathcal{O}(t)$$

The extra term can be traced to the Lorentzian part of the scalar constraint. Recall that

$$C = C_E + C_L$$

Two possibilities for C_L:

- "first reduce, then regularize" (as in LQC): at reduced level, $C_L \sim C_E$, so C_L can be regularized as C_E
- "first regularize, then reduce" (as in LQG): in full GR $C_L \not\sim C_E$, so one regularizes them differently

Alternative LQC

Everything boils down to the treatment of extrinsic curvature K^I_a in C_L:

$$K^I_a = \frac{1}{\gamma} A^I_a \qquad \text{vs} \qquad K^I_a = \frac{1}{8\pi G \gamma^3}\{A^I_a, \{C_E, V\}\}$$

Using Thiemann identity in the reduced case of flat cosmology, leads to a 4th order difference operator on the Hilbert space of LQC [Assanioussi, AD, Liegener, Pawlowski]:

$$\Theta = -\frac{3\pi G}{4}\gamma^2 \left[s f_8(v) N^8 - f_4(v) N^4 - 2(s-1) f_0(v) I - f_{-4}(v) N^{-4} + s f_{-8}(v) N^{-8} \right]$$

where $f_a(v) := \sqrt{|v(v+a)|}|v + a/2|$ and $s := (1+\gamma^2)/(4\gamma^2)$.

Quantum evolution of semiclassical states can be approximated by effective dynamics of $\langle \Psi^{\mu,t}_{(c,p)} | \hat{H}_{LQG} | \Psi^{\mu,t}_{(c,p)} \rangle |_{\mu=\bar{\mu}}$. The latter can be analytically solved:

$$V(\phi) = \sqrt{\frac{4\pi G \Delta p_\phi^2}{3}} \, \frac{1 + \gamma^2 \cosh^2(\sqrt{12\pi G}\phi)}{\sinh(\sqrt{12\pi G}\phi)}$$

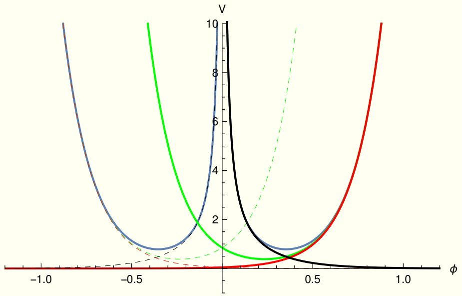

Figure: LQC (green), alt LQC (blue), classical FLRW (red), dS with Λ_{eff} (black).

Pre-bounce branch: contracting de Sitter with effective cosmological constant

$$\Lambda_{eff} = \frac{3}{\Delta(1+\gamma^2)}$$

The physics of this model (with inflaton field) has been studied in the contexts of effective dynamics **[Li, Singh, Wang 2018]** and primordial power spectrum **[Agullo 2018]**.

Main message: At least in the symmetry-reduced setting, $\langle \Psi_h | \hat{H} | \Psi_h \rangle$ can be thought of as an **effective Hamiltonian**, i.e., its dynamics captures the main feature of the quantum dynamics of semiclassical states.

Remark on μ_0 vs $\bar{\mu}$

In LQG, μ is a parameter of the complexifier coherent state Ψ_h, representing the *coordinate length* of an edge: μ_0-scheme seems therefore the natural choice.

$\Rightarrow$ several problems: in particular, bounce at sub-Planckian energy density

Possible way out: Recover $\bar{\mu}$-scheme via the following procedure **[Alesci, Cianfrani 2016]**. Consider the mixed state

$$\hat{\rho} = \sum_{N=1}^{N_{max}} c_N |\Psi^N_{(c,p)}\rangle\langle\Psi^N_{(c,p)}|$$

with $\Psi^N_{(c,p)}$ semiclassical state living on a graph with N vertices. Then, choosing N_{max} and c_N appropriately, one finds

$$\langle\hat{H}\rangle := \text{Tr}(\hat{\rho}\hat{H}) = \sum_{N=1}^{N_{max}} c_N \langle\Psi^N_{(c,p)}|\hat{H}|\Psi^N_{(c,p)}\rangle = \sum_{N=1}^{N_{max}} c_N H^{eff}_{LQC}|_{\mu=\frac{1}{N}} = H^{eff}_{LQC}|_{\mu=\bar{\mu}} + O(t)$$

Problem: N_{max} is fixed by p; since effective dynamics generated by $\langle\hat{H}\rangle$ changes p in time, then N_{max} must change in time. However, quantum Hamiltonian $\hat{H}$ is graph-preserving, so it cannot change the number of vertices!

Conclusion – Effective dynamics generated by $\langle\hat{H}\rangle$ cannot be a good approximation of the full quantum dynamics generated by $\hat{H}$.

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Generalized coherent states

Theorem

Let Ψ be a state of the form

$$\Psi(g) = \frac{1}{N} f(g) e^{-S(g)/t}$$

where t-dependent normalization N and f and S t-independent holomorphic functions. If S satisfies

- $\mathrm{Re}(S)$ has single minimum at g_o
- Hessian of S is non-degenerate at g_o

then we have (use saddle point method)

$$\langle\Psi|\hat{U}_e|\Psi\rangle = g_{o,e}\big[1 + \mathcal{O}(t)\big], \qquad \langle\Psi|\hat{E}^I_e|\Psi\rangle = \frac{1}{t}\big[P^I_{o,e} + \mathcal{O}(t)\big]$$

where $P^I_{o,e} := -32\pi\ell_P^2\gamma\big(R^I_e \mathrm{Im}(S)\big)(g_o)$.

For this reason, we say that

$$\Psi \text{ is a generalized coherent state peaked at } (g_o, P^I_o)$$

Example: complexifier coherent states.

Pseudodifferential operators and principal symbols

Consider a fixed graph with N edges. The phase space associated with it is T^*G with $G = SU(2)^N$.

To any smooth function $a \in T^*G$, we can associate an operator $\hat{A}$ on $L_2(G, d\mu)$: its kernel is

$$A(g_1, g_2) = \frac{1}{(2\pi)^{\dim G}} \int_{\mathrm{Lie}(G)} d\xi \; e^{i\xi^I X_I(g_1,g_2)} a(g_1, \xi)$$

with $X \in \mathrm{Lie}(G)$ s.t. $g_1^\dagger g_2 = e^{2X}$.
$\hat{A}$ is called a *pseudodifferential operator*, pdo. Note: polynomial of pdo's is pdo.

The *principal symbol* of $\hat{A}$, denoted $\mathcal{P}(\hat{A})$, is the leading order of the power series of a for large ξ. Some properties:

- $\mathcal{P}(\hat{A}\hat{B}) = \mathcal{P}(\hat{A})\mathcal{P}(\hat{B})$
- $\mathcal{P}([\hat{A}, \hat{B}]) = i\hbar\{\mathcal{P}(\hat{A}), \mathcal{P}(\hat{B})\}$

Example: $\hat{U}_e$ and $\hat{E}_e^I$ are pdo's with principal symbols

$$\mathcal{P}(\hat{U}_e)(g, \xi) = g_e, \qquad \mathcal{P}(\hat{E}_e^I)(g, \xi) = \xi_e^I$$

Theorem

Let Ψ be a generalized coherent state peaked at (g, P) and $\hat{A}$ a pdo. Then

$$\langle\Psi|\hat{A}|\Psi\rangle = \mathcal{P}(\hat{A})\left(g, \frac{P}{t}\right)[1 + \mathcal{O}(t)]$$

This in particular applies to polynomials $f(\hat{U}, \hat{E})$, for which we thus have

$$\langle\Psi|f(\hat{U}, \hat{E})|\Psi\rangle = f\left(g, P/t\right)[1 + \mathcal{O}(t)]$$

We would like to use this result to compute expectation value of $\hat{H}_{LQG}$ on Ψ.

Problem: $\hat{H}_{LQG}$ involves volume $\hat{V} = \sum_v \hat{V}_v$, which is not a polynomial:

$$\hat{V}_v = \sqrt{\frac{1}{48}\left|\sum_{e,e',e'' \text{ at } v} \epsilon(e, e', e'')\epsilon_{IJK}\hat{E}^I_e\hat{E}^J_{e'}\hat{E}^K_{e''}\right|}$$

Solution: *microlocal equivalence*.

Microlocal equivalence

Consider $(g_o, \xi_o) \in T^*G$ and the class of cunctions c s.t.

- $c(g, \lambda\xi) = c(g, \xi)$ for all $\lambda > 0$
- $c(g, \xi) = 1$ for (g, ξ) in a neighbourhood of (g_o, ξ_o)

We denote this class by $S_{(g_o, \xi_o)}$.

Two (not necessarily pdo's), $\hat{A}$ and $\hat{B}$, are *microlocally equivalent* at (g_o, ξ_o) iff there exists $c \in S_{(g_o, \xi_o)}$ such that $(\hat{A} - \hat{B})\hat{C}$ has smooth kernel. We write

$$\hat{A} \stackrel{(g_o, \xi_o)}{=} \hat{B}$$

Theorem

Let Ψ be a generalized coherent state peaked at (g, P). If $\hat{A} \stackrel{(g,P)}{=} \hat{B}$, then

$$\hat{A}|\Psi\rangle = \hat{B}|\Psi\rangle + \mathcal{O}(t^\infty)$$

Evaluation of $\langle \hat{H}_{LQG} \rangle$

It is possible to find a pdo $\hat{W}$ such that

$$\hat{W} \overset{(g,P)}{=} \hat{V}$$

for all (g, P) with $\mathcal{P}(\hat{W})(g, P) \neq 0$. Not surprisingly, the principal symbol of this $\hat{W}$ is

$$\mathcal{P}(\hat{W})(g, P) = \sqrt{\frac{1}{48}\left| \sum_{e,e',e'' \text{ at } v} \epsilon(e, e', e'')\epsilon_{IJK} P^I_e P^J_{e'} P^K_{e''} \right|}$$

which is the classical volume, $V_{class}(P)$.

Putting all together, we get

Theorem

If Ψ is generalized coherent state peaked on (g, P) with non-zero volume, then

$$\begin{aligned}\langle \Psi | \hat{H}_{LQG} | \Psi \rangle = \langle \Psi | H_{LQG}(\hat{U}, \hat{V}) | \Psi \rangle &= \langle \Psi | H_{LQG}(\hat{U}, \hat{W}) | \Psi \rangle + \mathcal{O}(t^\infty) \\ &= H_{LQG}(g, V_{class}(P/t)) \, \left[1 + \mathcal{O}(t)\right]\end{aligned}$$

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Let us go back to quantum mechanics.

Egorov's Theorem

Let $\hat{A}$ be a positive, self-adjoint, elliptic pdo. If $\hat{B}$ is a pdo, then

$$\hat{B}_s := e^{is\hat{A}/\hbar}\hat{B}e^{-is\hat{A}/\hbar}$$

is also a pdo.

IF this theorem can be applied to our case (phase space T^*G), using the result of previous section it follows

$$\langle\Psi|\hat{B}_s|\Psi\rangle = \mathcal{P}(\hat{B}_s)(g, P/t)\ \big[1+\mathcal{O}(t)\big] = b_s(g, P/t)\ \big[1+\mathcal{O}(t)\big]$$

where we denoted by b_s the principal symbol of $\hat{B}_s$.

Apply d/ds on both sides: at leading order in t we have

$$\begin{aligned}\frac{d}{ds}b_s(g,P/t) \approx \frac{d}{ds}\langle\Psi|\hat{B}_s|\Psi\rangle = \frac{i}{\hbar}\langle\Psi|[\hat{A},\hat{B}_s]|\Psi\rangle \approx \frac{i}{\hbar}\mathcal{P}([\hat{A},\hat{B}_s])(g,P/t) \\ = -\{\mathcal{P}(\hat{A}), b_s\}\end{aligned}$$

Conclusion (assuming that $\hat{H}_{LQG}$ satisfies Egorov's theorem requirements).

Conjecture

Let Ψ be a generalized coherent state peaked at (g, P). Then, for any pdo $\hat{B}$, the expectation value on the time-evolved state

$$b_s := \langle \Psi | e^{is\hat{H}_{LQG}/\hbar} \hat{B} e^{-is\hat{H}_{LQG}/\hbar} | \Psi \rangle$$

satisfies to leading order in t the effective Hamilton equation

$$\frac{d}{ds} b_s = \{b_s, H_{eff}\}, \qquad b_0 = \langle \Psi | \hat{B} | \Psi \rangle$$

where the *effective Hamiltonian* is

$$H_{eff} := \langle \Psi | \hat{H}_{LQG} | \Psi \rangle$$

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If the conjecture is true, then we have a *dynamical* confirmation that (discretized) GR is the classical limit of LQG.

Moreover, we:

- confirm that the alternative LQC presented at the beginning *is* the cosmological sector of LQG
- can apply this method to other symmetry-reduced systems, e.g. BH

Application: static, spheically symmetric black holes

System extensively studied in the reduced-symmetry context: **Ashtekar, Bodendorfer, Boehmer, Bojowald, Campiglia, Corichi, Gambini, Modesto, Olmedo, Pullin, Saini, Singh, Vandersloot, Alesci, Pranzetti, ...**

Summary – Schwartzschild interior can be recasted in Kantowski-Sachs form:

$$ds^2 = -dT^2 + f(T)^2 dR^2 + g(T)^2 d\Omega^2$$

Points of interest:

- horizon: $f(T_h) = 0$ and $g(T_h) = 2M$
- singularity: $g(T_s) = 0$

Ashtekar-Barbero variables:

$$A_R^R = -\gamma a, \quad A_\theta^\theta = -\gamma b, \quad A_\phi^\phi = -\gamma b \sin\theta, \quad A_\phi^R = \cos\theta$$

$$E_R^R = p_a \sin\theta, \quad E_\theta^\theta = \frac{p_b}{2}\sin\theta, \quad E_\phi^\phi = \frac{p_b}{2}$$

where $\{a, p_a\} = 2G = \{b, p_b\}$ are related to f, g by $p_a = g^2$ and $p_b = 2fg$.

Construction of Ψ_h: recall that, writing $h_e = e^{-i\tau_I p^I_e} u_e$, we have

$$\langle \Psi_h | \hat{U}_e | \Psi_h \rangle = u_e [1 + \mathcal{O}(t)], \qquad \langle \Psi_h | \hat{E}^I_e | \Psi_h \rangle = \frac{p^I_e}{\alpha} [1 + \mathcal{O}(t)]$$

Hence, for the current system we must choose

$$u_R = \exp[\gamma a \tau_1 \mu_1], \quad u_\theta = \exp[\gamma b \tau_2 \mu_2], \quad u_\phi = \exp[(\gamma b \tau_3 \sin\theta - \tau_1 \cos\theta)\mu_3]$$

$$p^1_R = \alpha p_a \mu_2 \mu_3 \sin\theta, \quad p^2_\theta = \frac{\alpha}{2} p_b \mu_3 \mu_1 \sin\theta, \quad p^3_\phi = \frac{\alpha}{2} p_b \mu_1 \mu_2$$

Since the state Ψ_h thus obtained in peaked on (u, p), from the previous results

$$H_{eff} := \langle \Psi_h | \hat{H}_{LQG} | \Psi_h \rangle = H_{LQG}(u, V_{class}(p/t)) \; [1 + \mathcal{O}(t)]$$

Computations are long (they involve a non-trivial sum over θ), but mechanical. The result is too long to fit a slide, and unfortunately *does not* look like the effective Hamiltonian found in literature by "polymerization".

Still, H_{eff} is analytical and Hamilton's equations can be integrated numerically.

Dynamics of f and g is particularly imporntant.

$$g = -dT^2 + f(T)^2 dR^2 + g(T)^2 d\Omega^2$$

Numerical solution for initial conditions at horizon with $M = \#$ of vertices:

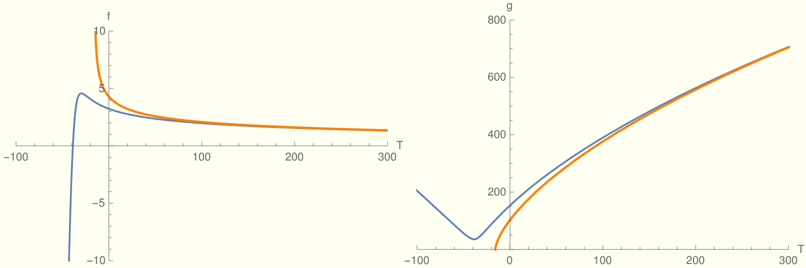

Figure: Metric components f and g in LQG (blue) and GR (orange).

BH $\rightarrow$ WH transition very non-symmetric (contrary to results in the literature).

Mass of the final state:

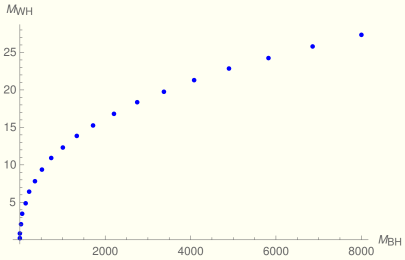

Figure: White Hole mass as a function of Black Hole mass M.

Approximately $M_{WH} \sim M^{\frac{1}{3}}$. Different from [Ashtekar, Olmedo, Singh 2018], where a linear relation is found.

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Summary of the results

- Proposal for generalized coherent states Ψ_h representing any discrete spatial geometry (g, P). Proof that, to leading order in t,

$$\langle \Psi_h | \hat{H}_{LQG} | \Psi_h \rangle \approx H_{LQG}(g, V_{class}(P/t))$$

- Dynamical conjecture: for any pdo $\hat{B}$, to leading order in t,

$$b_s := \langle \Psi | e^{is\hat{H}_{LQG}/\hbar} \hat{B} e^{-is\hat{H}_{LQG}/\hbar} | \Psi \rangle$$

  satisfies

$$\frac{d}{ds} b_s = \{b_s, H_{eff}\}, \qquad \text{with} \quad H_{eff} := \langle \Psi | \hat{H}_{LQG} | \Psi \rangle$$

  Therefore, LQG *dynamically* reduces to (discrete) GR in the classical limit!

- Applications:
  - ∗ cosmology: alternative LQC (non-symmetric bounce)
  - ∗ spherical BH: singularity replaced by BH → WH transition, $M_{WH} \sim \sqrt[3]{M}$

Open questions

- Proof of the conjecture!
- Role of discreteness scale μ: LQG with graph-preserving Hamiltonian suggests μ_0-scheme, but this seems to lead to unphysical predictions
- More applications:
  - ∗ general spherical models: stellar collapse and BH formation
  - ∗ cylindrical models: Kerr BH
  - ∗ develop code to perform $\langle \Psi_h | \hat{H}_{LQG} | \Psi_h \rangle$ for any given discrete geometry