Black Hole Entropy and Number Theory

International Loop Quantum Gravity Seminar
February 10th, 2009.

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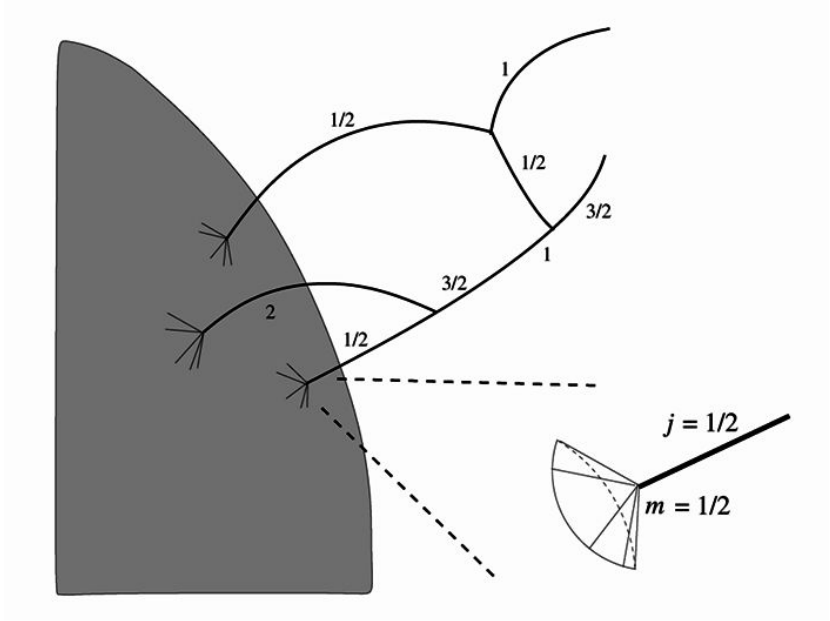
PLAN OF THE TALK

- Introduction (extra-brief review to BH entropy in LQG)
- Motivation
- Combinatorial problem
- How can Number Theory help?
- Improved computational analysis
- Generating Functions
- Asymptotics revisited
- New windows
  - Asymtotic expansion
  - CFT-related computation
  - Hawking radiation

INTRODUCTION

A. Ashtekar, J. Baez, A. Corichi, K. Krasnov. Phys.Rev.Lett 80 (1998) 904
A. Ashtekar, J. Baez, K. Krasnov. Adv.Theor.Math.Phys 4 (2000) 1

- Black hole entropy arises, in LQG, from the (isolated) horizon degrees of freedom
- Horizon described by a Chern-Simons theory on a sphere with N distinguishable punctures
- CS states labeled by quantized angle deficits a_i
- N spin network edges, with labels j_i and m_i, pierce the horizon
  - j_i labels the contribution to area
  - $m_i \in \{-j_i, -j_i+1, \ldots, j_i\}$ are the spin projections and match with a_i through IH boundary conditions



INTRODUCTION

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- For a given value of area A, entropy is given by

$$S(A) = k_B \ln N(A) \quad (1)$$

  - N(A) = number of independent CS states
- Horizon states must satisfy:

$$\sum_{i=1}^{N} a_i = 0 \quad (2) \qquad 2m_i = -a_i \mod k \quad (3)$$

$$m_i \in \{-j_i, -j_i + 1, ..., j_i\} \quad (4) \qquad 8\pi\gamma\ell_P^2 \sum_{i=1}^{N} \sqrt{j_i(j_i+1)} = A \quad (5)$$

- Goal: count all possible ordered lists of a_i satisfying the above constraints for a given A.

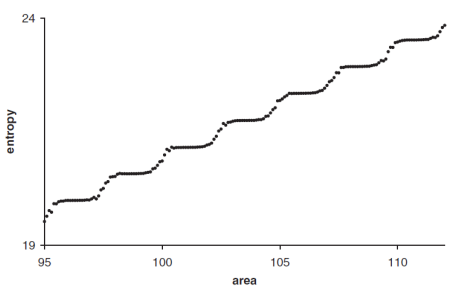
MOTIVATION

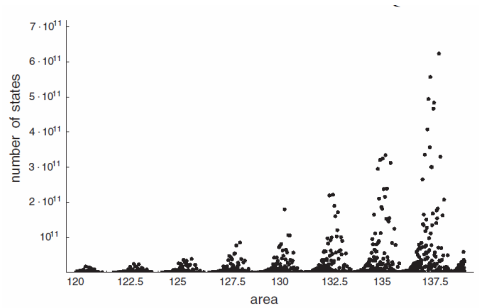
- Approximate solution of the problem for the asymptotic limit:

$$S(A) = \frac{\gamma_0}{\gamma}\frac{A}{4\ell_P^2} - \frac{1}{2}\ln\frac{A}{\ell_P^2} \qquad (6)$$

  - Linear behavior of entropy with area
  - Choosing the appropriate value of γ, obtain the Bekenstein-Hawking law
  - Additional logarithmic correction

- Exact computational solution for microscopic black holes
  - Discretization
  - Band structure





- Questions:
  - Is the discrete behavior a feature only of the low area regime?
  - Can we better understand its origin?
  - Can we extend the exact computations to the large area limit?

COMBINATORIAL PROBLEM

M.Domagała, J. Lewandowski. Class. Quantum Grav. 21 (2004) 5233

- State the combinatorial problem in a precise but simple way
- Express it in terms of only m_i labels

N(A) is 1 *plus the number of all the finite, arbitrarily long, sequences* $(m_i,\ldots,m_N)$ *non-zero half integers, such that*

$$\sum_{i=1}^{N} m_i = 0 \quad (7) \qquad\qquad \sum_{i=1}^{N} \sqrt{|m_i|(|m_i|+1)} \leq \frac{A}{8\pi\gamma\ell_P^2} \quad (8)$$

- This is equivalent to compute all CS states satisfying equations (2), (3), (4) and (5)

NUMBER THEORY SOLUTION

I. Agulló, J.F. Barbero G., J. Díaz-Polo, E. Fernández-Borja, E.J.S. Villaseñor. Phys.Rev.Lett.100 (2008) 211301

- In order to further simplify the notation, let us work with integer positive numbers, by defining $k_i = 2|m_i|$
- Define also the occupancy numbers n_k as the number of punctures in a given state with labels m such that 2|m|=k
- Use units such that $4\pi\gamma\ell_P^2 = 1$
- A set of n_k numbers characterizes a horizon state up to reorderings and sign assignments.
- We can solve the combinatorial problem in 4 steps:
  - Compute all n_k sets that satisfy $\sum_{k=1}^{k_{max}} n_k \sqrt{k(k+2)} = A$
  - Compute, for each of them, all possible ordered lists of k
  - Compute, for each of them, all possible ordered sign assignments
  - Sum the resulting degeneracy for all values of area lower than A

NUMBER THEORY SOLUTION

I. Agulló, J.F. Barbero G., J. Díaz-Polo, E. Fernández-Borja, E.J.S. Villaseñor. Phys.Rev.Lett.100 (2008) 211301

STEP 1: CHARACTERIZING THE AREA SPECTRUM

- Solve the equation $$\sum_{k=1}^{k_{max}} n_k \sqrt{k(k+2)} = A \quad (9)$$
- Any value $\sqrt{k(k+2)}$ can be written as a linear combination of square roots of square-free (SQSF) numbers p_i: $\sum q_i \sqrt{p_i}$
- Equation (9) can be rewritten: $$\sum_{k=1}^{k_{max}} n_k \sqrt{(k+1)^2 - 1} = \sum_{i=1}^{r} q_i \sqrt{p_i} \quad (10)$$
- For a given value of p, which values of k can contribute?
  - Solve: $\sqrt{(k+1)^2 - 1} = y\sqrt{p_i} \quad (11)$
  - Pell equation $x^2 - p_i y^2 = 1$ known in number theory, for a given p_i infinite known solutions (x^i_m, y^i_m): $x^i_m + y^i_m \sqrt{p_i} = (x^i_1 + y^i_1 \sqrt{p_i})^m$
- Then, equation (9) can be rewritten:

$$\sum_{i=1}^{r} \sum_{m=1}^{\infty} n_{k^i_m} y^i_m \sqrt{p_i} = \sum_{i=1}^{r} q_i \sqrt{p_i} \quad (12)$$

NUMBER THEORY SOLUTION

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- Taking into account the linear independence of SQSF numbers with integer coefficients, we get:

$$\sum_{m=1}^{\infty} y_m^i n_{k_m^i} = q_i, \qquad i = 1, 2, ..., r \quad (13)$$

- This set of uncoupled diophantine equations can be solved by standard methods
- In practice, all sums are finite
- For a given value of area (q_1,...,q_r):
  - Either we get no solutions → area does not belong to the spectrum
  - Or we get solutions of the kind $\{(k_m^i, n_{k_m^i})\}$ → all k that contribute to this area and the corresponding occupancy numbers → all n_k sets we were looking for

NUMBER THEORY SOLUTION

I. Agulló, J.F. Barbero G., J. Díaz-Polo, E. Fernández-Borja, E.J.S. Villaseñor. Phys.Rev.Lett.100 (2008) 211301

STEP 2: REORDERING DEGENERACY

- Given a $\{n_k\}$ set, the number of different ordered k sequences can be obtained from basic combinatorics:

$$R(\{n_k\}) = \frac{(\sum_{k=1}^{k_{max}} n_k)!}{\prod_{k=1}^{k_{max}} n_k!} \qquad (14)$$

NUMBER THEORY SOLUTION

I. Agulló, J.F. Barbero G., J. Díaz-Polo, E. Fernández-Borja, E.J.S. Villaseñor. Phys.Rev.Lett.100 (2008) 211301

STEP 3: SIGN ASSIGNMENTS

- Given an ordered sequence of k_i, look for all choices of signs for the corresponding m_i, such that the projection constraint (7) is satisfied
- This problem is analogous to the "partition problem":
  - Given a set of N natural numbers, which are the possible ways of splitting it into two disjoint subsets, such that the elements of both subsets sum up to the same value?
  - To find all solutions in general is a very hard problem in number theory
  - However, we just need the number of solutions. The answer to this question is known in general, and is given by:

$$P(\{n_k\}) = P(\{k_i\}) = \frac{2^N}{M} \sum_{s=0}^{M-1} \prod_{i=1}^{N} \cos\left(2\pi s k_i / M\right) \quad (15) \qquad M = 1 + \sum_{i=1}^{N} k_i$$

NUMBER THEORY SOLUTION

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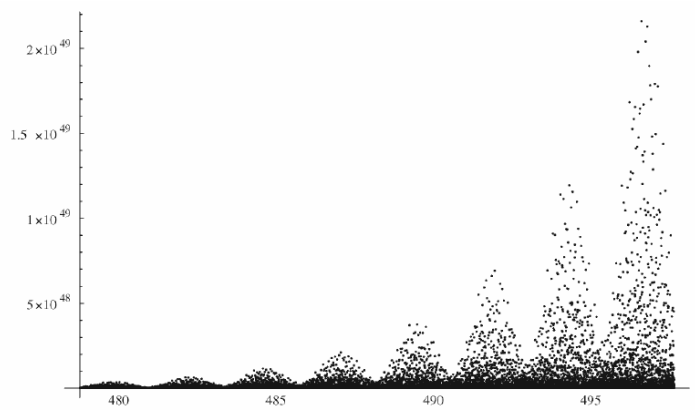
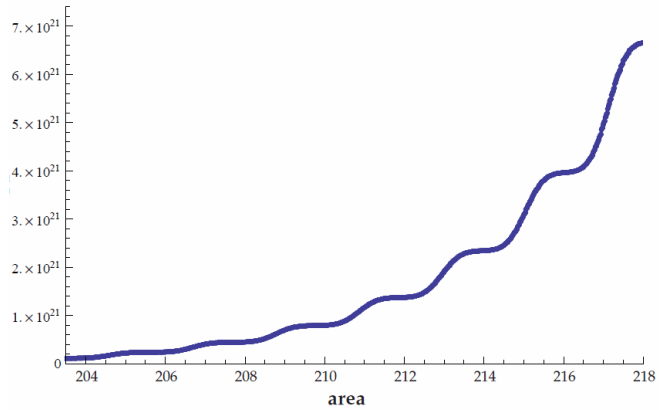
- With this three steps, we obtain the "degeneracy spectrum" of the black hole:



STEP 4: INTRODUCING THE INEQUALITY

- Summing for all the lower values of area we get the (exponentiated) stair behavior:



IMPROVED COMPUTATIONAL ANALYSIS

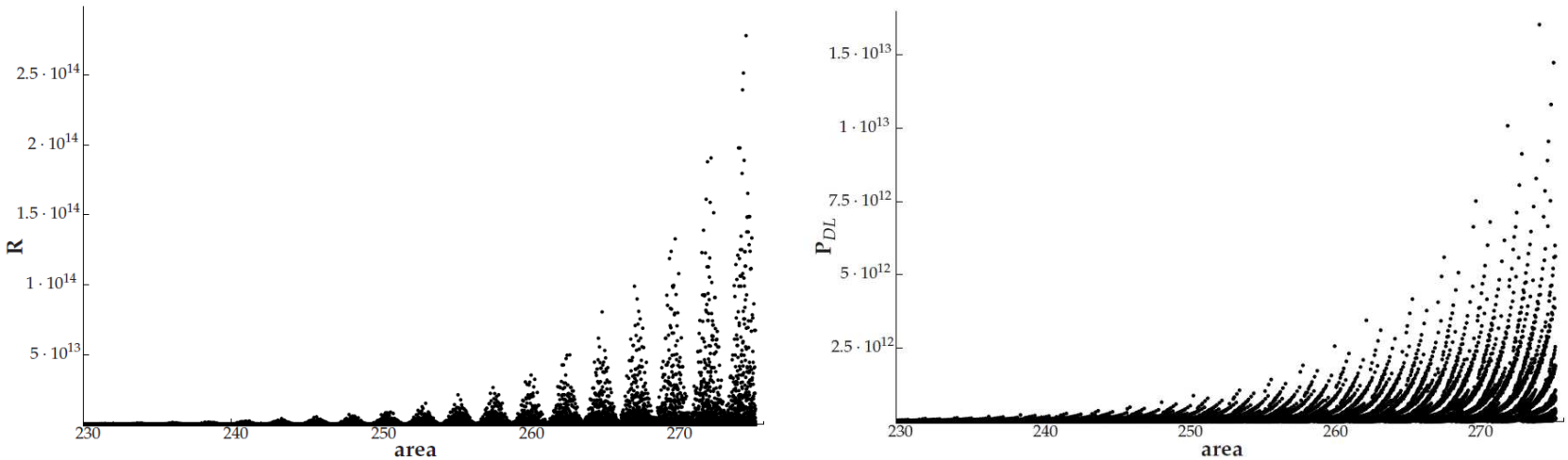
I. Agulló, J.F. Barbero G., E.F. Borja, J. Diaz-Polo, E.J.S. Villaseñor. *In progress*

- This 4-step procedure can be implemented in a computer
- Improve the existing results by allowing to:
  - Reach higher values of area (seen in the previous slide)
  - Weight the contribution of the different "spin" values k
  - Make a separate study of the different sources of degeneracy
  - Test the previous models that account for the origin of bands
- Contribution of different k values
  - Just by choosing the square free numbers p_i appearing in the values of area we control the values of k that can contribute
  - A detailed analysis shows that only the lower values of k are responsible for the highest degenerate states (the peaks)

IMPROVED COMPUTATIONAL ANALYSIS

I. Agulló, J.F. Barbero G., E.F. Borja, J. Diaz-Polo, E.J.S. Villaseñor. *In progress*

- Plot separately the two degeneracies R and P

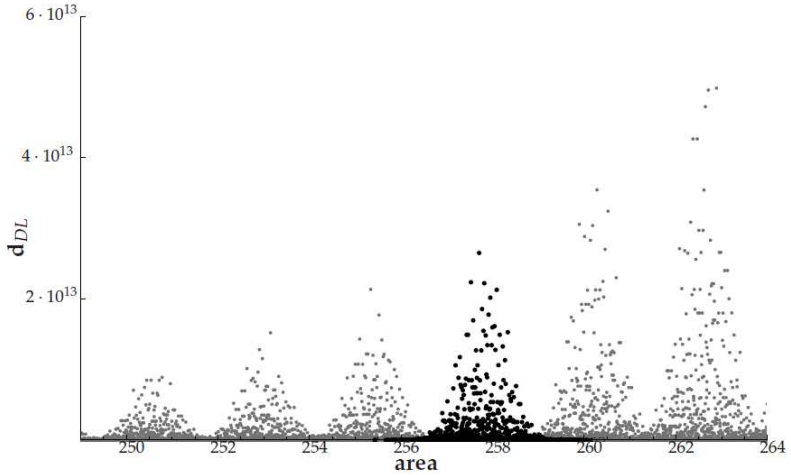


  - The band structure comes entirely from the reordering degeneracy
  - The sign configurations solving the projection constraint give an overall exponential contribution.

IMPROVED COMPUTATIONAL ANALYSIS

I. Agulló, J.F. Barbero G., E.F. Borja, J. Diaz-Polo, E.J.S. Villaseñor. *In progress*

- Analysis made in *Agulló, Diaz-Polo, Fernandez-Borja. Phys. Rev. D77 104024 (2008)* made use of a parameter K characterizing all states appearing in a given peak
- This characterization allowed to compute with good accuracy the area spacing between peaks
- We can test this analysis by plotting all states with a given value of that parameter K



- We observe that this K is a good parameter to characterize a given band
- This supports the reliability of the results in that paper

GENERATING FUNCTIONS

J.F. Barbero G., E.J.S. Villaseñor. Phys.Rev. D77 (2008) 121502

- It is possible to push one more step forward in this analytical solution of the combinatorial problem
- A generating function contains all the information about the solution to a combinatorial problem
- It allows to obtain closed analytical expressions for the solution, that are the starting point for an asymptotic analysis
- Can we find a generating function for our combinatorial problem?

YES!

- We will use functions of the variables x_i, such that the coefficient of the $(x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r})$ term in the Taylor expansion is the solution for the value of area $q_1\sqrt{p_1} + q_2\sqrt{p_2} + \cdots + q_r\sqrt{p_r}$

GENERATING FUNCTIONS

J.F. Barbero G., E.J.S. Villaseñor. Phys.Rev. D77 (2008) 121502

- Generating functions for the number of (positive) solutions for diophantine equations like (13) are given in text books

$$G^{\#sol}(x_1, x_2, \ldots) = \prod_{i=1}^{\infty} \prod_{m=1}^{\infty} \frac{1}{(1 - x_i^{y_m^i})} \qquad (16)$$

- If we want the solution to include the reordering degeneracy, we get

$$G^{R}(x_1, x_2, \ldots) = \left(1 - \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} x_i^{y_m^i}\right)^{-1} \qquad (17)$$

- We can also give a generating function for the solutions of the projection constraint (given a list of k_i)

$$G^{\pm}(z) = \prod_{i=1}^{N} (z^{k_i} + z^{-k_i}) \qquad (18)$$

  – This time one has to compute the Laurent expansion

GENERATING FUNCTIONS

J.F. Barbero G., E.J.S. Villaseñor. Phys.Rev. D77 (2008) 121502

- Putting all together, we get the generating function for the whole combinatorial problem

$$G^d(z, x_1, x_2, \ldots) = \left(\sum_{i=1}^{\infty} \sum_{m=1}^{\infty} (z^{k_m^i} + z^{-k_m^i}) x_i^{y_m^i} \right)^{-1} \qquad (19)$$

where the coefficient of the term $z^n x_1^{q_1} \cdots x_r^{q_r}$ in the expansion gives the number of solutions such that $\sum_i m_i = n$

- Closed integral expressions can be obtained for the general form of the coefficients in the expansions, using Cauchy's theorem

$$N(A) = N(q_1, \ldots, q_r) = \frac{1}{(2\pi i)^r} \oint_{\gamma_1} \frac{\mathrm{d}\zeta_1}{\zeta_1^{q_1+1}} \cdots \oint_{\gamma_r} \frac{\mathrm{d}\zeta_r}{\zeta_r^{q_r+1}} G^d(1; \zeta_1, \cdots, \zeta_r, 0, \ldots) \qquad (20)$$

ASYMPTOTICS REVISITED

J.F. Barbero G., E.J.S. Villaseñor. Class.Quantum Grav. 26 (2009) 035017

- We come back to the question: is the discrete behavior also present in the large area regime?
- The new available tools allow to explore this question
- Barbero and Villaseñor found an alternative way to find Meissner's result, with some subtleties:
  - There is a subtle point missing in the asymptotic analysis
  - The real pole is an accumulation point for the real parts of the poles of the Laplace transform:

$$P(s) = \frac{2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2}}{s(1 - 2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2})}$$

  - While still being true that the real pole drives the asymptotic behavior, one cannot exclude a contribution from the infinite many infinitely close (in real part) poles
- There is room for the discrete behavior to appear in the large area limit!

NEW WINDOWS

The powerful techniques presented here allow to address some problems from a new perspective:

- The asymptotic analysis that can determine the presence or not of the discrete behavior of entropy in for macroscopic black holes (Agullo, Barbero, Borja, Diaz-Polo, Villaseñor, *in progress*)
- A new way of computing black hole entropy using CFT-related techniques (Agullo, Borja, Diaz-Polo, *in progress*)
- A more detailed study of the effects in Hawking radiation spectrum, in the spirit of *J.Diaz-Polo, E.Fernandez-Borja. Class.QuantumGrav. 25 (2008) 105007*.