Recursive graphical construction of GFT Feyn. diagrams.

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work in collaboration with Dr. D. Oriti.

Max Planck Institute for Gravitational Physics,
Albert Einstein Institute (AEI)

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MAX-PLANCK-GESELLSCHAFT

Max-Planck-Institut
für Gravitationsphysik
(Albert-Einstein-Institut)

Plan of the Talk.

Part I.

- Introduction to GFTs.
- Divergences and Rad. Corr. in GFTs and SFs. *Why important? What can we learn?*

Two main challanges:

- *Computing the amplitudes' scaling.*
- *Generate the relevant classes of Feyn. graphs.*

Part II.

- Computing the amplitudes' (superficial) degree of divergence (*d.o.d*). Known methods.
- Short review of recent results on the Rad. Corr. (N-point functions, $N \leq 6$, order λ_5^2) for a new class of Simplicial GFT models (work with D. Oriti).

Part III. Main part of the talk.

- Recursive graphical construction of GFT Feynman diagrams. Warm up example from ordinary QFT: $\lambda\phi^k$ theories, $k \geq 3$.
- The same ideas can be adapted to GFTs. Mathematica notebook demonstration.
- Concluding remarks and outlook.

Divergences, Radiative corrections and Beyond.

Three parts. Three fundamental questions I will focus on in my talk.

Why divergences are an important issue to investigate?

How to compute the d.o.d of a given graph?

How to automatize the generation of Feynman graphs and what to do with it?

Many other current research topics in GFT I will not talk about.

Renormalization (P/NP). Continuum limit.
Carrozza.

Effective Macroscopic Physics.
Gielen, Oriti, Sindoni; Oriti, Sindoni, WE.
Pranzetti, Oriti, Sindoni. (to app.)

Entanglement Entropy and Holography.
Chirco, Mele, Oriti; Chirco, Oriti, Zhang.

Formal algebraic aspects of GFTs.
Kegeles, Oriti;
Kotecha, Oriti. (to app.)

GFTs are combinatorially non-linear and non-local QFT on group manifolds.

$$\Phi : \mathbb{G}^{\times d} \to \mathbb{K}, \qquad \mathcal{Z}_{\mathrm{GFT}}[\mathcal{J}] = \frac{1}{\mathcal{Z}_0} \int d\mu_{\mathcal{K}}(\Phi) e^{-S_{\mathrm{int}}[\Phi] - \Phi \mathcal{J}} \tag{1.1}$$

$$\Phi(G_i) = \Phi(G_1, \cdots, G_d); \quad \mathbb{G} = \mathrm{SU}(2),\ \mathrm{Spin}(4),\ \mathrm{SL}(2,\mathbb{C}); \quad \mathbb{K} = \mathbb{R},\, \mathbb{C} \tag{1.2}$$

Two classes of models: Simplicial and Tensorial GFTs. SGFT, example in $d = 4$.

$$\mathcal{K} : \quad \text{Covariance} \qquad S_{\mathrm{int}}[\Phi] = \frac{\lambda_5}{5!} \int [dG_{\mathrm{All}}]\ \Phi\,\Phi\,\Phi\,\Phi\,\Phi\,\mathcal{V}_5 \tag{1.3}$$

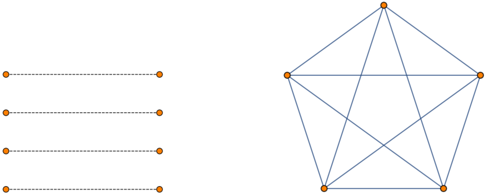


Figure: Stranded representations of the covariance and of the simplicial interaction.

Tensorial GFTs are a generalization of Tensor models. For rank-d models:

- The theory space is given by all possible d-bubble interactions terms.
- A d-bubble $b \in \mathcal{B}_p^d$ (p even) is connected bipartite d-colored graph with p d-valent nodes.

$$\mathcal{K}: \quad \text{Covariance} \qquad S_{\rm int}[\Phi, \bar{\Phi}] = \sum_{n=1}^{\infty} \sum_{b \in \mathcal{B}_{2n}} \frac{\lambda_b}{c_b} S_b[\Phi, \bar{\Phi}] \tag{1.4}$$

By construction $\mathcal{B}_p$ consists of all connected bipartite vacuum graphs of the ϕ^d with p vertices.

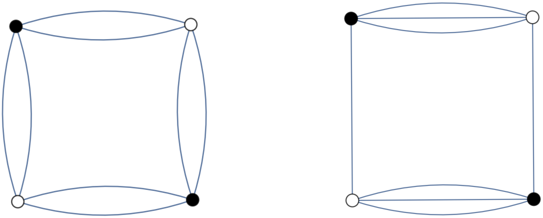


Figure: An example in $d = 4$. The two graphs of the $\mathcal{B}_4$ bubble.

GFTs, LQG and Spinfoams

- GFT yields a proposal for the dynamics of a class of LQG related quantum states.
- GFTs provides a QFT reformulation of all known SF models. $\mathcal{A}_{\mathcal{G}} = \mathcal{A}_{\Delta^* \simeq \mathcal{G}}$.
- It implements the sum over topologies as perturbative expansion of $\mathcal{Z}$ in Feyn. diagrams.
- More in general GFT provides powerfull QFT tools to study the P/NP renormalizability and the continuum limit of a large class of models for $4d$ Quantum Gravity.

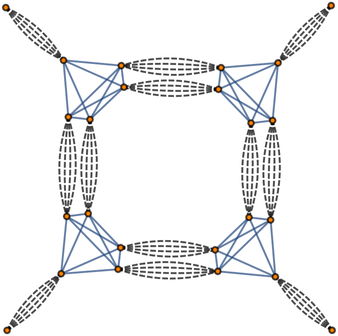


Figure: Example of GFT stranded Feynman graph.

Divergences and Radiative corrections. Why Important?

The analysis of the divergencies in the Radiative corrections allows us to:

- Understand the role of different prescriptions on the amplitudes' (UV) scaling.
- Put constraint on the model building choices, e.g. tuning the choice of the face weights to obtain the convergence of specific classes of bubbles.
- Formulate an euristic ansatz of the model's theory space (e.g. by keeping or adding those interactions required to concistently reabsorb the divergences).
- Extract an estimate of the d.o.d $\omega(\mathcal{G})$ for specif types of diagrams. Test the accuracy of abelian sub-optimal bounds on $\omega(\mathcal{G})$, derived by Heat Kernel methods.

$$\mathcal{A}_{\mathrm{BC}} = \int \Big[\prod_{v\in\mathcal{G}} \prod_{e\ni v} dH_{ve} \Big] \prod_{f\in\mathcal{F}_{\mathrm{cl}}} \mathcal{A}_f(H_{ve}) \quad \mathcal{A}_f = \int \Big[\prod_{e\in f} du_{ef} \Big] \delta \Big[\prod_{e,v\in f} H_{ve}^{-1} U_{ef} H_{v'e} \Big]$$

$$\delta(G) \to \mathscr{K}^{\eta}(G) \to \frac{e^{-\eta|G|^2}}{\sqrt{2\pi\eta}} \qquad G = \prod_i G_i \quad \Rightarrow \quad |G| = |\mathscr{B}(\vec{K}_1, \cdots, \vec{K}_n)|$$

$$G_1,\ G_2 \in \mathrm{Spin}(4) \quad G = G_1 G_2 = e^{\frac{i}{2}\vec{\mathscr{B}}(\vec{K}_1,\vec{K}_2)\cdot\vec{\sigma}} \quad \vec{\mathscr{B}}(\vec{K}_1, \vec{K}_2) \approx \vec{K}_1 + \vec{K}_2 \tag{1.5}$$

- Signal the presence of residual gauge symmetries [Dittrich, Bonzom].

Summary of Part I.

In Part I we have:

Introduced the GFT approach to Quantum gravity.

Motivated the analysis of the divergences as a tool to probe our models.

In Part II we will:

Outline and discuss known methods to compute the amplitude's d.o.d $\omega_{\mathcal{G}}$.

Review recent results on the leading order Rad. corr. for a new class of SGFTs.

GFT Amplitudes. General Structure.

Regardless of the models' details, the amplitudes $\mathcal{A}_{\mathcal{G}}$ have always the same structure. They differ only by the form and the n° of insertions of the coeff. encoding the simplicity constr.

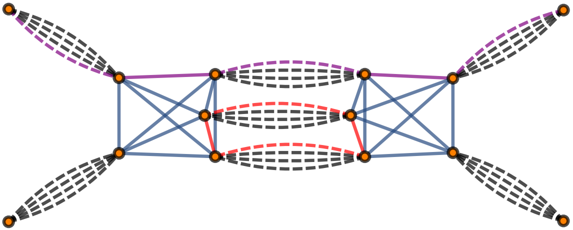


Figure: GFT stranded Feynman graph. Internal face in red, external face in purple.

$$\mathcal{A}^{\beta}(\mathcal{G},\Lambda) = \int \left[\prod_{v\in\mathcal{G}} \prod_{e\ni v} dH_{ve} \right] \prod_{f\in\mathcal{F}_{\mathrm{cl}}} \mathcal{A}^{\beta}_{f}(H_{ve}, k_e, \Lambda) \qquad \beta = \frac{\gamma-1}{\gamma+1} \tag{2.1}$$

$$\mathcal{A}^{\beta}_{f} = \sum_{J_f j_{ef}}^{\Lambda} d^{1+a}_{j_f^-} d^{1+b}_{j_f^+} \int \left[\prod_{e\in f} du_{ef} \right] \Theta^{J_f} \left[\prod_{e\in f} H_{ve} U_{ef} (H_{v'e})^{-1} \right] \left[\prod_{e\in f} d^{1+c}_{j_{ef}} \, w^{q}(J_f, j_{ef}, \beta) \chi^{j_{ef}}(u_{ef}) \right]$$

Before discussing the methods to compute the d.o.d $\omega_{\mathcal{G}}$ we need to introduce the following definitions: **Bulk amplitude**, **Contractible face** and **Master integral.**

Definition. For any GFT Feynman graph $\mathcal{G}$ the *bulk amplitude* $\mathcal{A}_{\rm bulk}$ is the amplitude obtained by setting: to zero all the spins labelling the external faces; j_f^-, j_f^+ $f \in \mathcal{F}_{\rm op}$.

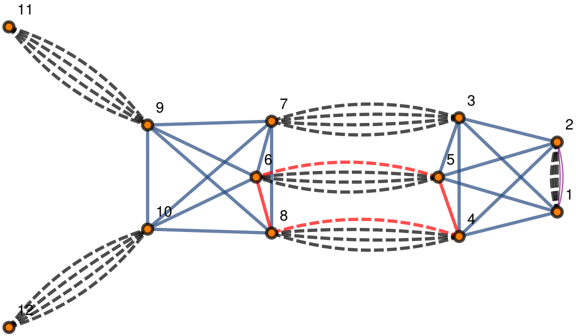


Figure: In red non-contractible internal face. In purple contractible internal face.

Definition. An internal face or loop $f \in \mathcal{F}_{cl}$ of is *contractible* if it does not share any internal edge with any other internal face of the GFT Feynman graph $\mathcal{G}$.

Proposition. The bulk amplitude $\mathcal{A}_{\text{bulk}}$ can be reduced by setting the face amplitudes $\mathcal{A}_f = 1$ for all contractible internal faces f. We call the remainder *"Master Integral"*, $\mathscr{I}(\mathcal{G}, \beta, \Lambda)$.

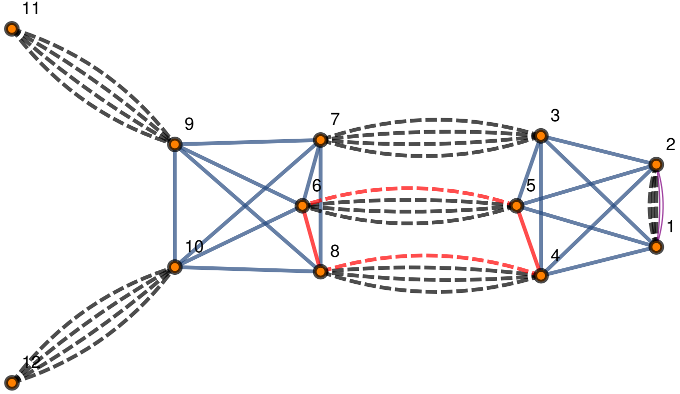


Figure: In red non-contractible internal face. In purple contractible internal face.

Corollary: If all the internal faces $f \in \mathcal{F}_{\text{cl}}$ of a GFT Feynman graph $\mathcal{G}$ are contractible, then its (reduced) Feynman amplitude $\mathcal{A}_{\mathcal{G}}$ is finite (regardless of the model).

GFT amplitudes. Computing the d.o.d. $\omega(\mathcal{G})$.

- Let $\mathcal{G}$ be a GFT stranded graph. Regularize $\mathcal{A}_{\mathcal{G}}$ (e.g. cutoff, heat kernels).
- Compute the (regularized) bulk amplitude $\mathcal{A}_{\rm bulk}(\mathcal{G},\Lambda)$. Gauge-fix $H \in \mathcal{T}_{\mathcal{G}}$.
- Reduce the bulk amplitude to the Master integral $\mathscr{I}(\mathcal{G},\beta,\Lambda)$.
- Evaluate the Master integral. Choices of variables and possible strategies largely depends on the model and on the problem's complexity. Some example:

Heat Kernel approach. Use (abelian) asymptotic formulas for the HK.	Spinorial-Holomorphic Approach.
Saddle point evaluation with Perelomov states.	Extract the scaling exponent from data fitting

- Ab initio brute force numerical evaluation in representation variables.

Radiative corrections in SFs and GFTs. Known results.

- Self-energy and Vertex (l.o.) of the SU(2) Ooguri and EPRL SF models. Self-energy (l.o.) of the Lorentzian EPRL SF model. [Perini,Rovelli,Speziale; Riello].
- Exact degree of divergence of arbitrary connected 2-complexes in the Riemannian Dupuis-Livine (holomorphic) Spinfoam model. [L.Q.Chen; Banburski,L.Q.Chen].
- Holonomy SF models. Single and multiple bubble contr. in the BC model. Divergences and the WFS analysis. [Dittrich,Bonzom, Dittrich,Bahr,Hellmann,Kaminski]
- Boulatov-Ooguri and EPRL GFT models (euclidean). [BenGeloun,Bonzom;BenGeloun,Gurau,Rivasseau].
- Scaling bounds, powercounting results and perturbative renormalizability of topological TGFTs in 3d and $4d$. [Oriti,Gurau,Freidel,BenGeloun,Carrozza]

A new SGFT model for euclidean QG.

Simplicity constraints (2nd class): $kx^- k^{-1} + \beta x^+ = 0$. *Translate this condition into restrictions on Spins, Perelomov states, Spinors, Fluxes. Several methods.* [Perez] .

- The new Model: the simplicity constraints are imposed directly on flux variables via star-multiplication with a non-commutative delta function $\delta^\star$.

$$\tilde{\Phi}^\beta(X_i) = \prod_{i=1}^{4} S^\beta(X_i) \star \Phi_k(X_i) \quad S^\beta(X_i) = \delta_{-x^-}\left(\beta x^+\right) = \int du\, E_u(x^-) E_u(\beta x^+) \quad (2.2)$$

- The NC delta encodes the choice of the Q-map via the plane wave E (here Duflo).
- S^β commutes with the closure constraint up to a rotation of $k \in S^3$,thus ensuring a covariant imposition of the constraints.
- S^β is not an orthogonal projector unless $\beta = 0, 1$. It does not impose any rationality condition on the Immirzi parameter.
- In flux variables the amplitudes are simplicial path integrals for constrained BF theory.
- The amplitudes have a closed integral form in the group formulation as LGT.

Upon group Fourier transform and Peter-Weyl decomposition we find:

A new SGFT model for euclidean QG.

Upon group FT and PW decomposition we find [Finocchiaro, Oriti, (to appear)]:

$$w_{\rm new}(j^-,j^+,j,\beta) = \frac{(-1)^{j^-+j^++j}}{\pi\sqrt{(2j^-+1)(2j^++1)}} \sum_{a=0}^{\lambda} (\mathrm{sign}(\beta))^a \begin{Bmatrix} a & j^- & j^- \\ j & j^+ & j^+ \end{Bmatrix} \mathcal{T}_a^{j^-j^+}(|\beta|) \quad (2.3)$$

By comparison the EPRL Model coefficient read:

$$w_{\rm EPRL}(j^-,j^+,j,\beta) = \Delta^j_{j^-j^+}\delta_{j^-|\beta|j^+}\delta_{j\,(1-\beta)j^+} \quad \Delta^j_{j^-j^+} = d^a_{j^-}d^b_{j^+}d^c_j \qquad |\beta| \leq 1 \quad (2.4)$$

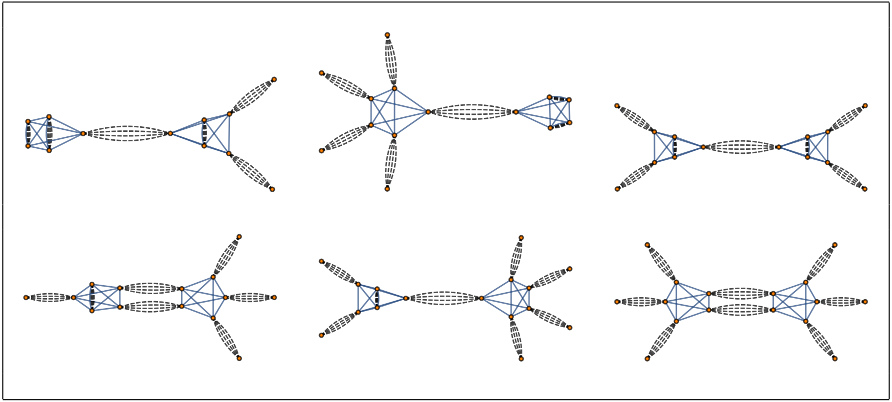
Particular cases:

$$w_{\rm new}(1) = \delta_{j^-j^+}\delta_{j0} \quad w_{\rm new}(-1) = \frac{(-1)^{2j^-+j}}{2j^-+1}\delta_{j^-j^+}\{j^-,j^+,j\} \quad w_{\rm new}(0) = \frac{2(-1)^{2j^-}}{(2j^-+1)^2} \quad (2.5)$$

Numerical analysis of $w_{\rm new}$. [Celoria, Finocchiaro, Oriti, (to appear)].

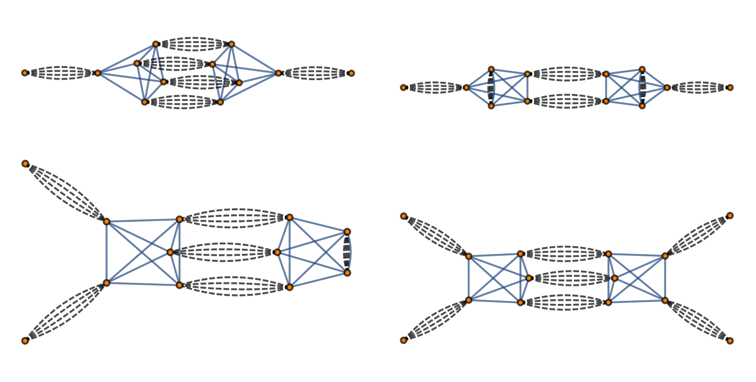
Radiative corrections. N-point functions, $N \leq 6$. Convergent graphs.

The GFT diagrams are labelled as follows: $\mathcal{G}_{n_e n_v p}$. In order: number of external legs, number of vertices, and position in the list of graph with the same first two indices.



According to the above stated convergence criterion the corresponding $\mathcal{A}_{\mathcal{G}}$ are finite.

Radiative corrections. N-point functions, $N \leq 6$. Divergent graphs.



Remarks:

- The graphs $\mathcal{G}_{2,2,1}$ $\mathcal{G}_{2,2,2}$ $\mathcal{G}_{2,2,3}$ $\mathcal{G}_{4,2,1}$ may diverge depending on the face weights choices $d^a_{j^-}$ $d^b_{j^+}$ d^c_j and on the power of $w^q(j^-, j^+, j, \beta)$.
- There are only two independent "Master integrals" to be studied $\mathscr{I}(\mathcal{G}_{2,2,1}, \beta, \Lambda)$ and $\mathscr{I}(\mathcal{G}_{2,2,2}, \beta, \Lambda)$. The graphs $\mathcal{G}_{2,2,2}$ $\mathcal{G}_{2,2,3}$ $\mathcal{G}_{4,2,1}$ reduces to the same "Master Integral".

L.O. Radiative corrections for the new model. Summary of the results.

[Finocchiaro, Oriti (*to appear*)].

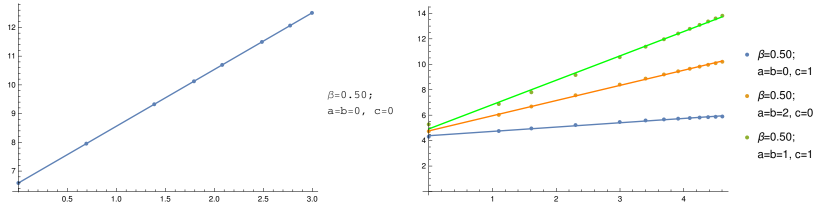


Figure: Log-Log plots. Left melonic amplitude. Right $\mathcal{G}_{2,2,2}$ $\mathcal{G}_{2,2,3}$ $\mathcal{G}_{4,2,1}$ amplitudes.

	$a = b = c = 0$	$a = b = 0,\ c = 1$	$a = b = 2,\ c = 0$	$a = b = c = 1$
EPRL $\mathcal{G}_{2,2,1}$	6	–	–	–
NM $q = 1$ $\mathcal{G}_{2,2,1}$	1.97615	–	–	–
EPRL $\mathcal{G}_{2,2,2}$ $\mathcal{G}_{2,2,3}$ $\mathcal{G}_{4,2,1}$	2	5	6	7
NM $q = 1$ $\mathcal{G}_{2,2,2}$ $\mathcal{G}_{2,2,3}$ $\mathcal{G}_{4,2,1}$	Finite	0.33957	1.19255	1.91360
NM $q = 2$ $\mathcal{G}_{2,2,2}$ $\mathcal{G}_{2,2,3}$ $\mathcal{G}_{4,2,1}$	Finite	Finite	Finite	Finite

Summary of Part II.

In Part II we have:

Outlined the main methods to compute the $\omega(\mathcal{G})$ of a given GFT graph.

Reviewed some recent results on the l.o. Radiative corrections for a new SGFT model.

In Part III we will:

Focus on the problem of generating specific sets of GFT Feynman graphs.

Discuss the ordinary scalar field theories case as a pivotal warmup example.

Show how basic QFT results can be easely adapted to both SGFTs and TGFTs.

Recursive construction of Feynman diagrams in QFT. Strategy.

Ordinary vacuum graphs can be seen as maximally decorated rooted trees. [Kleinert]

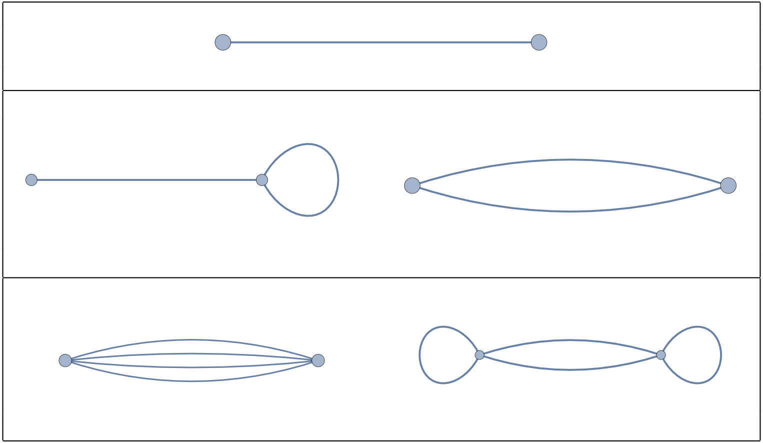
To generate all vacuum graphs with n vertices for the ϕ^k theory, we proceed as follows :

- Start with a tree graph (line graph) $\mathcal{T}_1$ with n vertices $\mathcal{V}_1$.
- Create the list of all pairs of vertices. Construct a list of edges $\mathcal{E}_1$ out of it.
- Create a new list of graphs $\mathcal{G}_1$ obtained by decorating the tree with the edge list $\mathcal{E}_1$.
- Iterate the above procedure untill all vertices are k-valent.
- At each step of the iteration delete topologically equivalent copies.
- At each step of the iteration delete graphs with at least one higher-valent vertex.

In the next slide we provide an example.

Recursive construction of Feynman diagrams in QFT. Example.

Example $\lambda\phi^4$, order λ^2. Tree graph. Result after one decoration. Final result.



Testing graph isomorphism.

Caveat: Test whether two graphs (with multiedges and selfloops) are isomorphic.

Solution: Promote the graph to an ordinary (bisected) graph by inserting (two) additional auxiliary bivalent vertices for each edge.

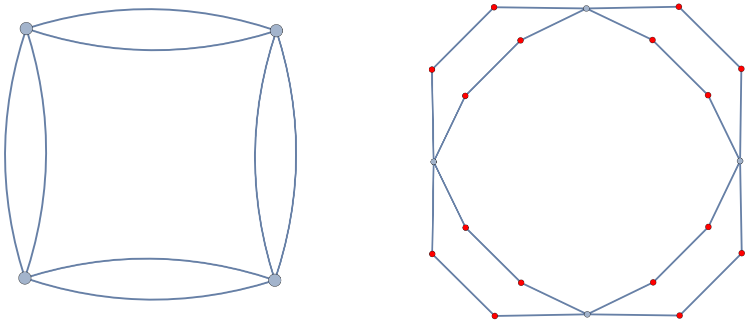


Figure: A vacuum (multi)graph and it bisected counterpart.

Graphical construction of the N-point functions.

The perturbative diagrammatic expansion of the connected N-point functions can be constructed from a set of Vacuum graphs as follows

- For (N, k) even cut $\frac{N}{2}$ internal lines for each Vacuum graph in the set in all ways.
- For (N, k) odd cut $\frac{N-1}{2}$ internal lines for each conn. 1-point function graph in all ways.

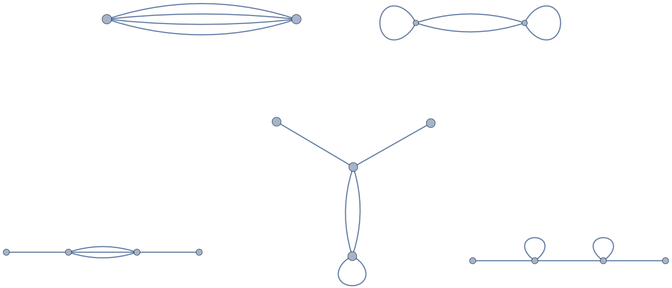


Figure: Diagrammatic expansion of the 2-point function at leading order.

The previous ideas can be applied to both SGFT and TGFT models.

We provide a practical demonstration with Mathematica.

Concluding Remarks.

The analysis of the divergences appearing in the radiative corrections for various classes of GFT and Spinfoam models brings two main challanges:

Evaluating the amplitudes (in particular their "UV" scaling).

Generate the required Feynman diagrams.

In my talk I:

Explained why it is an interesting and important issue to study.

Outlined the main methods to compute $\omega(\mathcal{G})$ for a given GFT graph.

Reviewed some recent results on the l.o. Radiative corrections for a new SGFT model.

Showed how to extend ordinary QFT diagrammatic techniques to GFT models.
(providing a Mathematica algorithm to recursively construct GFT stranded Feynman graphs.)

Future developments.

Improve the algorithms, (parallelize). Upgrade to C++. Alternative: Pachner moves.

Recursive graphical construction of Schwinger-Dyson equations

Appendix A. The kernels $\mathcal{K}^\beta$ and $\mathcal{V}^\beta$ of the new model.

$$\mathcal{K}^\beta(G_i, \tilde{G}_i, k, \tilde{k}) = \delta\left[k\tilde{k}^{-1}\right] \int \left[dHdH'\right] \left[\prod_{i=1}^{4} \mathcal{D}^\beta \mathscr{U}_i^{(n)}\right] \prod_{i=1}^{4} \delta\left[G_i H \mathscr{U}_i^{(n)} (H')^{-1} (\tilde{G}_i)^{-1}\right] \quad (3.1)$$

$$\mathcal{V}^\beta(G_{il}, \tilde{G}_{il}) = \int \left[\prod_{i=1}^{5} dH_i\right] \left[\prod_{i,\, l}^{5} \mathcal{D}^\beta \mathscr{U}_{il}^{(p)} \mathcal{D}^\beta \mathscr{U}_{il}'^{(p)}\right] \prod_{i,\, l}^{5} \delta\left[G_{il} \mathscr{U}_{il}^{(p)} H_i H_l^{-1} \mathscr{U}_{il}'^{(p)} (\tilde{G}_{il})^{-1}\right] \quad (3.2)$$

$$\mathscr{U}^{(n)} = \left(\prod_{a=1}^{n} u_a, \prod_{a=1}^{n} u_a^\beta\right) \qquad \mathcal{D}^\beta \mathscr{U}^{(n)} = \prod_{a=1}^{n} du_a\, \tilde{\Omega}(\beta, \psi_{u_a}) \qquad \tilde{\Omega}(\beta, \psi_u) = \frac{\sin|\beta|\psi_u}{|\beta| \sin\psi_u} \quad (3.3)$$

Appendix B. The new model and its coefficient $w_{\mathrm{new}}(j^-, j^+, j, \beta)$.

$$w_{\mathrm{FO}}(j^-, j^+, j, \beta) = \frac{(-1)^{j^- + j^+ + j}}{\pi\sqrt{(2j^- + 1)(2j^+ + 1)}} \sum_{a=0}^{\lambda} (\mathrm{sign}(\beta))^a \begin{Bmatrix} a & j^- & j^- \\ j & j^+ & j^+ \end{Bmatrix} \mathcal{T}_a^{j^- j^+}(|\beta|) \quad (3.4)$$

$$\mathcal{T}_a^{j^- j^+}(|\beta|) = (2a+1) \sum_{p=-j^-}^{j^-} \sum_{q=-j^+}^{j^+} C_{p\,0\,p}^{j^- a j^-} C_{q\,0\,q}^{j^+ a j^+} \Upsilon_{pq}(|\beta|) \quad (3.5)$$

$$\Upsilon_{pq}(|\beta|) = \int_0^{2\pi} d\psi \, \frac{1}{|\beta|} \sin\frac{\psi}{2} \sin\frac{|\beta|\psi}{2} e^{-i(p+|\beta|q)\psi} \quad (3.6)$$

$$= \begin{cases} \frac{i - ie^{2i\pi|\beta|} + 2\pi|\beta|(|\beta|-1)}{4\beta^2(|\beta|-1)} & \forall\, p, q; \quad 2(p+|\beta|q) = 1 - |\beta| \\ \frac{i - ie^{2i\pi|\beta|} - 2\pi|\beta|(|\beta|+1)}{4\beta^2(|\beta|+1)} & \forall\, p, q; \quad 2(p+|\beta|q) = -1 - |\beta| \\ -\frac{i - ie^{-2i\pi|\beta|} + 2\pi|\beta|(|\beta|+1)}{4\beta^2(|\beta|+1)} & \forall\, p, q; \quad 2(p+|\beta|q) = 1 + |\beta| \\ -\frac{i - ie^{-2i\pi|\beta|} - 2\pi|\beta|(|\beta|-1)}{4\beta^2(|\beta|-1)} & \forall\, p, q; \quad 2(p+|\beta|q) = -1 + |\beta| \\ \frac{8i|\beta|(p+|\beta|q)e^{-2i\pi(p+|\beta|q)} \cos|\beta|\pi - 2\left(-1+\beta^2+(p+|\beta|q)^2\right)e^{-2i\pi(p+|\beta|q)} \sin|\beta|\pi + 8i|\beta|(p+|\beta|q)}{|\beta|(2|\beta|q+|\beta|+2p-1)(2|\beta|q+|\beta|+2p+1)(2|\beta|q-|\beta|+2p-1)(2|\beta|q-|\beta|+2p+1)} & \end{cases} \quad (3.7)$$

Appendix C. The reduced amplitudes $\mathcal{A}(\mathcal{G}_{2,2,1})$ and $\mathcal{A}(\mathcal{G}_{2,2,2})$, New Model.

$$\mathcal{A}_{\rm bulk}(\mathcal{G}_{2,2,1},\beta) = \sum_{\{j_1^-,\cdots,j_6^-\}}^{\Lambda} \sum_{\{j_1^+,\cdots,j_6^+\}}^{\Lambda} \left[d_{j_1^-} d_{j_3^-} d_{j_3^+} d_{j_6^+}\right] \left[\Omega_{(0,q)}(j_1^-,j_1^+,\beta)\Omega^2_{(0,q)}(j_2^-,j_2^+,\beta)\right.$$
$$\times\Omega^2_{(0,q)}(j_4^-,j_4^+,\beta)\Omega^2_{(0,q)}(j_5^-,j_5^+,\beta)\Omega_{(0,q)}(j_6^-,j_6^+,\beta)\Omega^2_{(1,q)}(j_3^-,j_3^+,\beta)\Omega_{(1,q)}(j_1^-,j_1^+,\beta)$$
$$\left.\times\Omega_{(1,q)}(j_6^-,j_6^+,\beta)\right] \begin{Bmatrix} j_3^- & j_1^- & j_6^- \\ j_2^- & j_4^- & j_5^- \end{Bmatrix}^2 \begin{Bmatrix} j_3^+ & j_1^+ & j_6^+ \\ j_2^+ & j_4^+ & j_5^+ \end{Bmatrix}^2 \theta^q(\beta) \tag{3.8}$$

$$\mathcal{A}_{\rm bulk}(\mathcal{G}_{2,2,2},\beta) = \sum_{j_1^- j_1^+}^{\Lambda} \sum_{j_1 j_2 j_3} \frac{d_{j_1} d_{j_2} d_{j_3}}{d_{j_1^-} d_{j_1^+}} w^2(j_1^-,j_1^+,j_1,\beta) w^2(j_1^-,j_1^+,j_2,\beta) w^2(j_1^-,j_1^+,j_3,\beta) \tag{3.9}$$

$$\Omega^{(\alpha,\,q)}(j^-,j^+,\beta) = \sum_j d_j^\alpha w^q(j^-,j^+,j,\beta) \tag{3.10}$$