LQG as a TQFT with defects
New vacuum with a cosmological constant

Marc Geiller
Perimeter Institute

ILQG Seminar, February 9^{th} 2016

Based on work with B. Dittrich

Motivations

- ⋆ There are now different vacua (inequivalent quantum representations) for canonical LQG, spin foams, and group field theory (Ashtekar, Isham, Lewandowski; Koslowski, Sahlmann; Dittrich, MG)
- ⋆ How many vacua are there, how are they related, and what can they be used for?

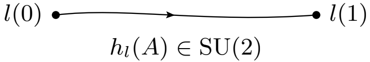
Results

- ⋆ We extend the construction of the flat curvature BF vacuum to the case of $(2+1)$ Euclidean dimensions for $\Lambda > 0$ (in a sense the most peculiar case)
- ⋆ The conjectured underlying mathematical structure is that

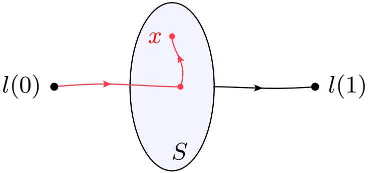
$$\text{vacuum} \;+\; \text{excitations} \;=\; \text{TQFT} \;+\; \text{defects}$$

Kinematical results

- ⋆ Boundary gravitational degrees of freedom: holonomies and fluxes encoding extrinsic and intrinsic spatial geometry



$$X_S^i(A,E) = \int_S h_{(0\to x)} E^i(x) h^{-1}_{(0\to x)}$$



- ⋆ Diffeomorphism-invariant Hilbert spaces supporting the holonomy-flux algebra
- ⋆ Derivation of quantum geometry (Ashtekar, Lewandowski; Rovelli, Smolin)
- ⋆ Vacuum tell us how to refine states and construct a kinematical continuum limit

Dynamical questions

- ⋆ How does a smooth diffeomorphism-invariant space-time emerge?
- ⋆ What is the phase diagram of our “discrete” gravity models?
- ⋆ What is a vacuum state?

Is LQG discrete or continuous?

- ⋆ Both, since we work on graphs or triangulations, but states live in $\mathcal{H}_\infty$
- ⋆ Field theory with arbitrary finite # of degrees of freedom = TQFT with defects

Classical kinematics as a TQFT with defects (Bianchi; Freidel, Ziprick, MG)

- ⋆ LQG phase space on a graph is $\mathcal{P}_\Gamma = \bigtimes_{\text{links}} \big(T^*\mathrm{SU}(2)\big)$
- ⋆ Same as that of gravity with almost-everywhere gauge-invariant flat connections

$$\mathcal{P}_\Gamma \simeq T^*\mathcal{A} \,/\!\!/\, \big(\mathcal{F} \ltimes \mathcal{G}\big)_\Gamma$$

Which TQFT and which defects?

- ⋆ In the AL representation, $E = 0$ and defects generate geometry
- ⋆ In the BF representation, $F(A) = 0$ and defects generate curvature
- ⋆ Classically, these geometries are gauge choices in the reconstruction

$$\mathcal{P}_\Gamma \ni \big(h_l, X_S\big) \mapsto \big(A(x), E(x)\big) \in T^*\mathcal{A}$$

- ⋆ In the quantum theory, these vacua lead to inequivalent representations

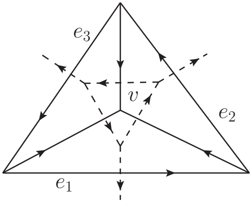
Contrasting the AL and BF representations

	AL	BF
background TQFT	$E = 0$	$F(A) = 0$
vacuum state	$\lvert\emptyset\rangle$ = nothing	$\lvert\emptyset\rangle = \prod_{\text{cycles}} \delta(g_c, \mathbb{1})$
excitations	holonomies $h_l \triangleright \lvert\emptyset\rangle = h_l$ dual graphs	exponentiated fluxes $R_i^h \triangleright \lvert\emptyset\rangle = \ldots \delta(g_i h, \mathbb{1}) \ldots$ $(d-1)$-simplices
defects	dual graphs	$(d-2)$-simplices
refinement	$j = 0$	flatness
measure	Haar	discrete
kin. C^∞ limit	✓	✓
generalization	background E_o	constant curvature $(\Lambda \neq 0)$

Other advantages of the BF representation

- ⋆ Deals for the first time successfully with the gauge-covariant fluxes $X(A,E)$
- ⋆ Diffeomorphisms as vertex displacement (in $2+1$)
- ⋆ Coarse-graining of the fluxes
- ⋆ Encode the interplay between curvature and torsion excitations

$$X_{e_3\circ e_2\circ e_1}\neq 0 \quad \text{if} \quad g_v\neq \mathbb{1}$$



Relation with point particles and quantum double structure

- ⋆ Exponentiated fluxes and gauge transformations respectively generate curvature and torsion excitations
- ⋆ In $(2+1)$ gravity coupled to point particles, curvature and torsion are related to mass and spin, and these label unitary irreps of the Drinfeld double DSU(2)

Constant curvature vacuum

Setup

- ⋆ $(2+1)$ spacetime dimensions with $\Lambda \neq 0$
- ⋆ It is known that quantum groups emerge in the quantum theory

What we will not (but could/should) do

- ⋆ Dynamical emergence: take flat SU(2) BF vacuum and solve $F[A] = \pm\Lambda(e \wedge e)$
- ⋆ Kinematical emergence: construct BF vacuum with non-commutative holonomies $h\big(A \pm \sqrt{\Lambda}e\big)$ (Noui, Perez, Pranzetti)

Strategy and goal

- ⋆ Assume quantum group structure from the onset
- ⋆ For Euclidean $\Lambda < 0$, we could construct a vacuum based on the deformed phase space $\mathrm{SL}(2,\mathbb{C}) \xrightarrow{\Lambda \to 0} \mathrm{ISU}(2)$ (Bonzom, Dupuis, Girelli, Livine)
- ⋆ For Euclidean $\Lambda > 0$, the quantum group is $\mathcal{U}_q\big(\mathfrak{su}(2)\big)$ at root of unity, the TQFT is known (Turaev, Viro), but there is no group picture!
- ⋆ We will show that the Turaev–Viro TQFT with defects is like a BF_Λ vacuum with excitations

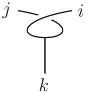
What we need to construct

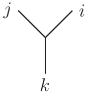
1) A way of writing down and manipulating states (since there is no group picture)
2) A Hilbert space which contains flat states (vacuum) and excited states
3) A creation/excitation operator

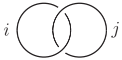
Constant curvature vacuum

1) Writing and manipulating states: graphical calculus

⋆ $\mathcal{U}_q\big(\mathfrak{su}(2)\big)$ at root of unity is a modular braided fusion category $\mathcal{C}$

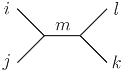
- Category: irreps given by spins $j \in \{0, 1/2, \ldots, \mathrm{k}/2\}$, with $\mathrm{k} = \big(G_{\mathrm{N}}\hbar\sqrt{\Lambda}\big)^{-1}$
- Fusion: there are fusion coefficients such that $i \otimes j = \bigoplus_k N_{ij}^k k$
- Braided: there is an R-matrix such that 

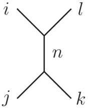
$= R_k^{ij}$ 

- Modular: $\det(S) \neq 0$ with $S_{ij} = \dfrac{1}{\mathcal{D}}$ 

and $\mathcal{D}^2 = \sum_j d_j^2$

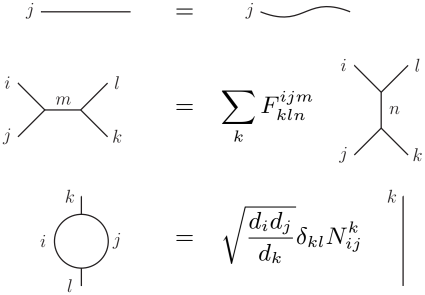
⋆ Dynamics (topological invariance) encoded in $\{6j\}_q$ symbol



$= \sum_k F_{kln}^{ijm}$


2) Ribbon graph Hilbert space

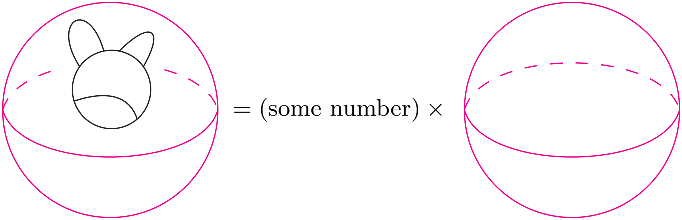
- Consider a p-punctured $2d$ manifold Σ_p
- A puncture is obtained by removing a disc and placing a marked point on $\partial\Sigma$
- Allow for graphs in Σ_p to have one link ending at each puncture •
- Define $\mathcal{H}_p$ as the span of such graphs modulo the local equivalence relations



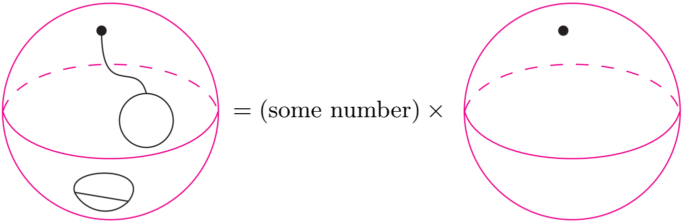
Constant curvature vacuum

Examples

⋆ 2-sphere with 0 punctures: $\dim \mathcal{H}_0 = 1$

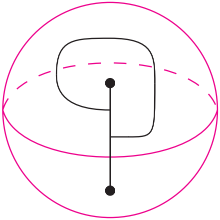


⋆ 2-sphere with 1 puncture: $\dim \mathcal{H}_1 = 1$



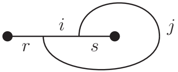
Examples

- 2-sphere with 2 punctures $\simeq$ topologically a cylinder
- Non-trivial states corresponding to allowed spin labelings of



- Introduce $\mathcal{Q}$ basis given by

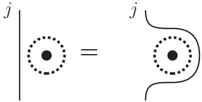
$$\mathcal{Q}^{ij}_{rs} =$$



- $\dim \mathcal{H}_2 = \sum_{ijrs} N^r_{ij} N^s_{ij}$ depends on the level k
- Punctures can carry curvature (non-contractible cycles) and torsion (open links)
- Graphs can be seen as dual to degenerate triangulations
- Basis found for arbitrary topology and p with pants and tree decomposition

Vacuum state

- The vacuum is selected by two projections imposing (Levin, Wen)
  - Gauge-invariance: $B_v \triangleright \text{vertex}(i,j,k) = N_{ij}^k \; \text{vertex}(i,j,k)$
  - Flatness: $B_p \triangleright \bullet = \frac{1}{\mathcal{D}} \sum_j d_j$  $:=$ 

- These dotted (vacuum) lines have the property that 

  i.e. make the punctures invisible, i.e. remove the curvature that they carry

- In the SU(2) case, B_p becomes $\sum_j \chi_j(g) = \delta(g)$

- If Γ is dual to a triangulation, one gets the Turaev–Viro invariant (Kirillov Jr.)

$$\langle \Gamma | \prod_{\text{faces}} B_p | \Gamma \rangle = \text{TV}\big(\Sigma \times [0,1]\big)$$

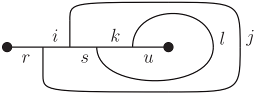
- The embedding map adds a puncture with no curvature, i.e. 

Constant curvature vacuum

The tube algebra

- ⋆ States $\mathcal{Q}$ on the cylinder (2-punctured 2-sphere) can be thought of as operators
- ⋆ Stacking two cylinders (matching the marked points) gives back a cylinder, and in terms of $\mathcal{Q}$'s this multiplication defines the tube algebra (Ocneanu; Müger)

$$\mathcal{Q}^{ij}_{rs}\mathcal{Q}^{kl}_{su} =$$



$$= \cdots = \sum_{mn} \alpha\big(F(\text{spins})\big)\,\mathcal{Q}^{mn}_{ru}$$

- ⋆ The $\mathcal{Q}$ states therefore define both a vector space and an algebra
- ⋆ The quasi-particle excitations we are looking for are representations of (or modules over) this algebra (Lan, Wen)
- ⋆ More precisely, we look for states ξ with a stability property $\mathcal{Q} \triangleright \xi = \xi$

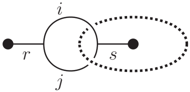
Remark: back to the group picture

- ⋆ Consider the idempotent (or projector) fluxes X_h such that $X_h X_{h'} = \delta_{hh'} X_h$
- ⋆ Gauge transformations act on the fluxes by conjugation, so $gX_h = X_{ghg^{-1}}g$
- ⋆ Then the algebra satisfies the multiplication rule of the Drinfeld double

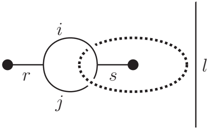
$$X_h g X_{h'} g' = \delta_{h,ghg^{-1}} X_h g g'$$

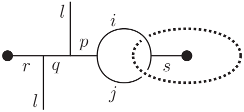
Quantum double

- ⋆ Irreducible representations of the tube algebra (or the category of modules) form a quantum double category $\mathcal{Z}(\mathcal{C})$, and because here $\mathcal{C}$ is modular, $\mathcal{Z}(\mathcal{C}) = \mathcal{C} \otimes \mathcal{C}^*$

- ⋆ Basis states $\mathcal{O}^{ij}_{rs} =$ 

satisfy the projection condition $\mathcal{O}\mathcal{O} \propto \mathcal{O}$, are orthonormal (König, Kuperberg, Reichardt), and are stable in the sense that



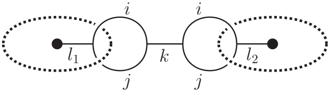
$= \sum_{pq} \Omega^{ij}_{rp,ql}$


- ⋆ The objects $\Omega^{ij}_{rs,kl} = \sum_{mn} \sqrt{\frac{d_m d_n}{d_r d_l^2}} R^{il}_m R^{lj}_n F^{nmr}_{ijl} F^{qlp}_{ijm} F^{rlq}_{jmn}$ are known as half-braiding tensors and label elements of the quantum double $\mathcal{Z}(\mathcal{C})$

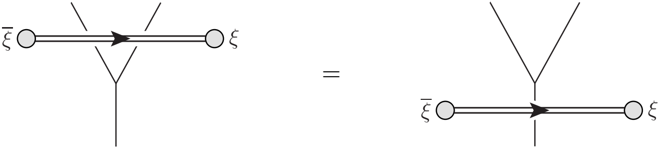
3) Excitations

- ⋆ The excitation operators are given by the following oriented ribbons

$$:= \frac{1}{\mathcal{D}^2} \sum_k \sqrt{\frac{d_k}{d_i d_j}}$$



- ⋆ By definition, the half-braiding tensors have to satisfy so-called naturality conditions $\leftrightarrow$ fixed point conditions for the $\mathcal{Q}$ algebra $\leftrightarrow$ sliding property



- ⋆ E.g., acting on the cylinder vacuum state   gives a basis state $\mathcal{O}^{ij}_{l_1 l_2}$
- ⋆ Open/closed ribbon operators correspond respectively to fluxes/holonomies
- ⋆ Automatically describes the Turaev–Viro model coupled to point particles

New framework and results

- ⋆ New physical BF vacuum for SU(2) in $d = 2, 3$ spatial dimensions
- ⋆ Full continuum Hilbert space supporting curvature excitations
- ⋆ Allows for geometrical coarse graining of the fluxes
- ⋆ New take on the dynamics and extraction of physics
- ⋆ Same structures found in wider class of TQFTs with defects (allows $\Lambda \neq 0$)
- ⋆ Establishes link with other areas (condensed matter, extended TQFTs)
- ⋆ Right mathematical framework to describe fixed points of coarse-graining flow

Generalizations and applications

- ⋆ Other quantum groups, signature, and signs of Λ
- ⋆ Extension to $(3 + 1)$
- ⋆ Dynamics of the defects (ILQGS by Wolfgang Wieland on April 5$^{\text{th}}$)
- ⋆ Non-commutative flux representation
- ⋆ Non-compact gauge groups
- ⋆ Cosmology
- ⋆ Black holes