# Graph-changing dynamics in Loop quantum gravity

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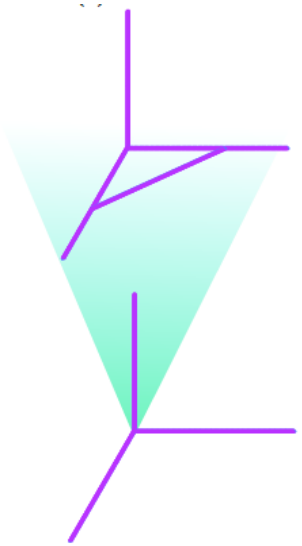

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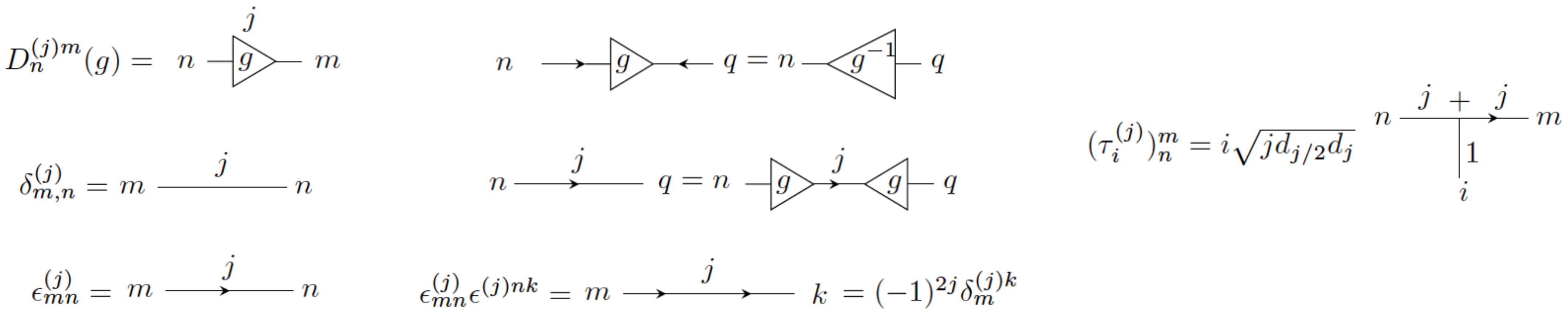
# Why graph-changing dynamics?

- The partition used in Thiemann's derivation of the Hamiltonian constraint allows to express the curvature as holonomies along (sub-partition-sized) loops
- Loop holonomies couple to spinnetworks around their nodes, but change the graphs by adding (or removing*) links
- Few graph-changing calculations present in the literature, mostly for 3-valent nodes, with little data for analysis
- Coarse-grained graph-preserving approximations preferred due to reduced complexity, but not yet entirely well justified
- Can one generate dynamics? How does it compare to covariant LQG? What happens to observables?

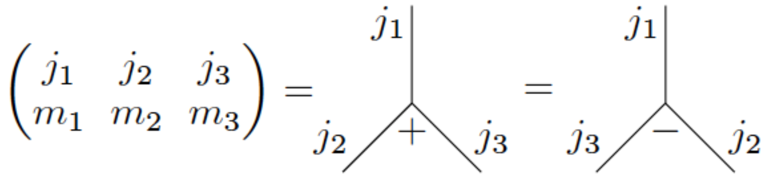
# Recoupling theory in a nutshell

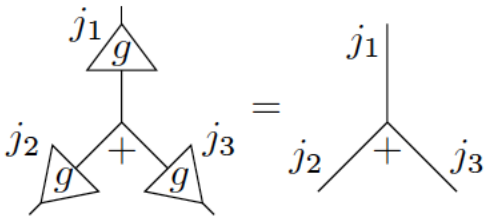
- Elements of SU(2) in given representations represented graphically
- Wigner matrices represented as "hollow" arrows
- Specific elements, like the identity and the invariant antidiagonal tensor, represented respectively by line and arrow
- Simple rules for inversion and contraction of elements in the same representation
- Generators of the su(2) algebra represented as "grasps"

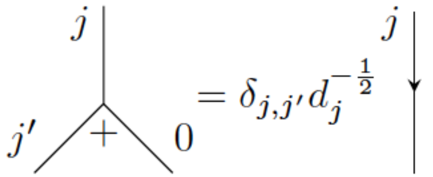


# Recoupling theory in a nutshell

- Intertwiners [invariant tensors of SU(2)] couple links forming spin singlets in the gauge-invariant case
- 3-valent intertwiners as building block of higher-valency intertwiners
- Invariance allows to effectively reduce some of the elements in the spinnetwork to identities

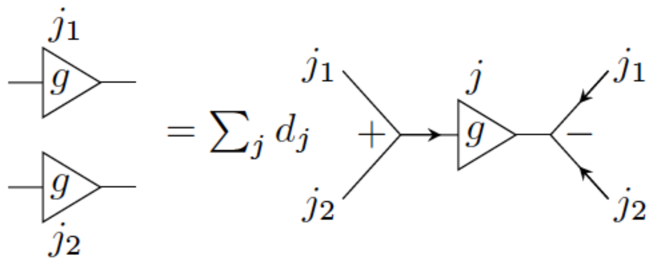


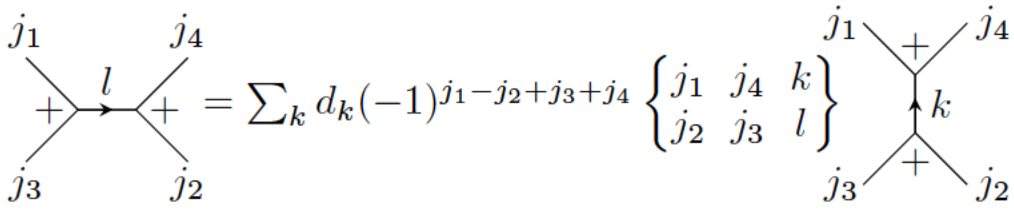



# Recoupling theory in a nutshell

- Coupling of Wigner matrices respects the Clebsch-Gordan relations (note Wigner 3j), with a dimensional factor for each representation
- Intertwiners form bases, and changes of bases are implemented through 2-2 Pachner moves

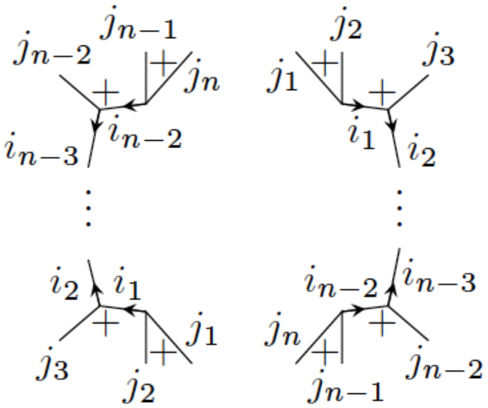


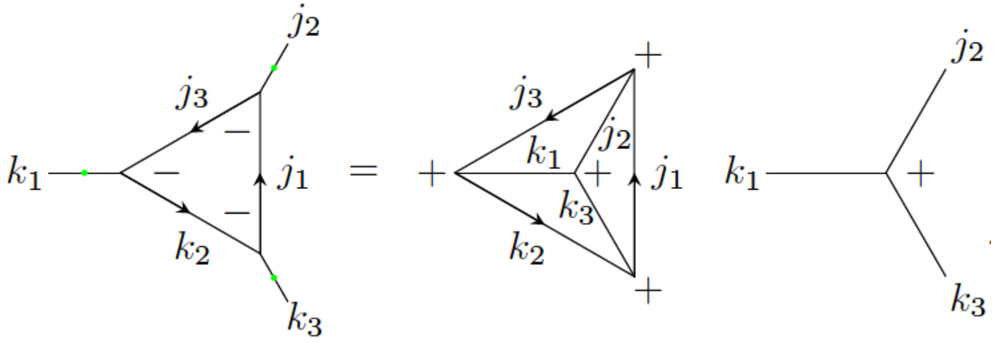



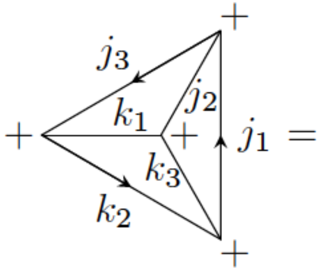
# Recoupling theory in a nutshell

- Resolution of identity in terms of intertwiners allows one to factor out closed "fractions" of spinnetworks

$$\mathbb{1} = \sum_{\{i\}} \left( \prod_{l=1}^{n-2} d_{i_l} \right)$$



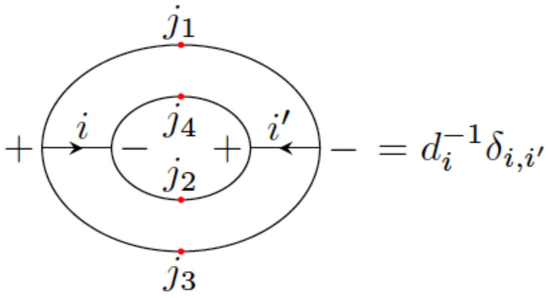



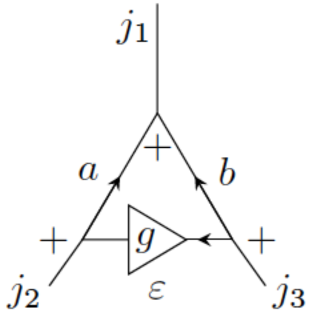



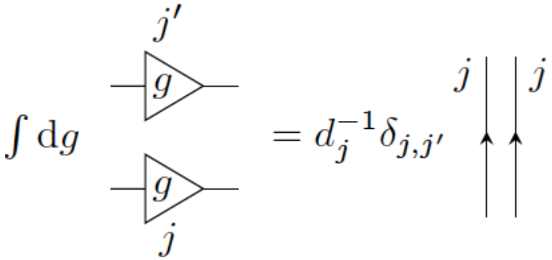

$$= \begin{Bmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{Bmatrix}$$

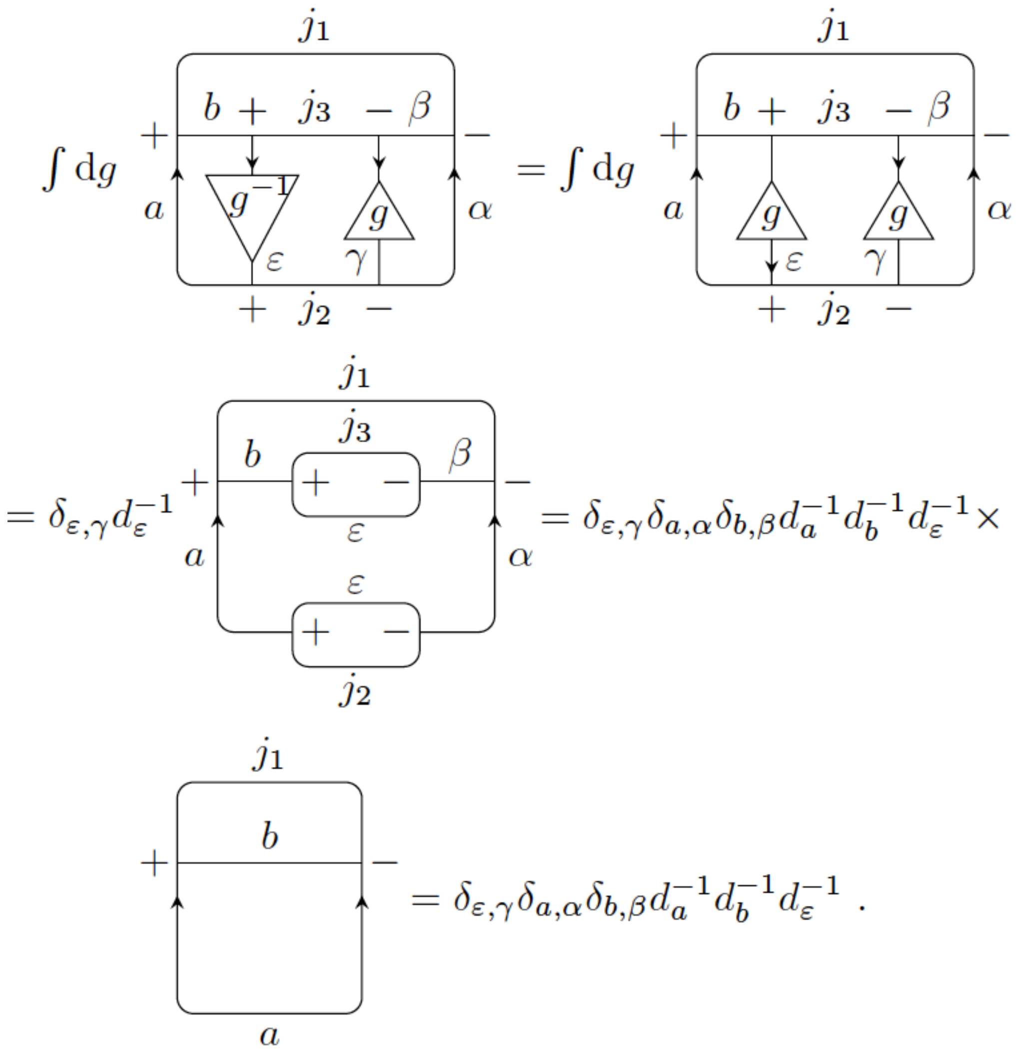
# Recoupling theory in a nutshell

- Spinnetworks require normalization: normalization of intertwiners is known, but once graphs are modified by the constraint, how does the normalization changes?



# Thiemann's Euclidean scalar constraint

- Alongside gauge- and diff-invariance constraints, Thiemann's formulation includes the scalar constraint, associated with time translation by a lapse
- Gause invariance is naturally impelemented by the Wigner 3j symbols
- Diffeomorphism invariance is satisfied by considering equivalence classes of (dual) spinnetworks with respect to diffeomorphisms: all graphs related to each other by analytic deformations should be superposed
- The scalar constraint is built from holonomies and the volume, both of which are "problematic": holonomies along loops change the graphs, while the volume requires building and diagonalizing matrices of variable sizes

$$\hat{C}_s = -\frac{i}{3l_0^2} \lim_{\Box \to 0} \sum_{\Box} N_{\Box} \epsilon_{ijk} \mathrm{tr} \left\{ \hat{h}[\alpha_{ij}] - \hat{h}[\alpha_{ji}], \hat{h}[p_k] \hat{V} \hat{h}^{-1}[p_k] \right\}$$

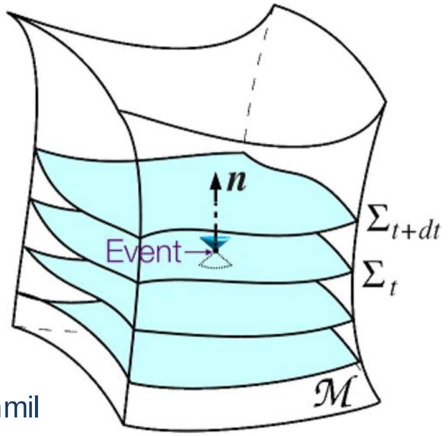
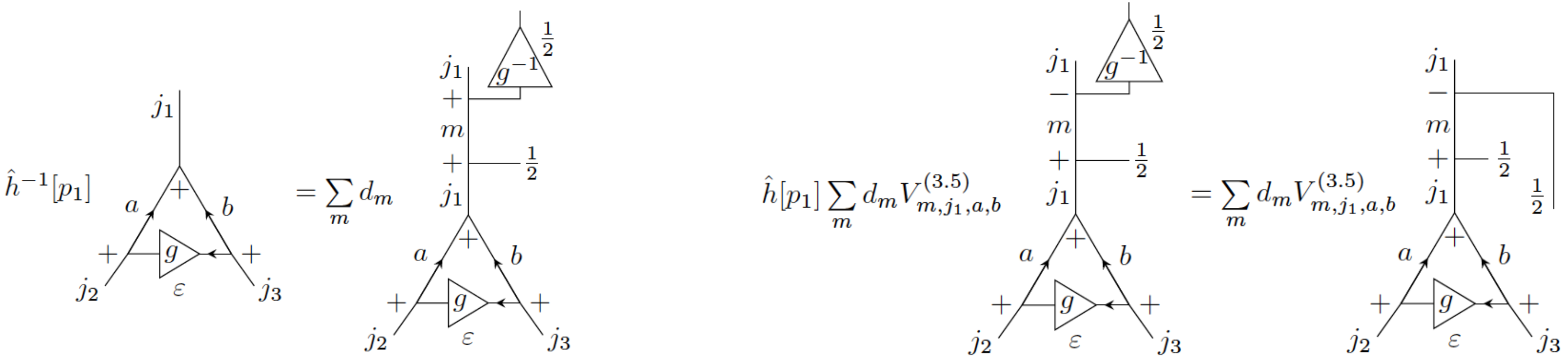



Figure by Muhammad Muzammil

# Thiemann's Euclidean scalar constraint

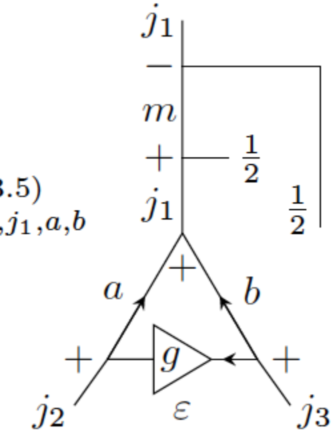
- Graph changes around 3-valent nodes



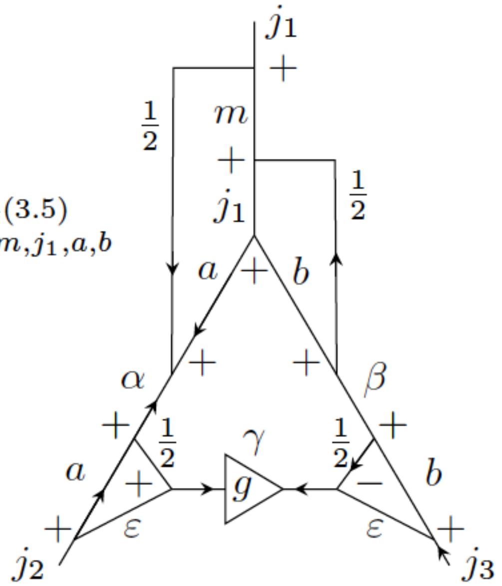
# Thiemann's Euclidean scalar constraint

- Graph changes around 3-valent nodes

$$\hat{h}[\alpha_{ij}] \sum_m d_m V^{(3.5)}_{m,j_1,a,b}$$



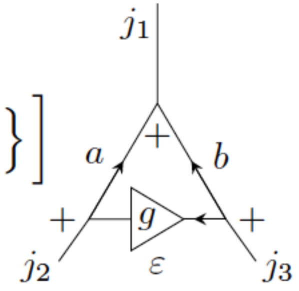

$$= \sum_{m,\alpha,\beta,\gamma} d_m d_\alpha d_\beta d_\gamma V^{(3.5)}_{m,j_1,a,b}$$



# Thiemann's Euclidean scalar constraint

- Graph changes around 3-valent nodes

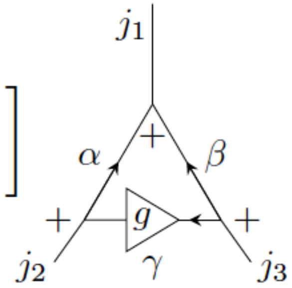
$$i\left[\mathrm{tr}\left\{\frac{\hat{h}[\alpha_{23}]-\hat{h}[\alpha_{32}]}{2}\hat{h}[p_1]\hat{V}\hat{h}^{-1}[p_1]\right\}+\mathrm{tr}\left\{\hat{h}[p_1]\hat{V}\hat{h}^{-1}[p_1]\frac{\hat{h}[\alpha_{23}]-\hat{h}[\alpha_{32}]}{2}\right\}\right]$$




$$=\frac{i}{2}\sum_{\alpha,\beta,\gamma,m,q} d_\alpha d_\beta d_\gamma d_m \times$$

$$\begin{Bmatrix} a & j_2 & \varepsilon \\ \gamma & \frac{1}{2} & \alpha \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & \gamma & \varepsilon \\ j_3 & b & \beta \end{Bmatrix}\Bigg[(-1)^{1/2-a+2b+\varepsilon+\gamma+j_1+j_2+j_3+m+2\alpha-\beta}\left(V^{(3.5)}_{m,j_1,a,b}+V^{(3.5)}_{m,j_1,\alpha,\beta}\right)\begin{Bmatrix} \frac{1}{2} & m & j_1 \\ a & b & \beta \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & \alpha & a \\ \beta & m & j_1 \end{Bmatrix}$$

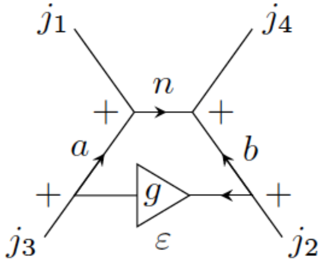
$$-(-1)^{-1/2-b-\varepsilon-\gamma-j_1+j_2+j_3-m-\alpha}\left(V^{(3.5)}_{m,j_1,\alpha,\beta}+V^{(3.5)}_{m,j_1,a,b}\right)\begin{Bmatrix} a & b & j_1 \\ m & \frac{1}{2} & \alpha \end{Bmatrix}\begin{Bmatrix} m & \frac{1}{2} & j_1 \\ \beta & \alpha & b \end{Bmatrix}\Bigg]$$



# Thiemann's Euclidean scalar constraint

- Graph changes around 4-valent nodes

$$\frac{i}{2}\mathrm{tr}\left\{\hat{h}[\alpha_{23}]-\hat{h}[\alpha_{32}],\hat{h}[p_1]\hat{V}\hat{h}^{-1}[p_1]\right\}$$

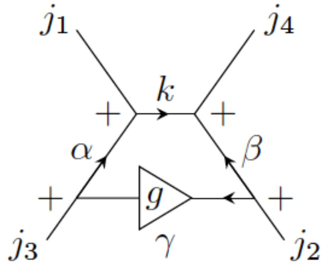



$$= \frac{i}{2}\sum_{\alpha,\beta,\gamma,m,k,l,p,q} d_\alpha d_\beta d_\gamma d_m d_k \begin{Bmatrix}\frac{1}{2} & \alpha & a\\ j_3 & \varepsilon & \gamma\end{Bmatrix}\begin{Bmatrix}b & \beta & \frac{1}{2}\\ \gamma & \varepsilon & j_2\end{Bmatrix}\Bigg[\sum_{p,l} d_l\big(V^{(4.5)p,q}_{l,j_1;j_4,b,a,m}\sqrt{d_p d_q d_l^{-1} d_{j_1}^{-1}}\big)$$

$$\begin{Bmatrix}j_1 & b & l\\ j_4 & a & n\end{Bmatrix}\Bigg((-1)^{1/2+2j_1+j_2+j_3-\varepsilon-\gamma+2q-l-2p+2a-n+2\alpha-2\beta}\begin{Bmatrix}p & m & \beta\\ \frac{1}{2} & b & q\end{Bmatrix}\begin{Bmatrix}p & m & \beta\\ k & j_4 & a\end{Bmatrix}\begin{Bmatrix}\frac{1}{2} & \alpha & a\\ k & m & j_1\end{Bmatrix}-\sum_u d_u(-1)^{1-b-j_1+j_2+j_3+j_4+2q+m-n-l-p+2u-\varepsilon-\gamma+\alpha+\beta}$$

$$\begin{Bmatrix}b & j_4 & u\\ a & q & p\end{Bmatrix}\begin{Bmatrix}a & u & q\\ m & \frac{1}{2} & \alpha\end{Bmatrix}\begin{Bmatrix}j_1 & \alpha & k\\ u & \frac{1}{2} & m\end{Bmatrix}\begin{Bmatrix}u & \frac{1}{2} & k\\ \beta & j_4 & b\end{Bmatrix}\Bigg)-\sum_{p,l} d_l d_q d_{j_1}^{-1}\big(V^{(4.5)p,j_1}_{l,q;j_4,\beta,\alpha,m}\sqrt{d_p d_{j_1} d_l^{-1} d_q^{-1}}\big)\begin{Bmatrix}\beta & j_4 & k\\ \alpha & j_1 & p\end{Bmatrix}\Bigg((-1)^{1/2+2j_1+j_2+j_3+2j_4+2a+2b+\varepsilon+\gamma+2m-p-k}$$

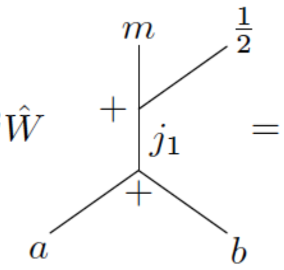
$$\begin{Bmatrix}\frac{1}{2} & \alpha & a\\ n & j_1 & m\end{Bmatrix}\begin{Bmatrix}b & m & l\\ \alpha & j_4 & n\end{Bmatrix}\begin{Bmatrix}m & \frac{1}{2} & q\\ \beta & l & b\end{Bmatrix}-\sum_u d_u(-1)^{1+j_1+j_2+j_3+j_4+a-b+\varepsilon+\gamma+l+2q-m+2n-p+2u-k+2\alpha-\beta}\begin{Bmatrix}\beta & q & l\\ \alpha & j_4 & u\end{Bmatrix}\begin{Bmatrix}m & a & u\\ n & \frac{1}{2} & j_1\end{Bmatrix}\begin{Bmatrix}n & \frac{1}{2} & u\\ \beta & j_4 & b\end{Bmatrix}\begin{Bmatrix}u & \alpha & q\\ \frac{1}{2} & m & a\end{Bmatrix}\Bigg)\Bigg]$$

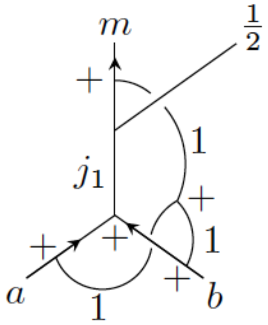


# Quantum volume

- Volume operator “2 ways”: intertwiner volume vs volume post holonomy coupling
- Volume defined by “grasping” all trios of edges
- Volume of 3-valent spinnetworks is zero and of “3.5”-valent ones is diagonal (no matrix construction required)

$$\hat{W}^{(v)}_{\{e_\alpha,e_\beta,e_\gamma\}} = \eta^{ijk}\,\hat{J}_i^{(e_\alpha,v)}\,\hat{J}_j^{(e_\beta,v)}\,\hat{J}_k^{(e_\gamma,v)} \qquad \hat{Q} = 48^{-1}\sum_{\{e_\alpha,e_\beta,e_\gamma\}}\kappa(\{e_\alpha,e_\beta,e_\gamma\})\hat{W}^{(v)}_{\{e_\alpha,e_\beta,e_\gamma\}} \qquad \hat{V} = V_0'\sqrt{|\hat{Q}|}$$

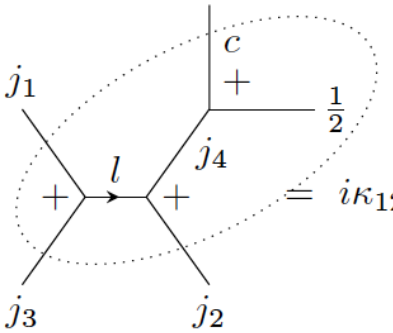
$6^{-1/2}\hat{W}$ 

$= -i\,[m(m+1)(2m+1)a(a+1)(2a+1)b(b+1)(2b+1)]^{\frac{1}{2}}$ 

# Quantum volume

- Volume operator “2 ways”: intertwiner volume vs volume post holonomy coupling
- Volume defined by “grasping” all trios of edges
- Volume of 4- or higher-valency spinnetworks requires proper convention for link orientation
- Matrices of variable sizes generating maps within equivalence classes

$$\hat{W}^{(v)}_{\{e_\alpha,e_\beta,e_\gamma\}} = \eta^{ijk}\,\hat{J}_i^{(e_\alpha,v)}\,\hat{J}_j^{(e_\beta,v)}\,\hat{J}_k^{(e_\gamma,v)} \qquad \hat{Q} = 48^{-1}\sum_{\{e_\alpha,e_\beta,e_\gamma\}}\kappa(\{e_\alpha,e_\beta,e_\gamma\})\hat{W}^{(v)}_{\{e_\alpha,e_\beta,e_\gamma\}} \qquad \hat{V} = V_0'\sqrt{|\hat{Q}|}$$

$\frac{8}{\sqrt{6}}\hat{Q}$ 


$$= i\kappa_{123}L_{j_1}L_{j_2}L_{j_3}W_{123} - i\kappa_{134}L_{j_1}L_{j_3}L_cW_{134} - i\kappa_{124}L_{j_1}L_{j_2}L_cW_{124} - i\kappa_{234}L_{j_2}L_{j_3}L_cW_{234}$$

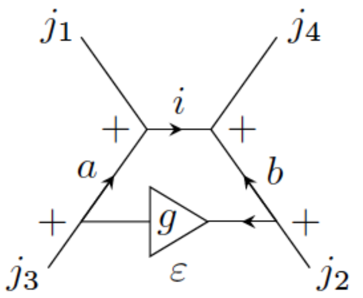
$$\kappa(\{e_2,e_3,e_4\}) = \kappa(\{e_1,e_2,e_4\}) = 6 = -\kappa(\{e_1,e_2,e_3\}) = -\kappa(\{e_1,e_3,e_4\})$$

# Scalar constraint in numerics

- Back to wavefunctions: use programming language "loopholes", i.e., a function that lacks definition (ghost function), like $\Psi$ in quantum-mechanics textbooks
- The ghost function has an arbitrarily large argument: a set of coordinates and spins, as well as a very large pool of zeros
- The only property of $\Psi$ is that inner products give 1 when arguments match

# Scalar constraint in numerics

- Scalar constraint implemented as a linear functional that reads and changes the arguments of the ghost functions
- Graph changes handled by using pool of zeros: old information (lower depth relative to intertwiner) are shifted to the right of argument sets (filling the pool), while updated information about spin networks (newly created inner loops) are encoded in the entries of the argument set

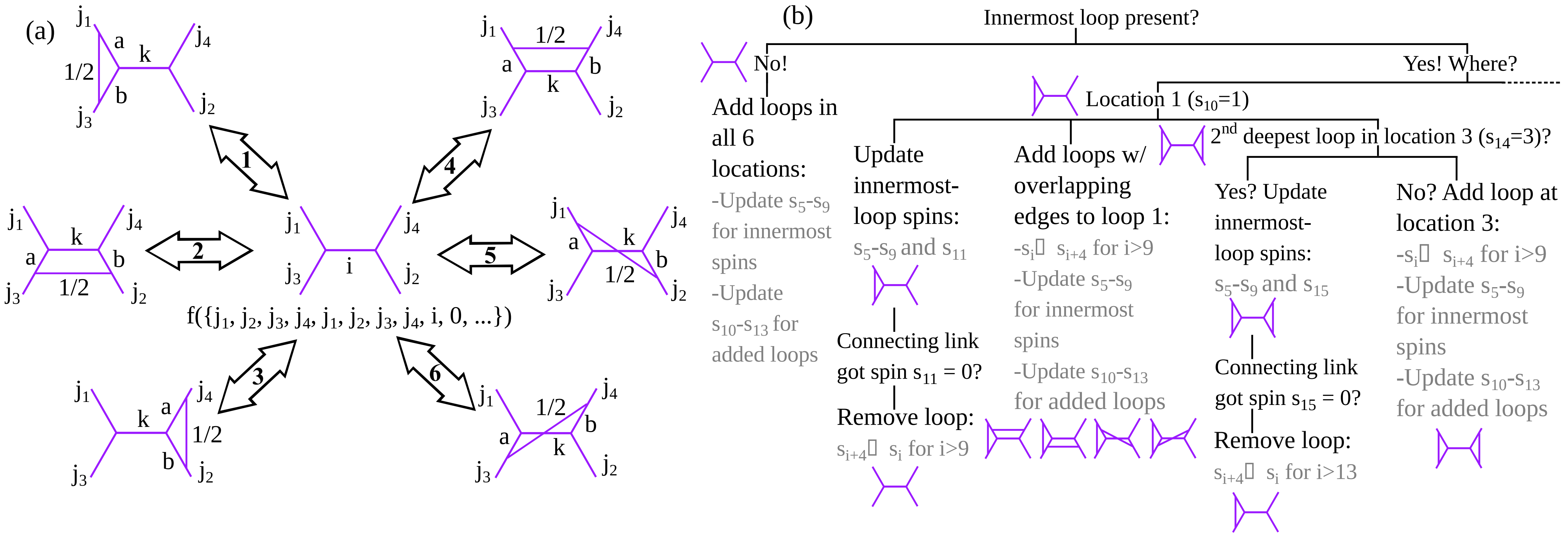



$$\{j_1, j_2, j_3, j_4, j_1, b, a, j_4, i, 2, \varepsilon, j_2, j_3, 0, \ldots, 0\}$$

$$\{j_1, j_2, j_3, j_4, c, b, d, j_4, k, 1, \gamma, j_1, a, 2, \varepsilon, j_2, j_3, 0, \ldots, 0\}$$

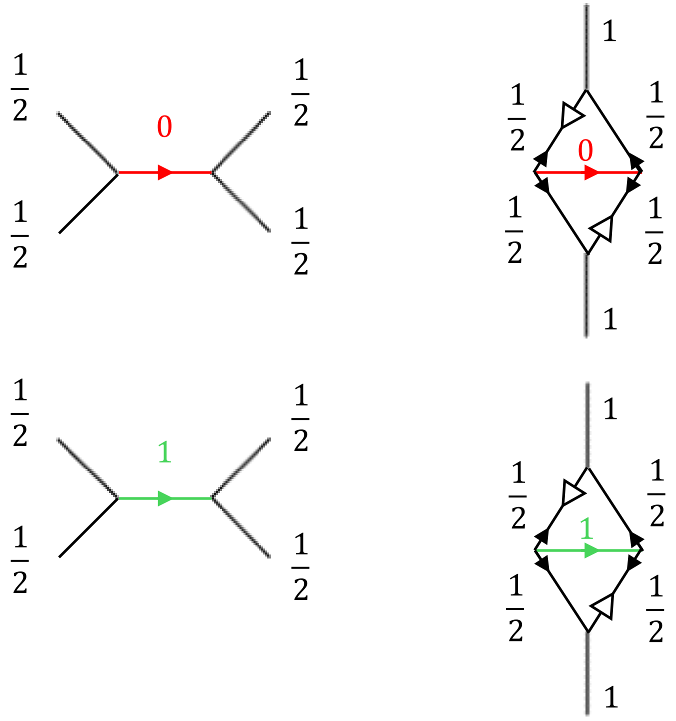
# Scalar constraint in numerics

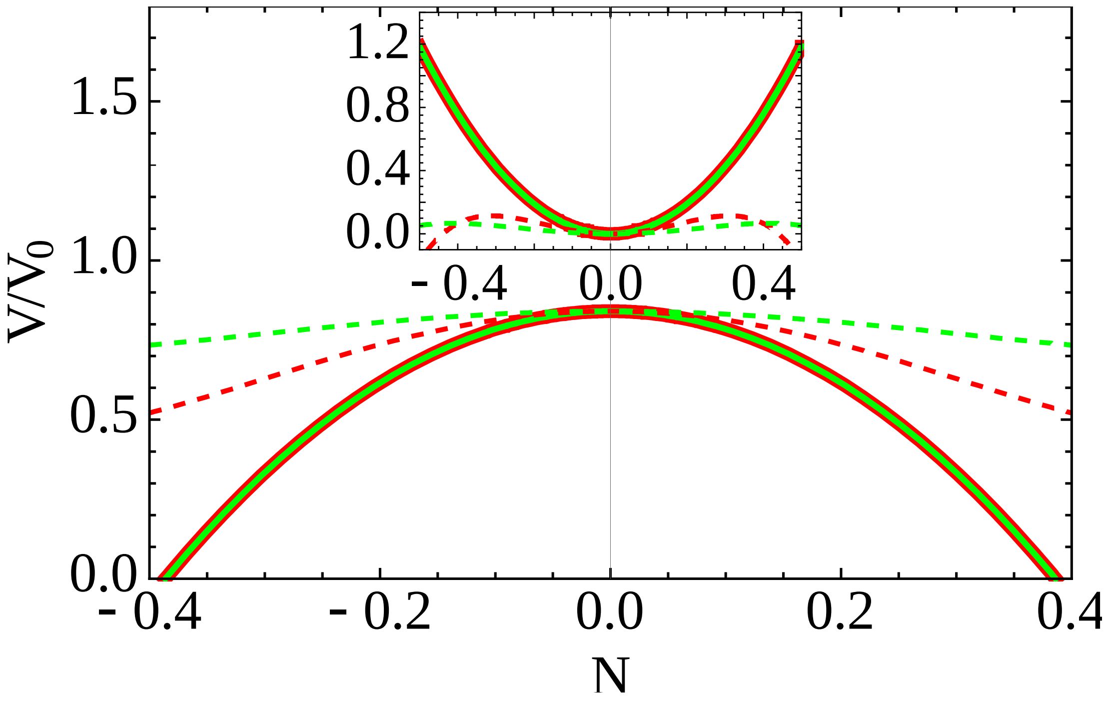
- Loop-insertion/-removal hierarchy



# Graph changes vs graph preservation

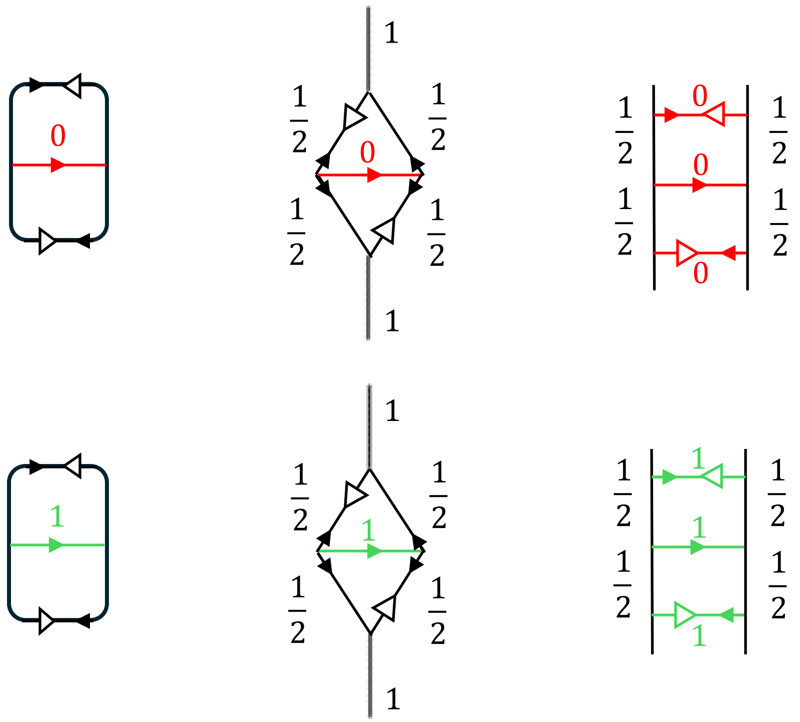
- 3rd-order (changing) and 4th-order (preserving) expansion of $\hat{U} = \exp(-iN\hat{C}_s)$ in the (perturbative) lapse
- Volume expectation values from graph-changing (solid) and graph-preserving (dashed) dynamics

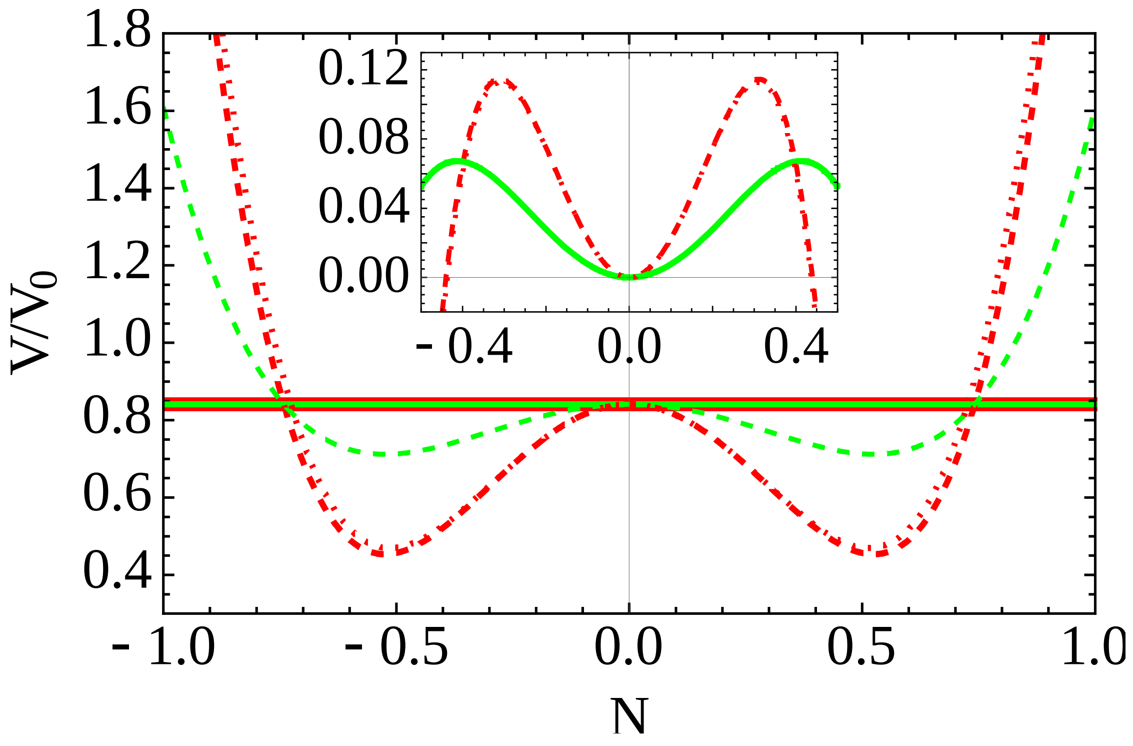


# Graph changes vs graph preservation

- 4th-order expansion of $\hat{U} = \exp(-iN\hat{C}_s)$ in the (perturbative) lapse for different graph-preserving scenarios
- Volume expectation values from graph-changing dynamics "well behaved"





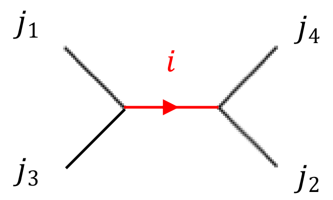
# Search for eigenstates

- Using the encoded constraint functional, can eingestates be found? Yes!
- Two types of eigenstates: one simple, another one requires infinite sums

$$|E_0\rangle = |s_0\rangle - \frac{\langle s_1|\hat{C}_s|s_0\rangle}{\langle s_1|\hat{C}_s|s_2\rangle}|s_2\rangle + \frac{\langle s_1|\hat{C}_s|s_0\rangle\langle s_3|\hat{C}_s|s_2\rangle}{\langle s_1|\hat{C}_s|s_2\rangle\langle s_3|\hat{C}_s|s_4\rangle}|s_4\rangle + \ldots = \sum_{i\,\text{even}} (-1)^{i/2}\frac{\langle s_1|\hat{C}_s|s_0\rangle}{\langle s_1|\hat{C}_s|s_2\rangle}\cdots\frac{\langle s_{i-1}|\hat{C}_s|s_{i-2}\rangle}{\langle s_{i-1}|\hat{C}_s|s_i\rangle}|s_i\rangle$$

# Computational times

- Computational times scale with spins considered and with perturbation order
- Single node computations unpractical on the long term




| $\{j_1, j_2, j_3, j_4, i\}$ | Time (seconds) |
|---|---|
| $\{1/2, 1/2, 1/2, 1/2, 0\}$ | 168.8 |
| $\{1/2, 1/2, 1/2, 1/2, 1\}$ | 187.5 |
| $\{1, 1/2, 1/2, 1, 1/2\}$ | 668.8 |
| $\{1, 1/2, 1/2, 1, 3/2\}$ | 687.9 |
| $\{1/2, 1, 1/2, 1, 0\}$ | 822.4 |
| $\{1/2, 1, 1/2, 1, 1\}$ | 826.0 |
| $\{1, 1, 1, 1, 0\}$ | 6543.8 |
| $\{1, 1, 1, 1, 1\}$ | 6744.7 |
| $\{1, 1, 1, 1, 2\}$ | 6482.7 |

| Functional | $\{j_1, j_2, j_3, j_4, i\}$ | Time (seconds) | | | |
|---|---|---|---|---|---|
| | | 1 | $C_s$ | $C_s^2$ | $C_s^3$ |
| Hamiltonian | $\{1/2, 1/2, 1/2, 1/2, 0\}\|_{gc}$ | NA | 168.8 | 10058.5 | |
| | $\{1/2, 1/2, 1/2, 1/2, 1\}\|_{gc}$ | NA | 187.5 | 10438.1 | |
| | $\{1/2, 1/2, 1/2, 1/2, 0\}\|_{gp}$ | NA | 184.7 | 9657.3 | 113261.4 |
| | $\{1/2, 1/2, 1/2, 1/2, 1\}\|_{gp}$ | NA | 173.2 | 9606.7 | 109962.8 |
| Collector | $\{1/2, 1/2, 1/2, 1/2, 0\}\|_{gc}$ | NA | 0.009 | 11.1 | |
| | $\{1/2, 1/2, 1/2, 1/2, 1\}\|_{gc}$ | NA | 0.008 | 11.7 | |
| | $\{1/2, 1/2, 1/2, 1/2, 0\}\|_{gp}$ | NA | 0.011 | 3.1 | |
| | $\{1/2, 1/2, 1/2, 1/2, 1\}\|_{gp}$ | NA | 0.009 | 3.1 | |
| Normalizer | $\{1/2, 1/2, 1/2, 1/2, 0\}\|_{gc}$ | 0.00010 | 0.0013 | 0.617 | |
| | $\{1/2, 1/2, 1/2, 1/2, 1\}\|_{gc}$ | 0.00009 | 0.0021 | 0.408 | |
| | $\{1/2, 1/2, 1/2, 1/2, 0\}\|_{gp}$ | 0.00083 | 0.0023 | 0.165 | |
| | $\{1/2, 1/2, 1/2, 1/2, 1\}\|_{gp}$ | 0.00003 | 0.0005 | 0.007 | |
| Volume | $\{1/2, 1/2, 1/2, 1/2, 0\}\|_{gc}$ | 0.092 | 1.39 | 27.8 | NA |
| | $\{1/2, 1/2, 1/2, 1/2, 1\}\|_{gc}$ | 0.086 | 1.40 | 27.7 | NA |
| | $\{1/2, 1/2, 1/2, 1/2, 0\}\|_{gp}$ | 0.085 | 1.46 | 9.1 | NA |
| | $\{1/2, 1/2, 1/2, 1/2, 1\}\|_{gp}$ | 0.081 | 1.38 | 8.7 | NA |

# Conclusions

- Updated derivations of action of Euclidean scalar constraint on 3-valent and 4-valent spin networks
- Encoded graph-changing (and graph-preserving) dynamics numerically, as well as volume operator
- Studied change in the volume expectation value as a function of lapse
- Searched for eingenstates of the scalar constraint
- First step towards systematic numerical studies of graph-changing Hamiltonian constraints

# Obrigado!