$SL(2,\mathbb{C})$ Chern-Simons theory and spinfoam gravity with a cosmological constant

Hal Haggard & Aldo Riello

with Muxin Han & Wojciech Kamiński

(to appear)

November 25th, 2014

International Loop Quantum Gravity Seminar

Motivations

The cosmological constant is non-zero: $\Lambda = 2.90 \times 10^{-122}\,\ell_{\mathrm{P}}^{-2}$

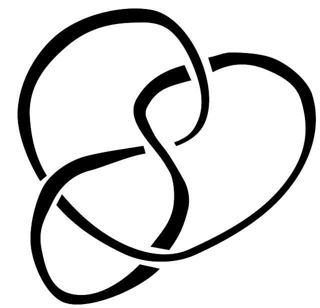
Why quantum groups in 4D?
Seek a (possibly more general) constructive route

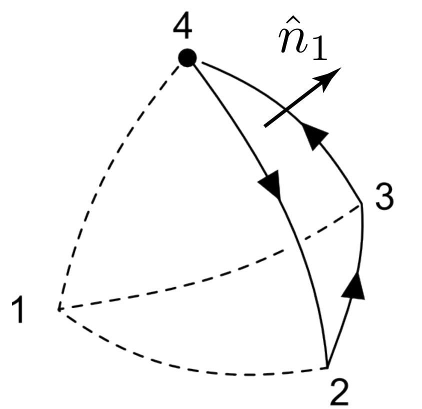
We are lead to couple Loop Quantum Gravity to Chern-Simons theory,
the result has strong relations with previous Hamiltonian studies

A more geometric language that casts light on the asymptotics

What do we gain?

We enlarge the usual framework,
taking tools from Chern-Simons theory into spinfoams





We develop a new description of
curved simplices in 3 and 4d,
where holonomies also encode fluxes

We obtain both $\Lambda \lessgtr 0$,
the sign being determined dynamically
at the semiclassical level

$$\Lambda$$

We find that Λ must be quantized

Main results from the asymptotics

Disclaimer: for now the construction is at the single 4-simplex level only

The equations of motion define non-perturbatively curved 4-simplices, of positive and negative curvature

The Regge action for curved 4-simplices augmented by the cosmological term is recovered exactly (work in progress on an extra term that we seem to obtain)

$$S_{\text{Regge}} = \sum_{\text{triangles}} a_t \Theta_t - \Lambda V_4$$

Plan of the talk

1. Construction and definition of the model ΛEPRL
2. The equations of motions (EoM)
3. Focus on 3d curved geometries, and their reconstruction from the EoM
4. Towards a deformed phase space for curved quantum geometries

EPRL philosophy

Regge's philosophy:
construct a manifold out of flat building blocks $+ \, (\mathrm{d}-2)$-dimensional defects

At the quantum level,
flatness is implemented via BF dynamics in the bulk of building blocks, while defects are created by using only geometric boundary states
(thus breaking BF symmetries)

GR = BF + geometricity constraints

EPRL mathematics

At the level of a single building block,
the EPRL amplitude of the 3d spin-network boundary state ψ_Γ is

holonomy of $\mathcal{A}$

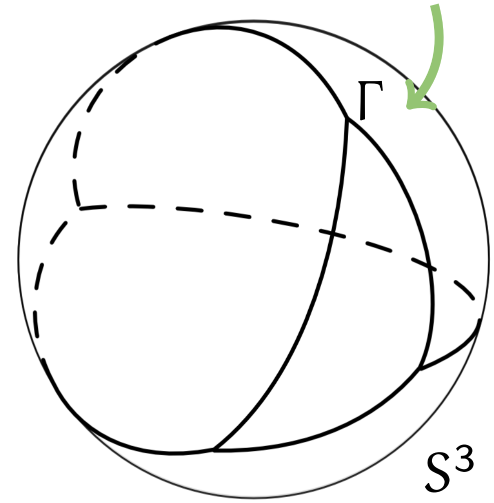
$$Z_{\mathrm{EPRL}}(\psi_\Gamma) := \int \mathcal{D}B\mathcal{D}\mathcal{A}\, e^{\frac{i}{2\ell_P^2}\int B\wedge\mathcal{F}[\mathcal{A}]}\,(f_\gamma\psi_\Gamma)(G[\mathcal{A}]) = (f_\gamma\psi_\Gamma)(\mathbb{1})$$

$SL(2,\mathbb{C})$
spin connection

B is the 'bivector' field
($B = \star e \wedge e$ on geometric states)

f_γ is the Dupuis-Livine map,
it embeds ψ into spacetime

dual to 4-simplex boundary



*(drawing is in one dimension less)

ΛEPRL philosophy

ΛRegge's philosophy:
construct a manifold out of **homogeneously curved** building blocks $+$ $(d-2)$-dimensional defects (Bahr & Dittrich)

At the quantum level,
the **homogenous curvature** is implemented via $BF - \frac{\Lambda}{6}BB$ dynamics, and defects are created as in the flat case

Λ-GR = $BF - \frac{\Lambda}{6}BB$ + geometricity constraints

ΛEPRL mathematics

$$Z(\psi_\Gamma) := \int \mathcal{D}B\,\mathcal{D}\mathcal{A}\; e^{\frac{i}{2\ell_P^2}\int B\wedge\mathcal{F}[\mathcal{A}]-\frac{\Lambda}{6}B\wedge B}\;\big(f_\gamma\psi_\Gamma\big)(G[\mathcal{A}])$$

$$= \int \mathcal{D}\mathcal{A}\; e^{\frac{3i}{4\Lambda\ell_P^2}\int \mathcal{F}[\mathcal{A}]\wedge\mathcal{F}[\mathcal{A}]}\;\big(f_\gamma\psi_\Gamma\big)(G[\mathcal{A}])$$

$$= \int \mathcal{D}\mathcal{A}\; e^{\frac{3\pi i}{\Lambda\ell_P^2}CS[\mathcal{A}]}\;\big(f_\gamma\psi_\Gamma\big)(G[\mathcal{A}])$$

where the Chern-Simons functional is

$$CS[\mathcal{A}] := \frac{1}{4\pi}\oint_{S^3} d\mathcal{A}\wedge\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}$$

For boundary connection functionals,
ΛBF in the bulk is equivalent to CS on the boundary

(Baez)

The ΛEPRL 4-simplex amplitude

Twisting the previous construction by using the γ-Holst action finally gives

$$Z_{\Lambda\text{EPRL}}(\psi_\Gamma) := \int \mathcal{D}A\mathcal{D}\overline{A}\, e^{i\frac{h}{2}CS[A]+i\frac{\overline{h}}{2}CS[\overline{A}]}\,(f_\gamma\psi_\Gamma)(G[A,\overline{A}])$$

where $(A,\overline{A})$ are the self- and antiself-dual parts of $\mathcal{A}$

$h := \frac{12\pi}{\Lambda\ell_P^2}\left(\frac{1}{\gamma}+i\right)$ is the complex CS level

Remark

$Z_{\Lambda\text{EPRL}}$ involves only quantities living on the boundary of the building block

ΛEPRL = $SL(2,\mathbb{C})$-CS evaluation of a specific Wilson graph operator

Two immediate consequences

$$\mathrm{h} := \frac{12\pi}{\Lambda \ell_P^2}\left(\frac{1}{\gamma} + \mathrm{i}\right)$$

The CS level h is complex, hence there is
no (known) quantum group structure associate to the graph evaluation

(Fairbairn & Meusburger, Han)

Invariance of the amplitude under large gauge transformation $\mathcal{A} \mapsto \mathcal{A}^g$ implies $\Re(\mathrm{h}) \in \mathbb{Z}$, i.e.

$$\frac{12\pi}{|\Lambda|} \equiv 4\pi R_\Lambda^2 \in \gamma \ell_P^2 \mathbb{N}$$

(Kodama, Randono, Smolin, Wieland)

Three interesting limits

$$\mathrm{h} := \frac{12\pi}{\Lambda \ell_P^2}\left(\frac{1}{\gamma} + \mathrm{i}\right)$$

Vanishing cosmological constant $\Lambda \to 0$:
$\mathrm{h} \to \infty$, thus CS is projected onto its classical solutions $\rightsquigarrow$ flat EPRL

q-deformed Lorentzian Barrett-Crane amplitude:
when $\gamma \to \infty$, the EPRL graph operator goes into Barrett-Crane's,
while h becomes $\in \mathrm{i}\mathbb{R}$, giving $q = \exp\left(-\ell_P^2/R_\Lambda^2\right)$
(Noui & Roche)

Semiclassical ΛRegge limit: more about this in a second

The semiclassical ΛRegge limit

$$\ell_P \to 0, \quad j \to \infty, \qquad \text{with} \quad a_{\text{phys}} \equiv \gamma \ell_P^2 j = cnst$$

$\ell_P \to 0$ means $h \to \infty$, which corresponds to CS classical flat limit,

however

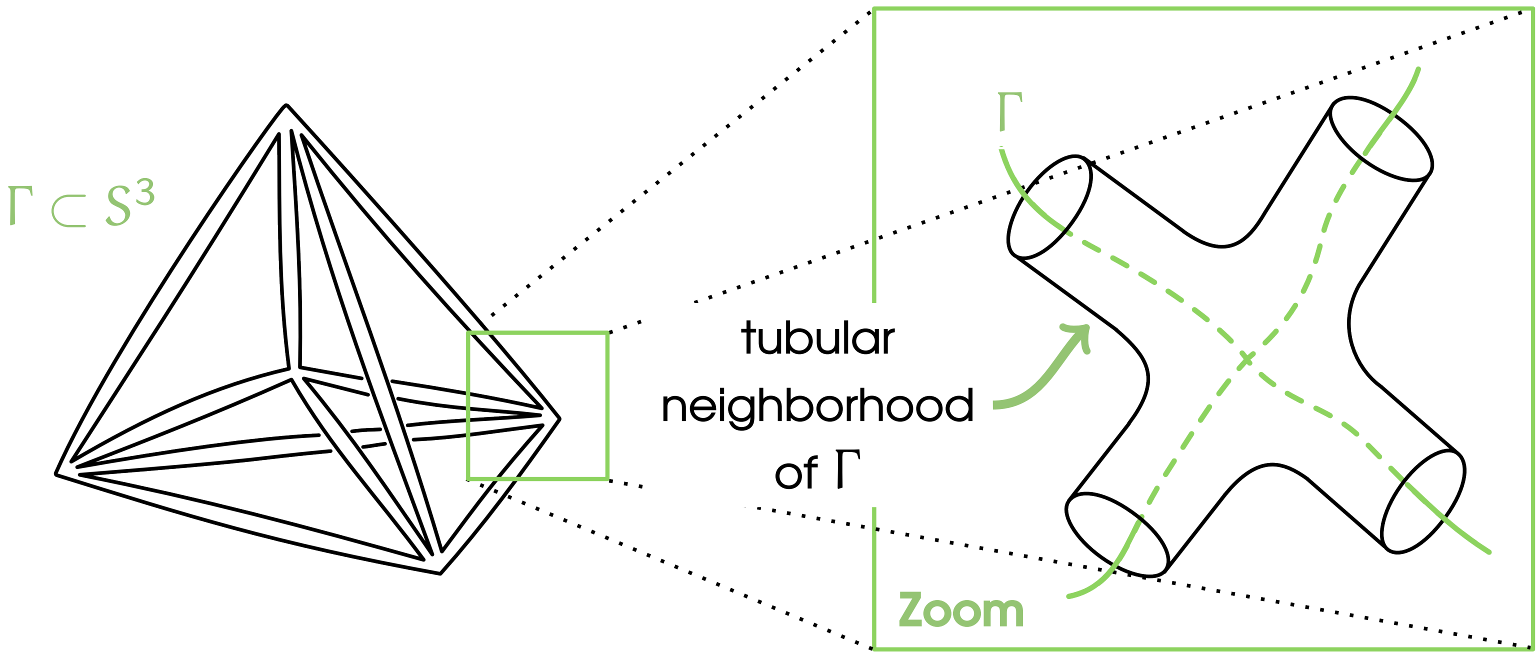
$j \to \infty$ makes the Wilson graph operator stand out and act as a distributional source for $(A, \overline{A})$,

thus avoiding flatness

Semiclassical limit = study of flat connections on the graph complement $S^3 \setminus \Gamma$

The graph complement $S^3 \setminus \Gamma$

Γ is the graph dual to the 4-simplex boundary



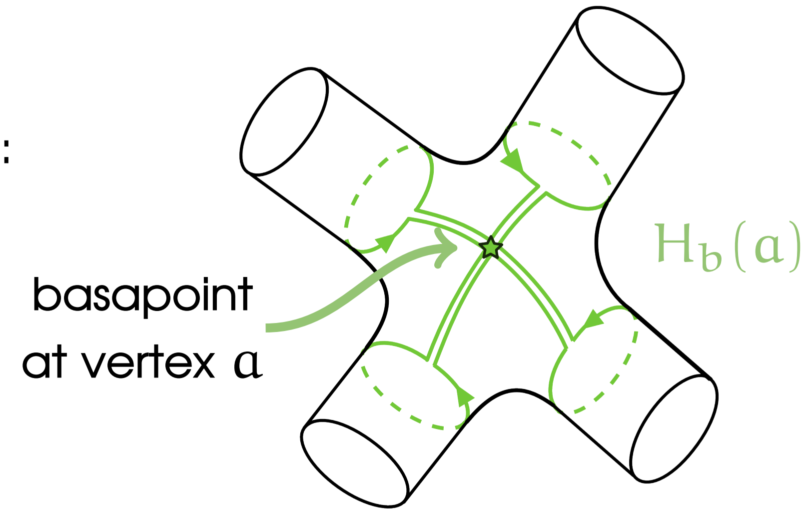
It is obtained by removing a tubular neighborhood of Γ from S^3
Its boundary is a genus 6 surface

Framing of the graph

There are two types of holonomies in $S^3 \setminus \Gamma$:

- transverse $H_b(a)$
- longitudinal G_{ba}

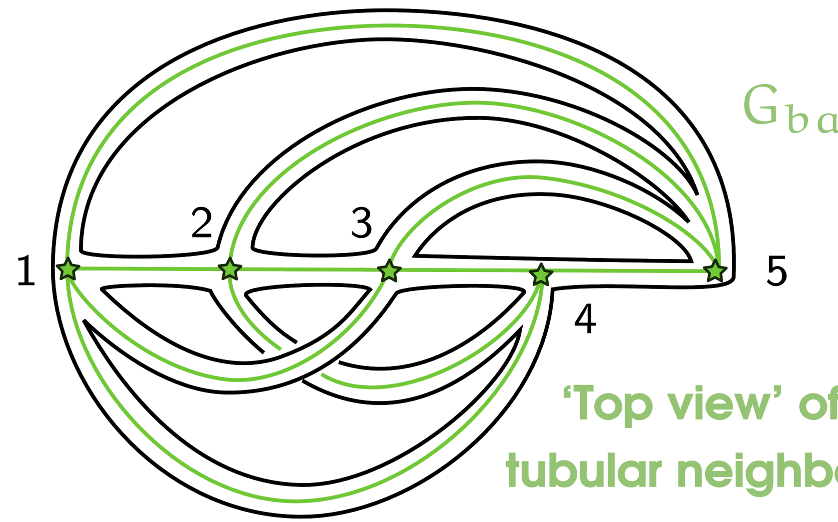
where $a, b, \ldots$ label the graph vertices



Zoom on vertex a

We need to specify the exact paths
This is called a **choice of framing for Γ**

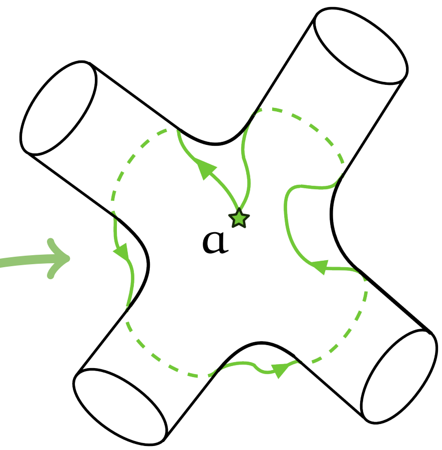
longitudinal paths
run on the
'top' of the tubes



'Top view' of the tubular neighborhood

Equations of motion

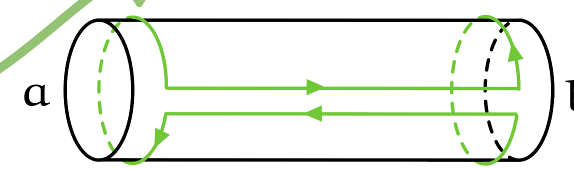
The connection on the graph complement is flat, hence holonomies along contractible paths are trivial:



closures $\overleftarrow{\prod}_b H_b(a) = \mathbb{1}$

parallel transports

$$G_{ba} H_b(a) G_{ab} = H_a(b)^{-1}$$

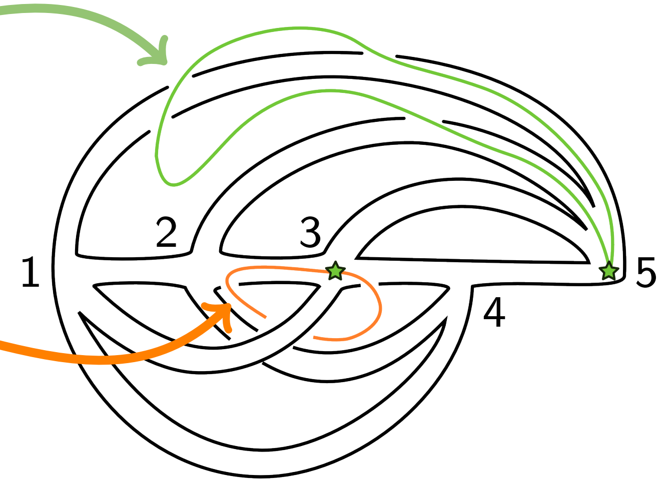


around 5 out of the 6 independent ‘faces’

$$G_{ac} G_{cb} G_{ba} = \mathbb{1}$$

while, around the last independent ‘face’:

$$G_{34} G_{42} G_{23} = H_1(3)$$

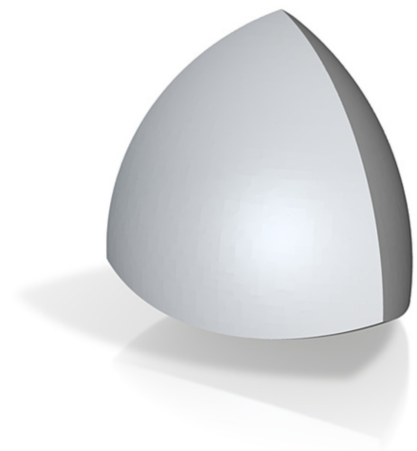


These equations of motion are enough to reconstruct the full 4D geometry of the 4-simplex.

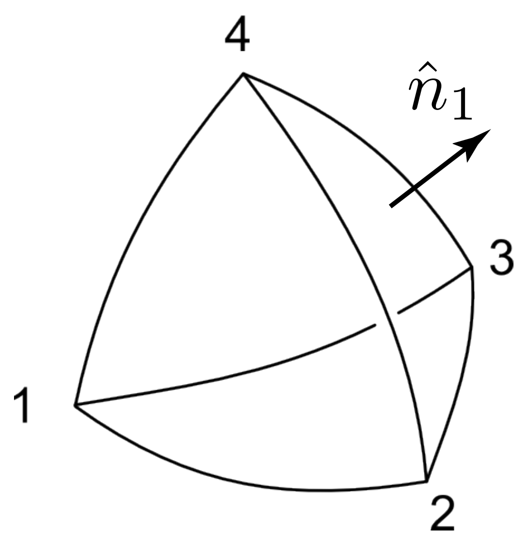
The reconstruction is more transparent in 3D $\rightsquigarrow$ will show you this with emphasis on the connections to 4D

A spherical tetrahedron is 4 points of S^3 connected by geodesics

Each face is a triangular portion of a great 2-sphere

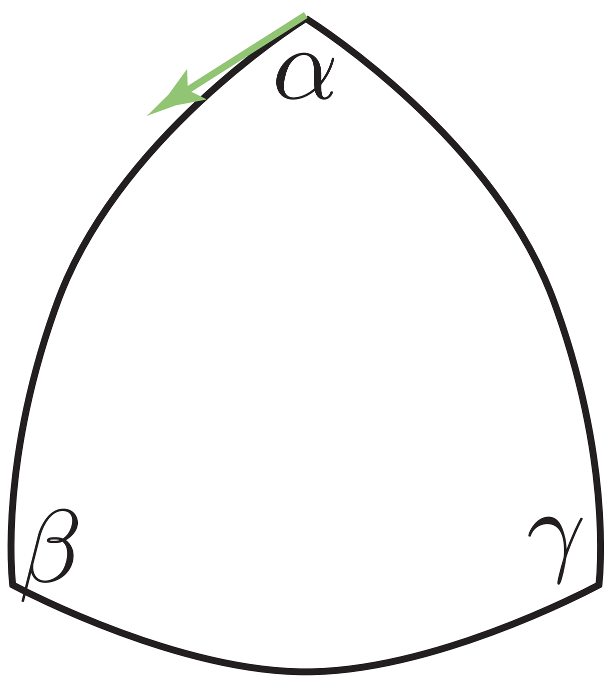


♦ Great spheres are flatly embedded in S^3 (i.e. $K_{ij} = 0$)

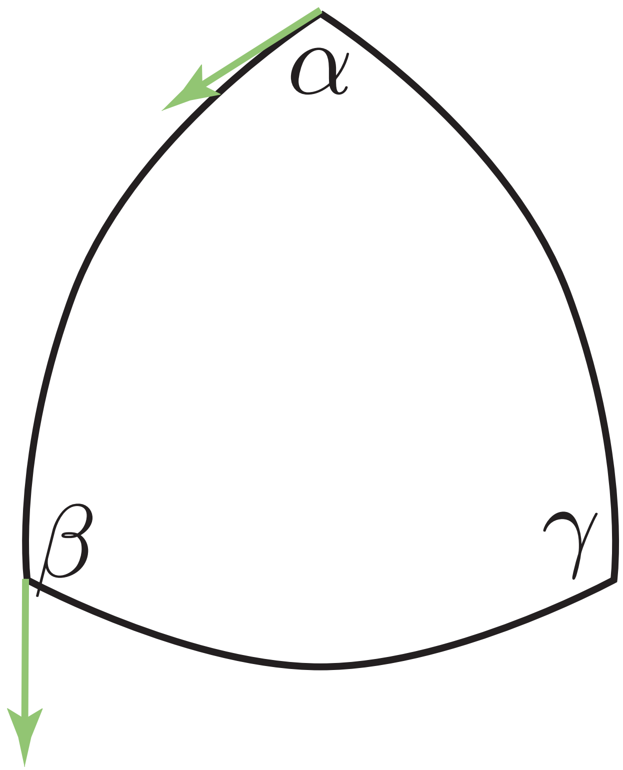


The normal to a face is well-defined and invariant under parallel transport

The holonomy around a curved triangle is equal to the area of the triangle

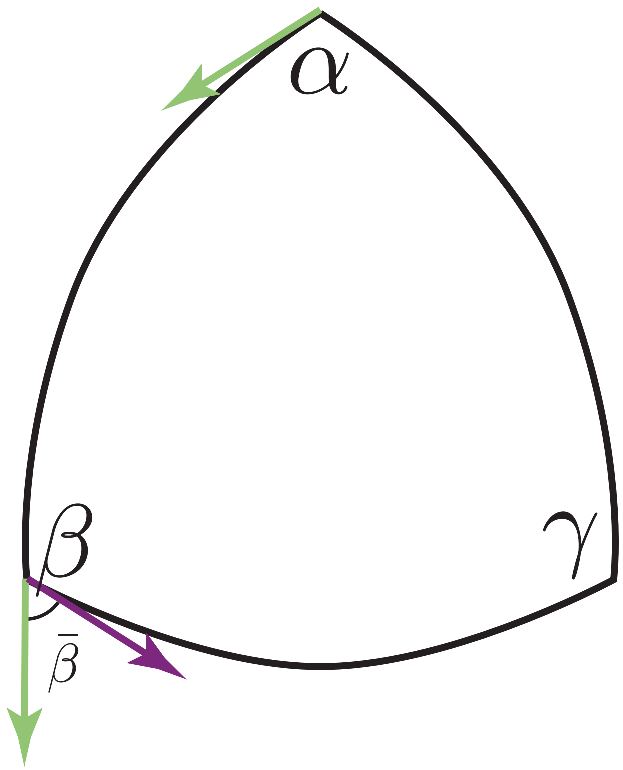


The holonomy around a curved triangle is equal to the area of the triangle



Parallel transport of the tangent vector is easy

The holonomy around a curved triangle is equal to the area of the triangle



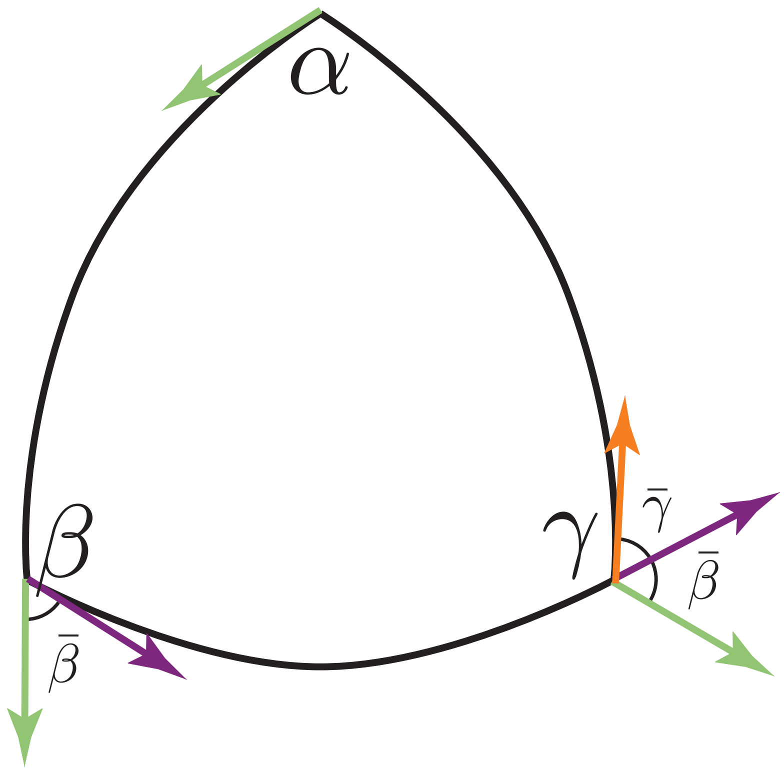
Pick up the next tangent vector

and introduce the complement

$$\bar{\beta} = \pi - \beta$$

This angle is preserved under transport

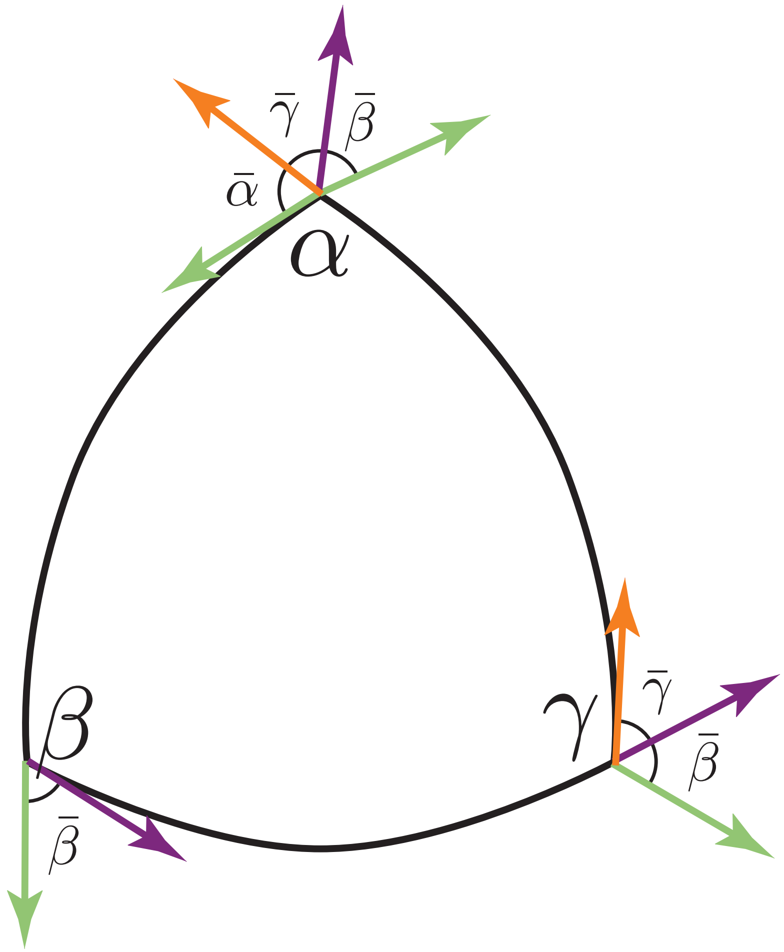
The holonomy around a curved triangle is equal to the area of the triangle



Repeat

$$\bar{\beta} = \pi - \beta$$
$$\bar{\gamma} = \pi - \gamma$$

The holonomy around a curved triangle is equal to the area of the triangle



$$\bar{\alpha} = \pi - \alpha$$
$$\bar{\beta} = \pi - \beta$$
$$\bar{\gamma} = \pi - \gamma$$

The full holonomy is a counterclockwise rotation about the normal with angle

$$\begin{aligned} a &= 2\pi - \bar{\alpha} - \bar{\beta} - \bar{\gamma} \\ &= \alpha + \beta + \gamma - \pi, \end{aligned}$$

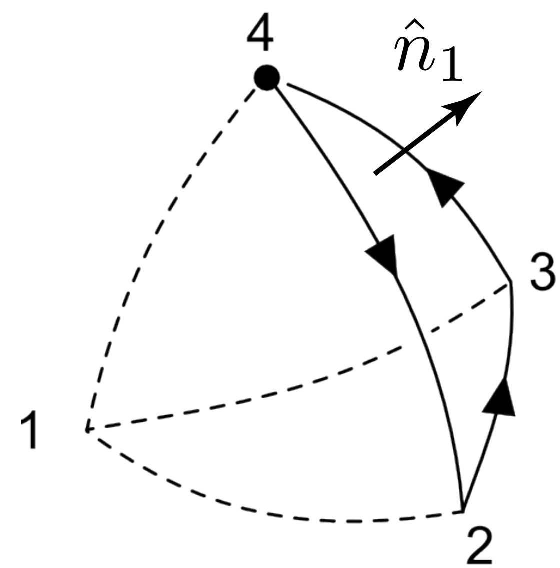
the area of the spherical triangle!

We have converted flux variables to 'transverse' holonomies

Our calculation shows that the holonomy can be cast as:

$$O = \exp\left(\frac{a}{R^2}\,\hat{n}\cdot\vec{J}\right), \quad O \in SO(3)$$

(Sahlmann, Dittrich & Geiller)



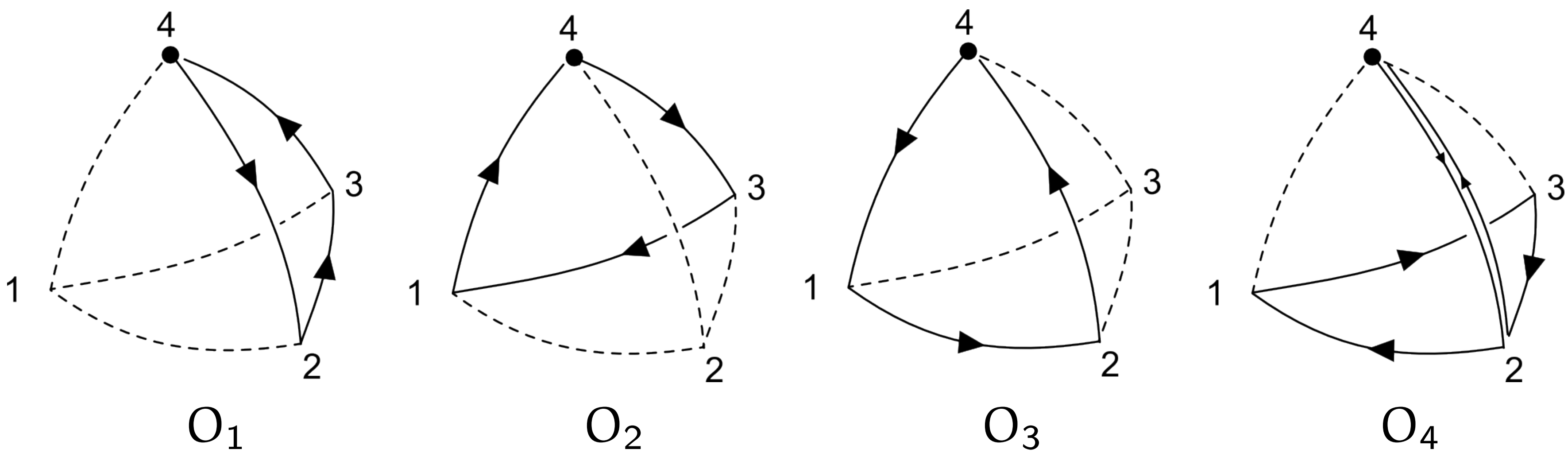
Idea: the closure relation should be replaced by the automatic homotopy constraint (Bonzom, Charles, Dupuis, Girelli, Livine)

$$O_4 O_3 O_2 O_1 = \mathbb{1}$$

This is the $SO(3)$ version of our eqn of motion, slide 15. For $R \to \infty$

$$O_4 O_3 O_2 O_1 = \mathbb{1} + R^{-2}(a_1\hat{n}_1 + a_2\hat{n}_2 + a_3\hat{n}_3 + a_4\hat{n}_4)\cdot\vec{J} + \cdots = \mathbb{1}$$

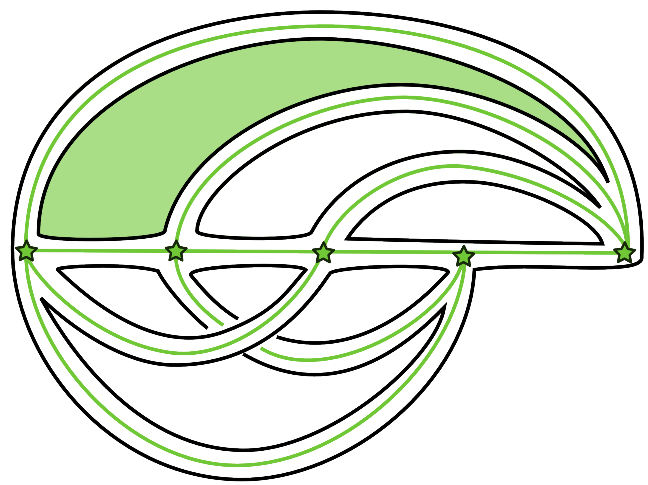
How do you access the global geometry? We use 'simple' paths.



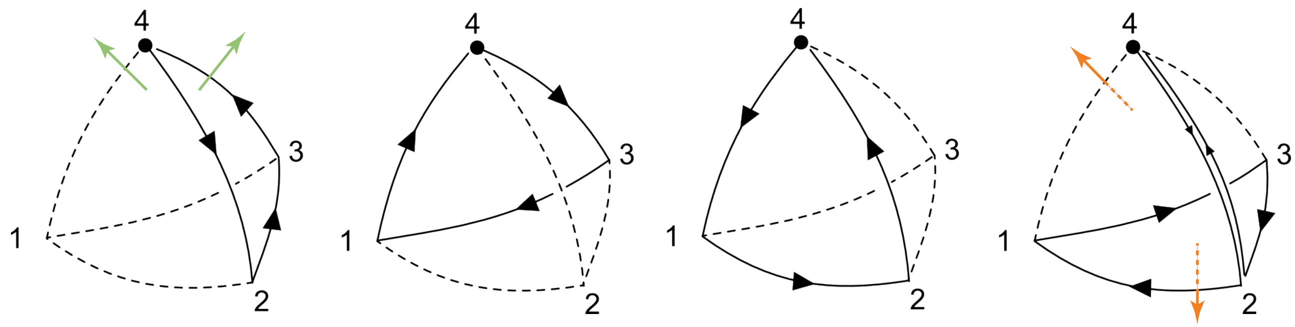
This particular choice appears arbitrary, but from the 4D perspective it is not.

Requiring that the faces of Γ only intersect along its edges...

...and using the framing discussed above uniquely fixes the paths.



The simple paths determine a geometrically meaningful curved Gram matrix.



The geometrical dot product $\hat{n}_1 \cdot \hat{n}_3$ is well defined at vertex 4,

but we have to rotate $\hat{n}_4$ to give a meaningful dot product with $\hat{n}_2$ at 4.

The Gram matrix is

$$\text{Gram} = \begin{pmatrix} 1 & \hat{n}_1 \cdot \hat{n}_2 & \hat{n}_1 \cdot \hat{n}_3 & \hat{n}_1 \cdot \hat{n}_4 \\ * & 1 & \hat{n}_2 \cdot \hat{n}_3 & \hat{n}_2 \cdot \mathbf{O}_1 \hat{n}_4 \\ * & * & 1 & \hat{n}_3 \cdot \hat{n}_4 \\ \text{sym} & * & * & 1 \end{pmatrix};$$

it is obtained by tracing, $\langle O_\ell O_m \rangle_C = \frac{1}{2}\text{Tr}(\mathbf{O}_\ell \mathbf{O}_m) - \frac{1}{4}\text{Tr}(\mathbf{O}_\ell)\text{Tr}(\mathbf{O}_m)$,

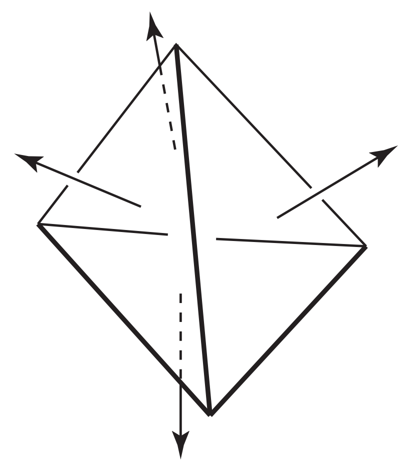
$$\hat{n}_\ell \cdot \hat{n}_m = \frac{\langle O_\ell O_m \rangle_C}{\sqrt{1 - \langle O_\ell \rangle^2}\sqrt{1 - \langle O_m \rangle^2}}$$

What determines the sign of the curvature?
The holonomies do it directly, through Gram.

$$\begin{cases} \det \mathrm{Gram} > 0 & \text{spherical geometry} \\ \det \mathrm{Gram} < 0 & \text{hyperbolic geometry} \end{cases}$$

Consider a flat
(Euclidean) tetrahedron

Its four vectors are
linearly dependent
$\rightsquigarrow \det \mathrm{Gram} = 0.$



The general claim follows from a special case and continuity in the curvature.

There is no need for another group.

♣ Finally we introduce a spin lift,

$$O_\ell \longrightarrow H_\ell, \quad H_\ell \in SU(2)$$

Result:

a full constructive proof of the Minkowski theorem for curved tetrahedra

Conjecture: There exists a convex constant curvature polyedron with N faces whenever (HMH, Freidel & Livine, Speziale)

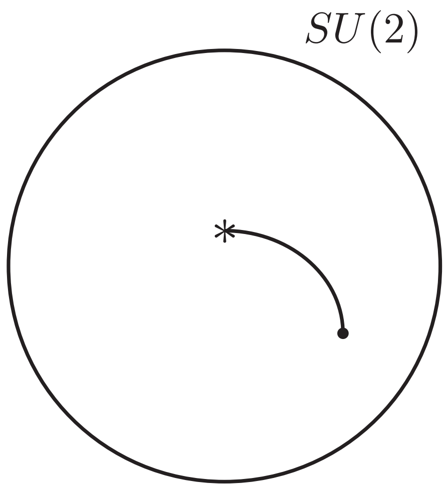
$$H_N \cdots H_1 = \mathbb{1}, \quad H_\ell \in SU(2)$$

Focused here on 3D; same techniques allow reconstruction of the 4-simplex.

The G_{ab} tell us how to assemble the tetrahedra. In fact, there is a sense in which H and G are conjugate (like Fenchel-Nielsen coords).

Turn to 3D again to explore simplest phase space structures.

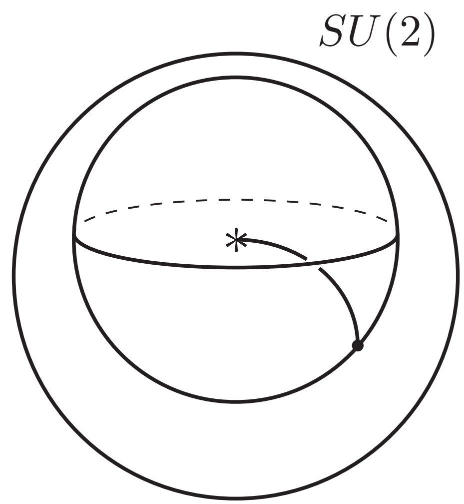
Transverse holonomies are, like standard fluxes, phase space variables



Each holonomy acts like a curved vector; really a geodesic segment from $\mathbb{1}$ to the group element.

$\rightsquigarrow$ $SU(2)$ plays nice role: $H = e^{-i\frac{a}{2}\hat{n}\cdot\vec{\sigma}}$

The conjugacy class of an holonomy sweeps out a 2-sphere in $SU(2) \cong S^3$.

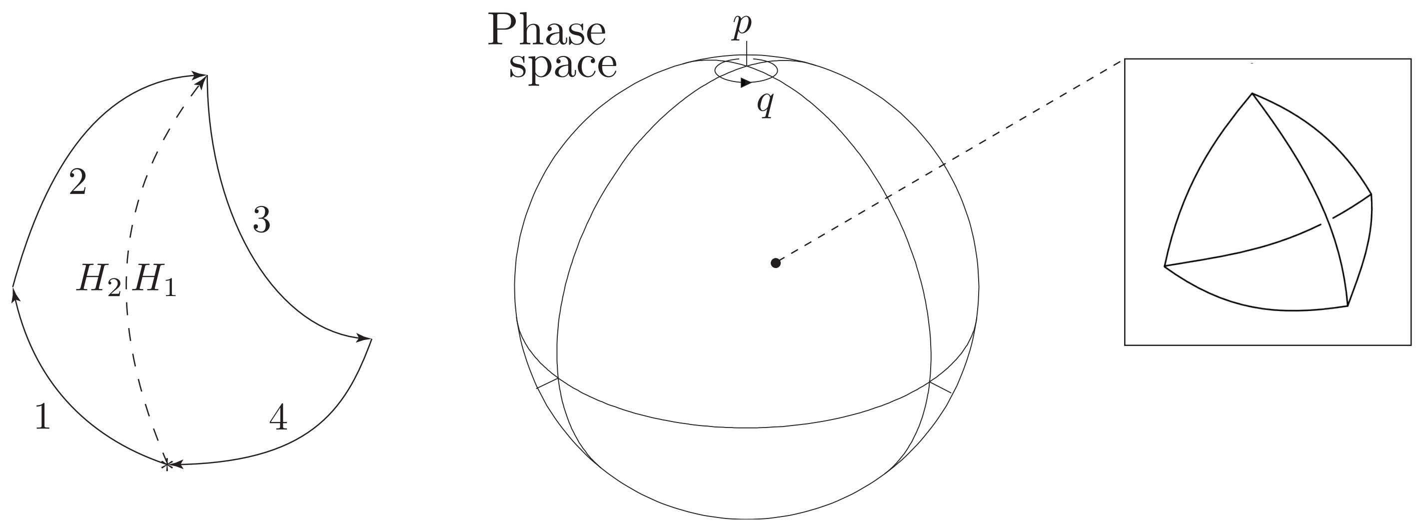


This 2-sphere is symplectic, an orbit of the dressing action of a quasi Poisson-Lie group ($\sim$ q-def.) $\Longrightarrow$ can use symplectic tools!

(Amelino-Camelia, Freidel, Kowalsky-Glikman, Smolin)

Like the flat case, we can construct a phase space of shapes

Form product of 4 fixed conjugacy class spheres (fixed areas of tet faces) and symplectically reduce by overall rotations (Ditrrich & Bahr, Treloar)



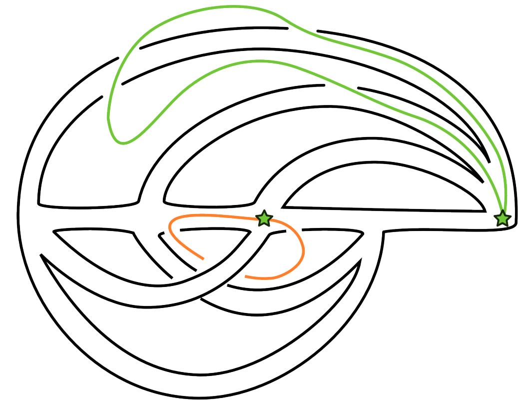
Distinct polyhedra (intertwiners for quantum theory with cosmological const.) correspond to different shapes of a spherical polygon

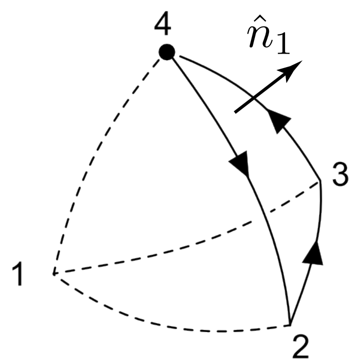
- Immediately conclude the volume of curved tetrahedra quantized

(HMH & Bianchi)

Conclusions

$SL(2,\mathbb{C})$ Chern-Simons theory is a tool for substituting BF theory with BF-$\frac{\Lambda}{6}$BB and leads to a quantized cosmological constant





Can reconstruct curved geometries
in 3 and $4\mathrm{D}$ from the resulting eqs of motion

- Phase space and quantized curved geometries

Connect Chern-Simons theory
and the cosmological constant in 4D

♦ clarifies the role and origin of quantum groups