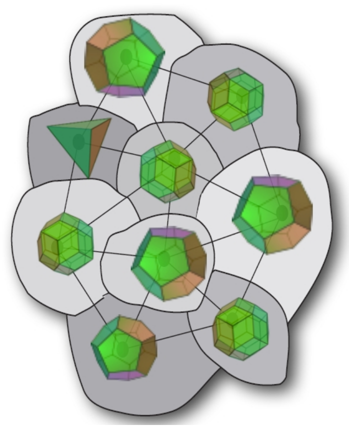
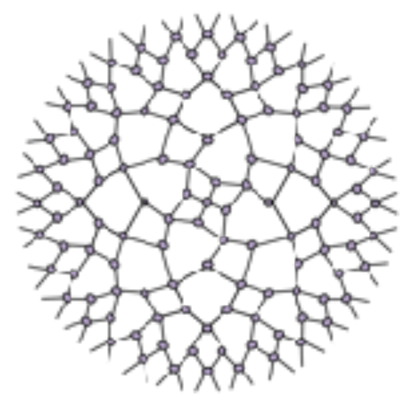
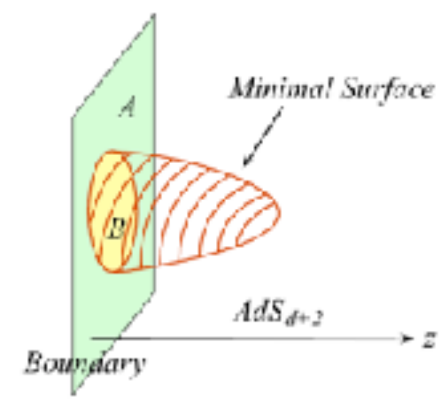
Loop Quantum Gravity,

Tensor Network,

and Holographic Entanglement Entropy







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Based on: MH and Ling-Yan Hung, arXiv:1610.02134

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Outline

AdS/CFT and Holographic Entanglement Entropy (Ryu-Takayanagi formula)

Tensor Network and Application to Holography, Need for LQG

Introduction to Loop Quantum Gravity (LQG) and Spin-Network State

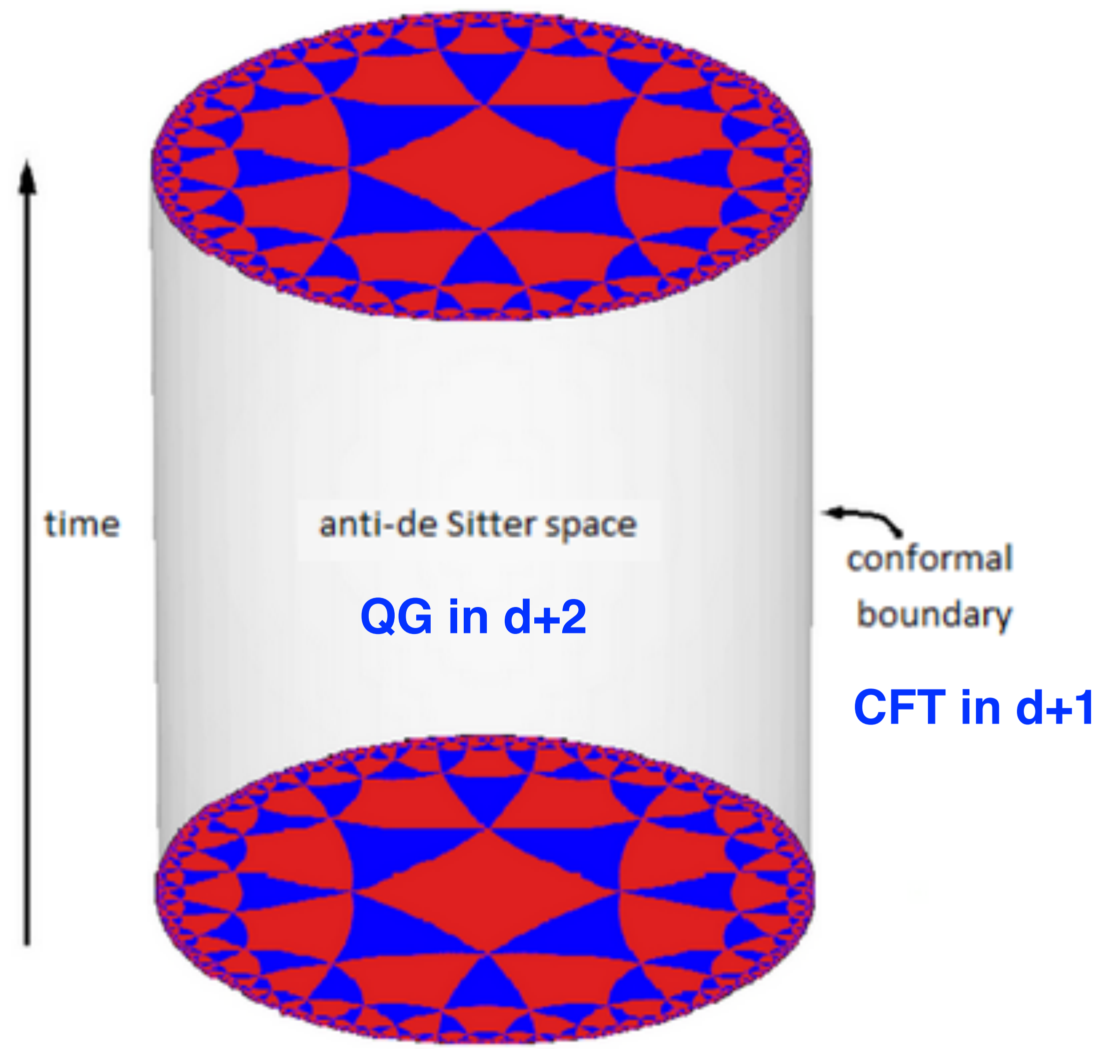
Emerging Tensor Network as Effective Theory from LQG

Derive Holographic Entanglement Entropy (RT formula) From LQG

Recent works with similar aims: Smolin 1608.02932, Chirco, Oriti, Zhang 1701.01383

AdS/CFT Correspondence

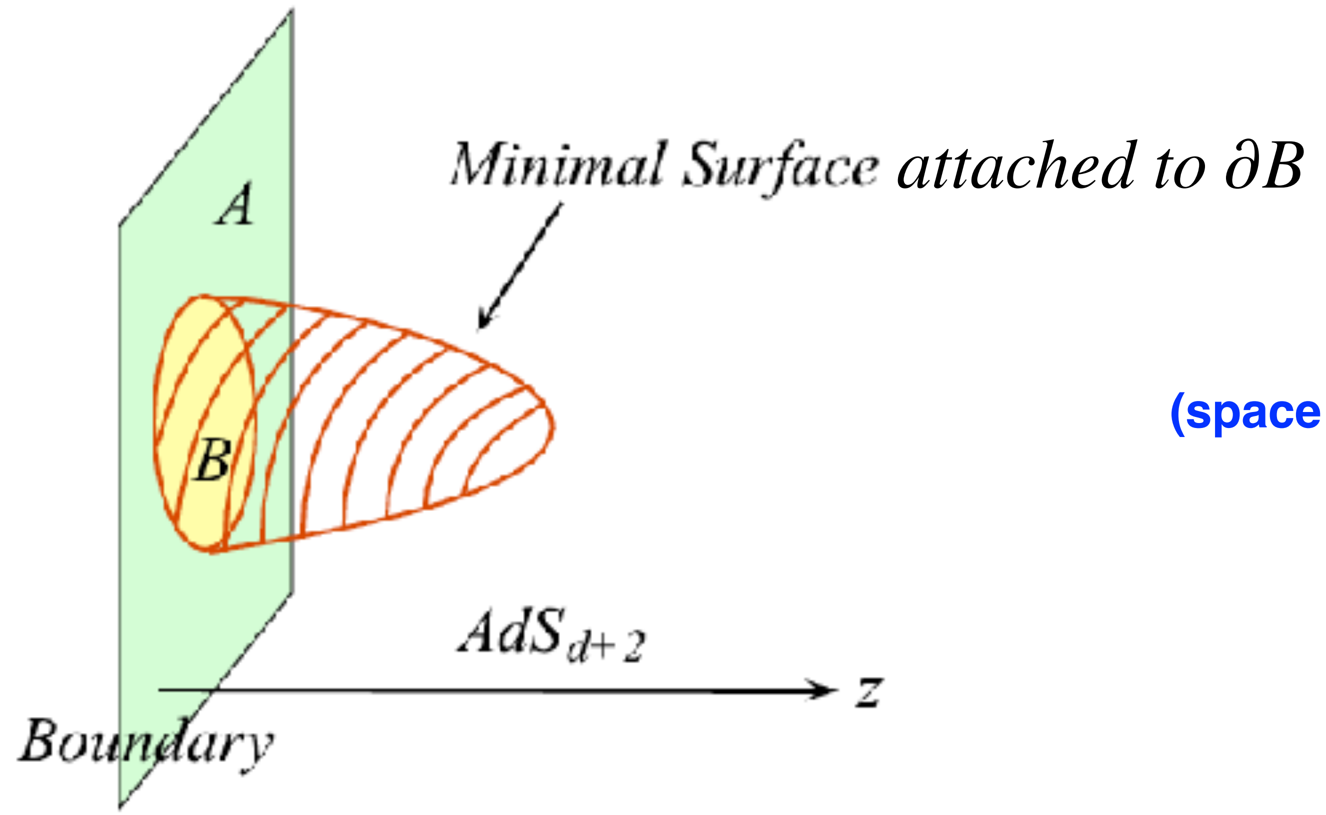
(d+2) dimensional AdS bulk spacetime (Asymptotical AdS), with (d+1) dimensional boundary



Conjecture of AdS/CFT Correspondence: $$Z_{QG_{d+2}} = Z_{CFT_{d+1}}$$

Conjecture of Holographic Entanglement Entropy

As a consequence of AdS/CFT:



(space diagram)

Ryu-Takayanagi Formula:

Boundary Entanglement Entropy = Bulk Minimal Surface Area

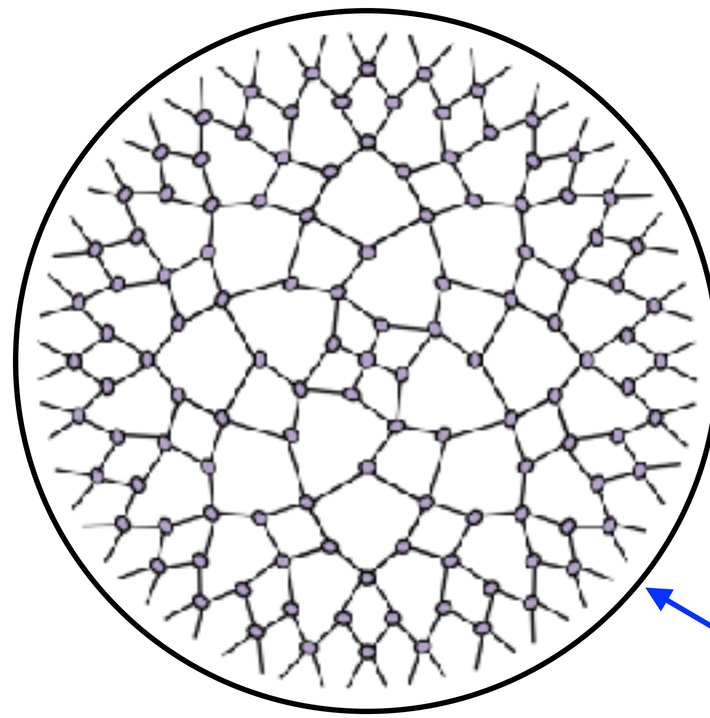
$$S_{EE}(B) = \frac{\mathbf{Ar}_{\min}}{4G_N},$$

The conjecture is also applied to more general bulk geometry

Tensor Network

Idea: realize the AdS/CFT from gapless (d+1) dim quantum system

(space diagram)



CFT_{d+1}

circle: spatial slice of d+1 dim

n-valent node: tensor with n indices $T_{a_1,\cdots,a_n}$

link connecting 2 nodes: contract a pair of indices from 2 tensors

the open legs shows that it is a quantum state in the Hilbert space of CFT_{d+1}

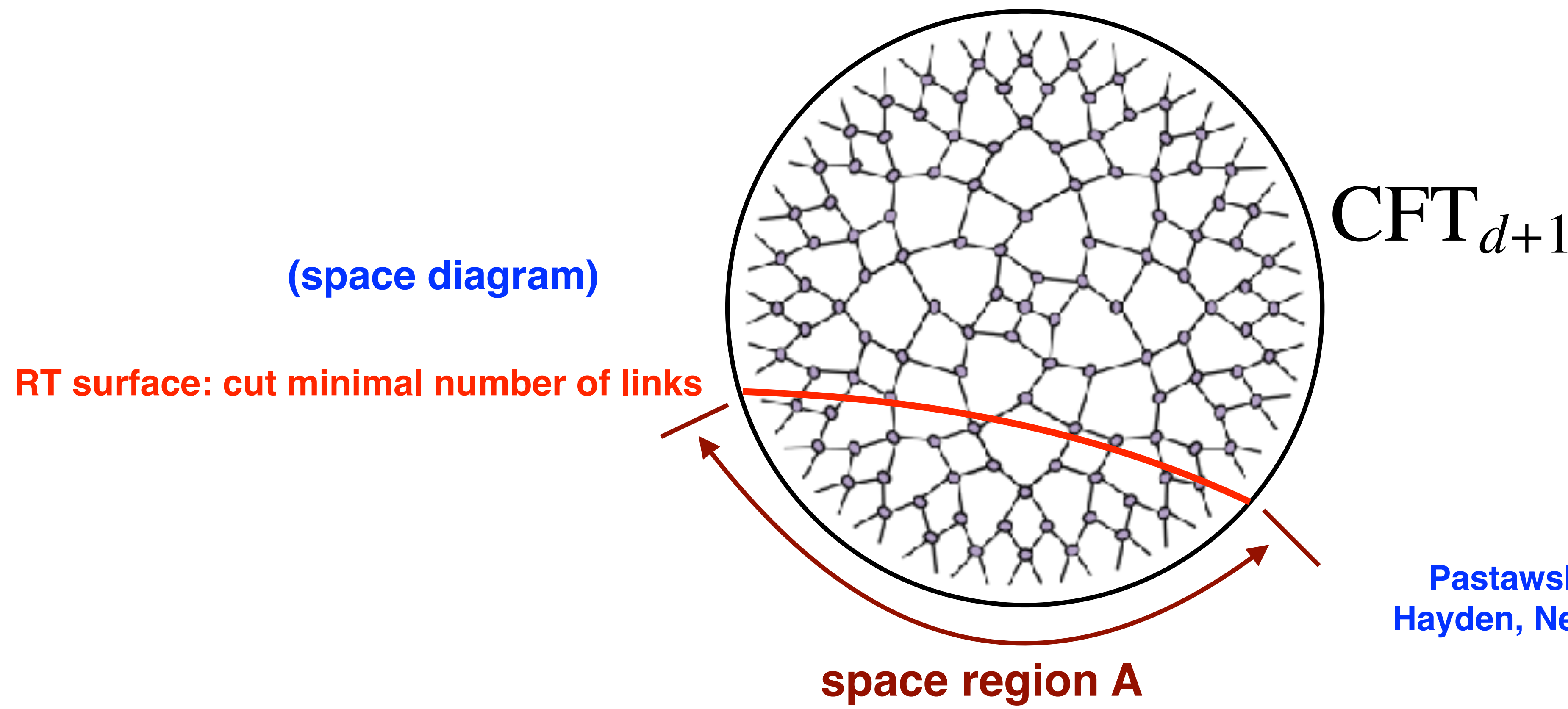
tensor network state: $$|\Psi\rangle = \sum_{\{a_i\}} f_{a_1,a_2,\cdots a_N}|a_1,a_2,\cdots,a_N\rangle = \sum_{\{a_i,\gamma_l\}} T_{\gamma_1\,\gamma_2\,\ldots\,a_i\ldots}|a_1,a_2,\cdots a_N\rangle$$

internal links open legs

- is understood as the ground state of CFT in (d+1) dim, exhibits long-range correlation
- suggests a bulk-boundary duality, and an emergent bulk dimension
- the tensors are bulk DOF, representing the bulk locality

Tensor Network

realize the Ryu-Takayanagi (RT) formula of holographic entanglement entropy



Pastawski, Yoshida, Harlow, Preskill, 2015,
Hayden, Nezami, Qi, Thomas, Water, Yang 2016

Entanglement entropy of tensor network:

Bond dimension: range of tensor indices

$$S_{EE}(A) = (\text{minimal number of cuts}) \cdot \ln D$$

Compare to RT formula $\quad S_{EE}(A) = \dfrac{\mathbf{Ar}_{\min}}{4G_N}, \quad \Longrightarrow \quad \text{minimal number of cuts} \sim \mathbf{Ar}_{\min}$

How does tensor network entropy relate to bulk quantum gravity (bulk geometry)?

In order to derive RT formula from tensor network entanglement entropy, it requires the relation between cut and bulk area. This relation requires to understand how tensor network encodes the bulk geometry.

A Candidate: Loop Quantum Gravity inside the Bulk

We consider the (3+1) dim bulk with LQG and (2+1) dim boundary

The (kinematical) Hilbert space of LQG is a quantization of the phase space of General Relativity, which is formulated as a gauge theory

Holst action of GR $$S_{GR} = \frac{1}{8\pi G_N}\int_{M_4} e^I \wedge e^J \wedge \left(*F + \frac{1}{\gamma}F\right)_{IJ}$$ **variational principle gives *Einstein equation***

The Hamiltonian analysis gives the canonical variables and Poisson brackets

canonical variables: **Position: Wilson lines (holonomy) along a curve** $h_c \in \mathrm{SU}(2)$

Momentum: Geometrical flux across a 2d surface $E^a(S),\ a = 1, 2, 3$ **(to be defined)**

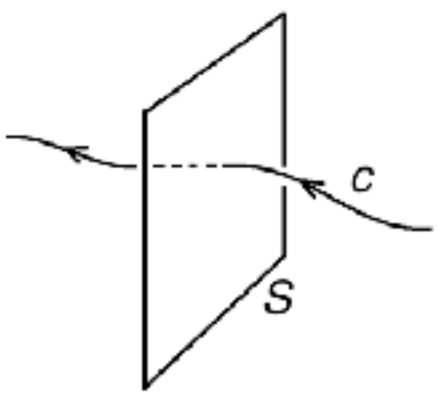
Poisson brackets:

$$\{h_c, h_{c'}\} = 0$$

$$\{h_c, E^a(S)\} = 8\pi G_N \gamma \frac{i\sigma^a}{2} h_c$$

$$\left\{E^a(S), E^b(S)\right\} = 8\pi G_N \gamma \varepsilon^{abc} E^c(S)$$

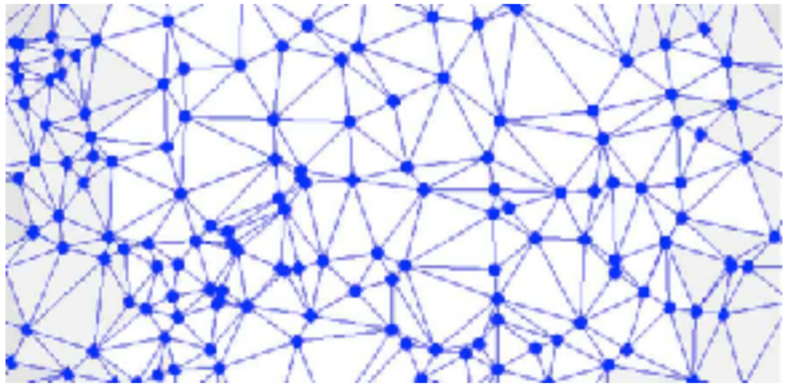
γ is Barbero-Immirzi parameter



Wave functions (LQG states):

functions of Wilson lines $\psi(h_1, \cdots, h_n)$

$h_1, \ldots, h_n$ **give a network of Wilson lines**

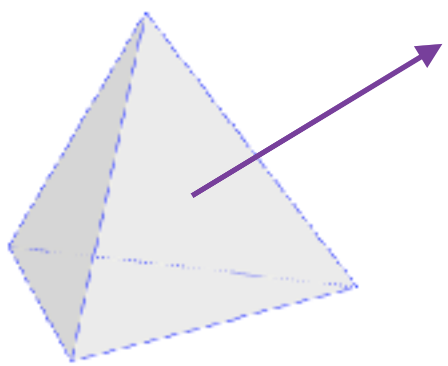


Geometrical Flux and Quantum Polyhedron

Gravity = Geometry $\longrightarrow$ **LQG states present a quantization of the geometry of 3d space**

3d geometry can be triangulated by a (large) number of geometrical tetrahedra

Geometrical Flux:



$\vec{E}(S_i)$ **oriented area vector** $\quad |\vec{E}(S_i)| = \mathbf{Ar}(S_i)$

$\vec{E}(S_i)$ **normal to** S_i

Closure constraint: $\displaystyle\sum_{i=1}^{4} \vec{E}(S_i) = 0$

Quantization and non-commutativity: classical vectors are promoted to be quantum operators

$$\left[\hat{E}^a(S_i), \hat{E}^b(S_j)\right] = i\ell_P^2\gamma\, \varepsilon^{abc}\, \delta_{ij}\, \hat{E}^c(S_i) \qquad i = 1, \cdots, 4 \qquad (\ell_P^2 = 8\pi G_N \hbar)$$

- **Different faces correspond to independent DOF**
- **For a given face, the same operator algebra as angular momentum** $\left[\hat{J}^a, \hat{J}^b\right] = i\,\varepsilon^{abc}\,\hat{J}^c$

The states are in tensor product of SU(2) irreps: $V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \otimes V_{j_4} \qquad \hat{E}^a(S_i) = i\ell_P^2\gamma\, \hat{J}^a_{(i)}$

subject to the quantum closure constraint: $\displaystyle\sum_{i=1}^{4} \hat{J}^a_{(i)}\, |\psi\rangle = 0$

The solutions are the invariant tensors in the invariant subspace $\mathrm{Inv}_{\mathrm{SU}(2)}\left(V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \otimes V_{j_4}\right)$

The invariant subspace is the Hilbert space of a quantum tetrahedron (polyhedron)

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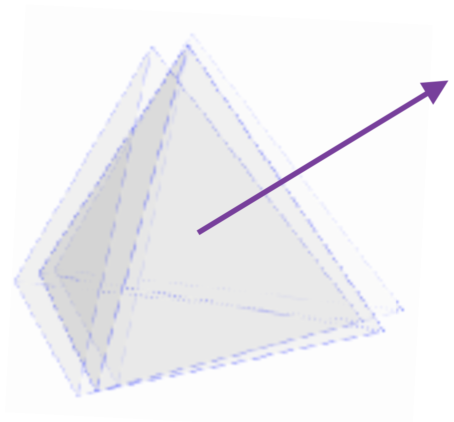

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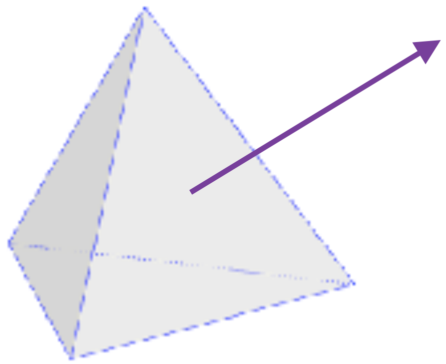


$\vec{E}(S_i)$ **oriented area vector** $\quad |\vec{E}(S_i)| = \mathbf{Ar}(S_i)$

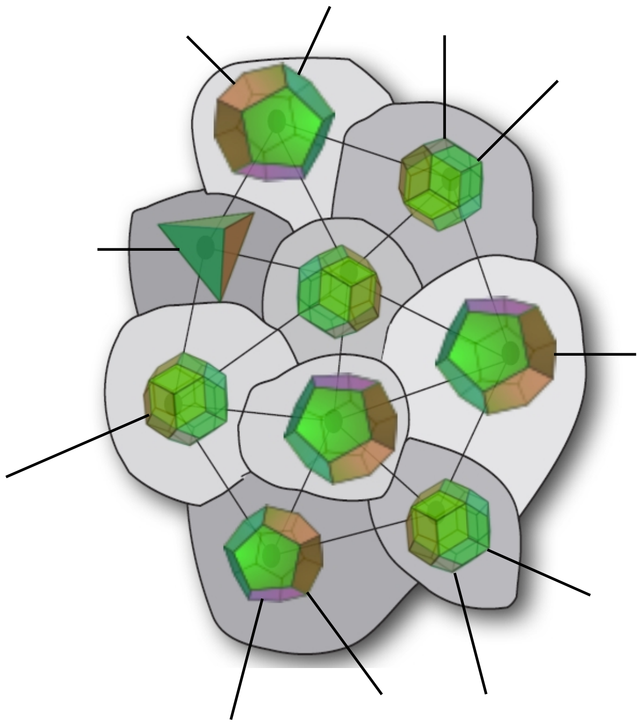
$$\hat{\mathbf{Ar}}(S) = \sqrt{\hat{E}^a(S)\hat{E}^a(S)} = \ell_P^2 \gamma \sqrt{\hat{J}^a \hat{J}^a} = \ell_P^2 \gamma \sqrt{j(j+1)}$$

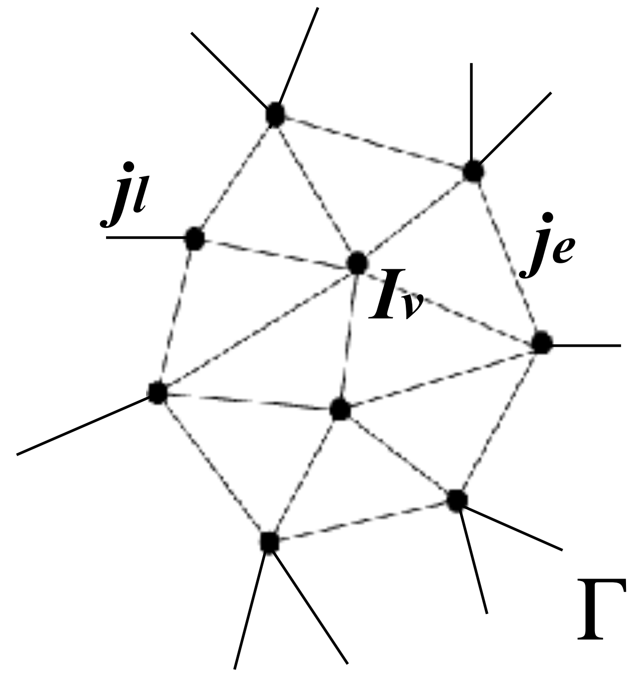
In LQG, the quantum area is fundamentally discrete at Planck scale, j is the area quantum number

Gluing of polyhedra and larger quantum geometry:

each edge carries a quantum area of Planck scale: Spin j_e j_l

each vertex carries a quantum chunk of space of Planck scale: Invariant tensor I_v





$$= \otimes_v |I_v^{\vec{j}}\rangle \otimes_l |j_l, m_l\rangle$$

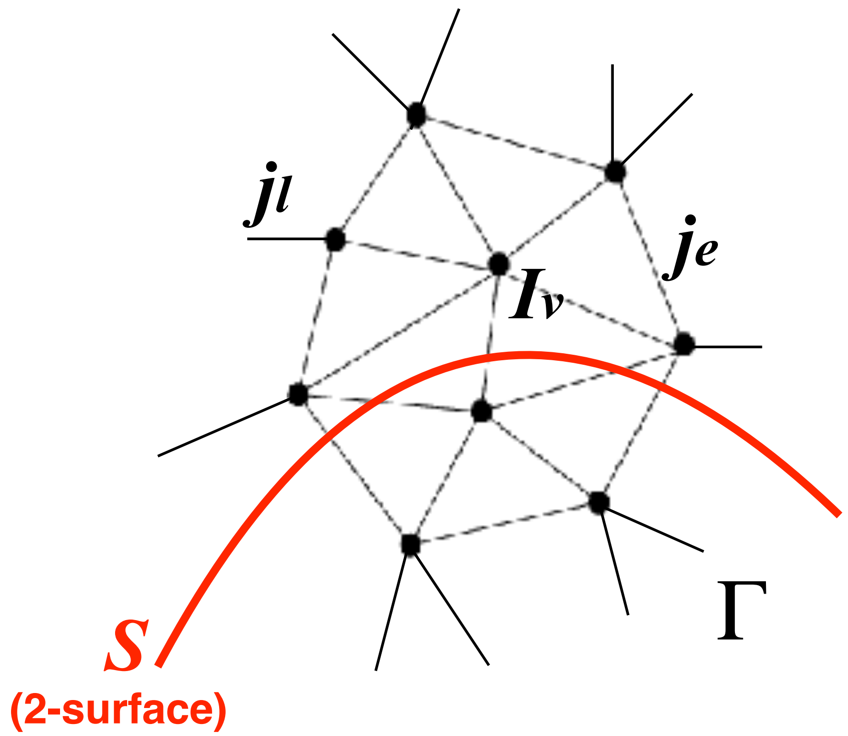
invariant tensor at vertices

spin states at dangling edges

orthonormal basis in LQG Hilbert space

"Spin-Network State" $\left|\Gamma, \vec{j}, \vec{I}, \vec{m}\right\rangle$ **The basis state of bulk quantum geometry**

Spin-network state is the eigenstate of area operator



eigenvalue $$\mathbf{Ar}(S) = \ell_P^2 \gamma \sum_{\text{cuts}} \sqrt{j_e(j_e+1)}$$

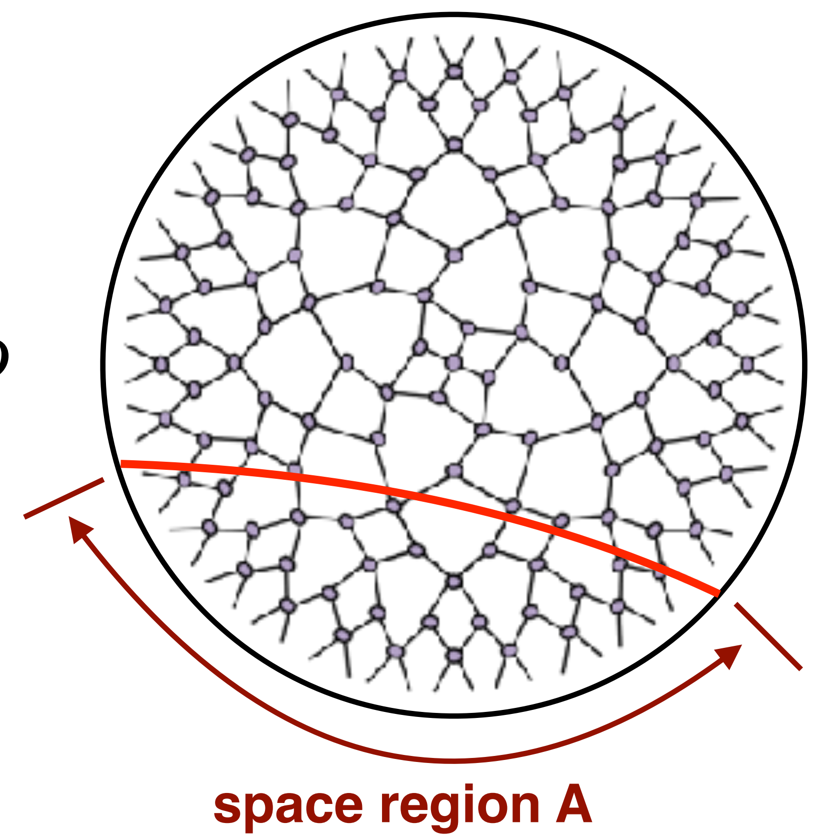
Now we relate the surface area to the number of cuts, weighted by the quantum area carried by the cut edge.

The picture is similar to

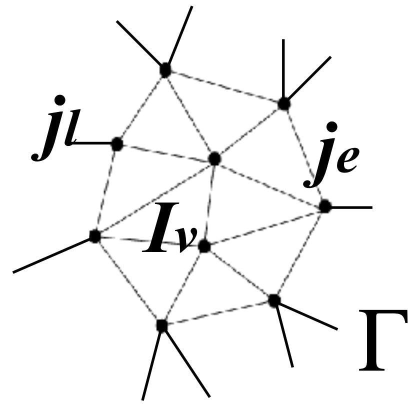
$$S_{EE}(A) = (\text{minimal number of cuts}) \cdot \ln D$$

V.S.

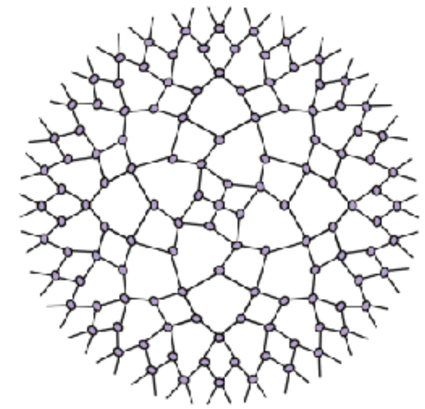
$$S_{EE}(A) = \frac{\mathbf{Ar}_{\min}}{4G_N},$$



What is the relation between Spin-Network and Tensor Network?



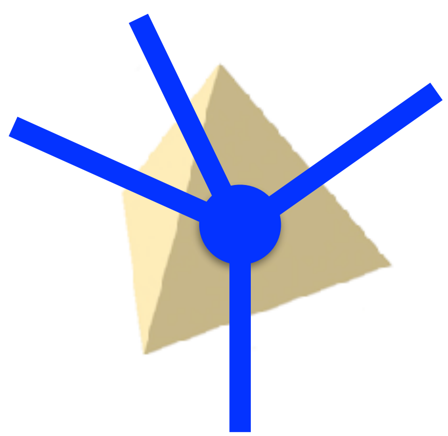
v.s.



The tensor network is emergent from spin-network via coarse graining.
The tensor network is an effective theory of spin-network at larger scale.

A tensor network vertex = a large number of spin-network vertices

zoom out
to a larger scale



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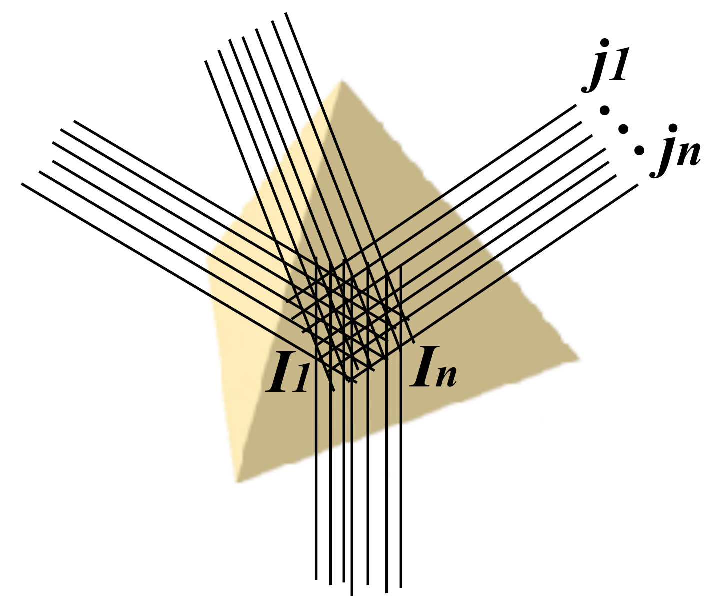
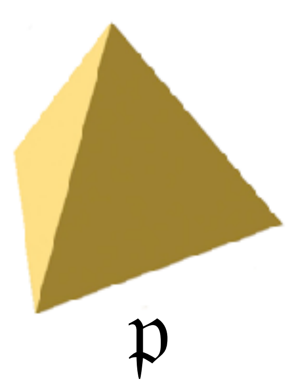


zoom in
Planck scale

Equivalently, A semiclassical polyhedron = a large number of Planck size quantum polyhedron

zoom out
to a larger scale



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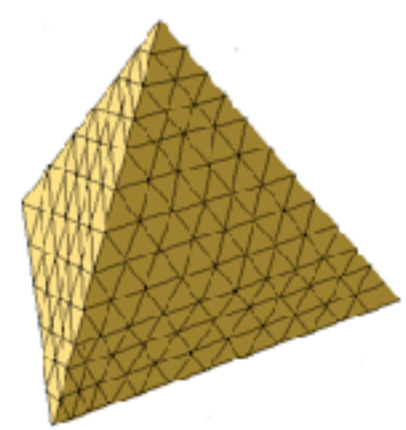
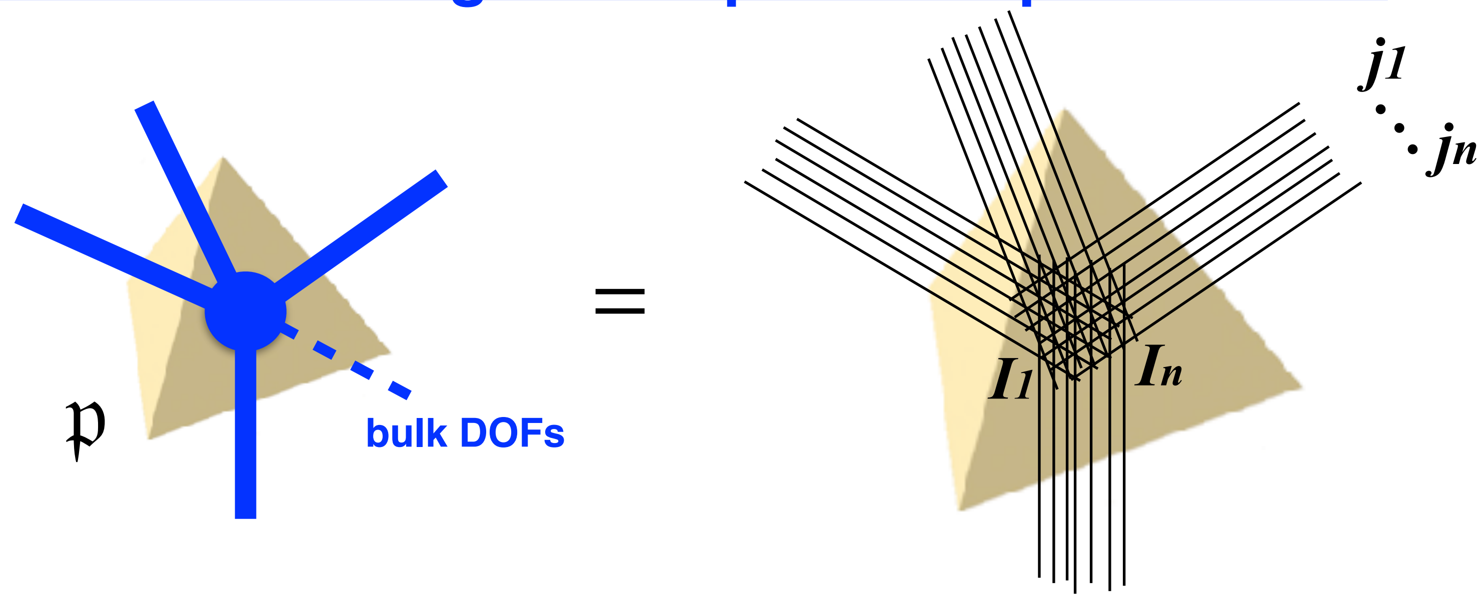


zoom in
Planck scale

Coarse-Graining Prescription of Spin-Network



An elementary tensor: $$|\mathcal{V}_\mathfrak{p}\rangle = \sum_{\mu_f,\xi_\mathfrak{p}} \mathcal{V}_{\mu_f,\xi_\mathfrak{p}} |\xi_\mathfrak{p}^{\vec{\mu}}\rangle \otimes |\mu_f\rangle = \sum_{\Gamma_\mathfrak{p},\vec{j},\vec{I},\vec{n}} \mathcal{V}_{\Gamma_\mathfrak{p},\vec{j},\vec{I},\vec{n}} \bigotimes_{v\in V(\Gamma_\mathfrak{p})} |I_v^{\vec{j}}\rangle \bigotimes_{\text{boundary } l} \langle j_l, n_l|$$

$|\xi_\mathfrak{p}\rangle$ **and** $|\mu_f\rangle$ **label the bulk and boundary states**

We coarse-grain the spin-network microstates, since we are interested the semiclassical physics but not interested in the Planck scale quantum details.

Practically, we random sample the coefficients, i.e. $|\mathcal{V}_\mathfrak{p}\rangle$ **is a random state**

The tensor has a "bulk leg"

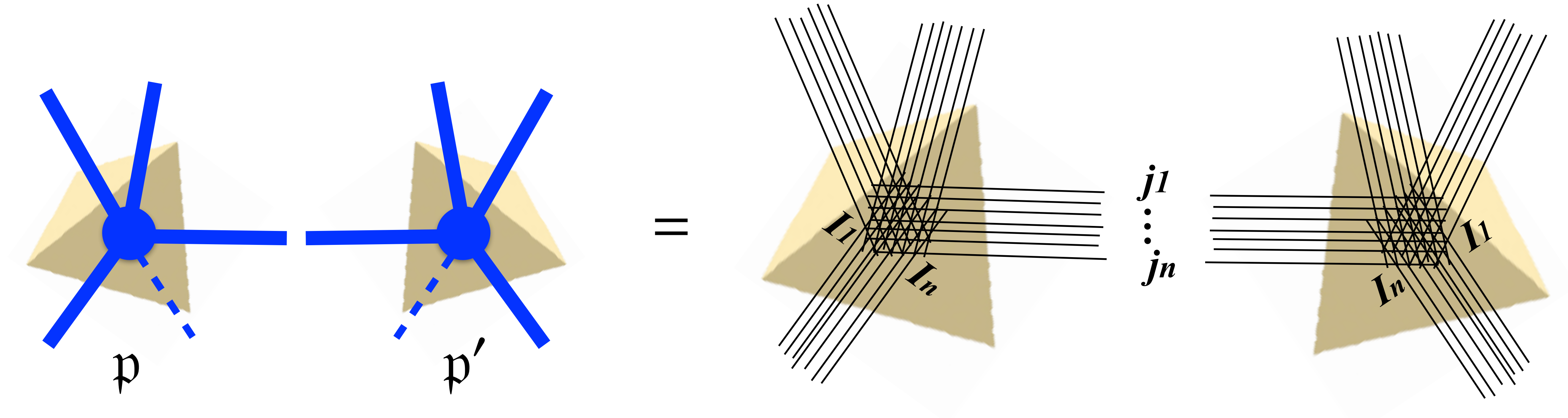
The quantum state of this type is an Exact Holographic Mapping firstly proposed in [Qi 2010]

On the other hand, $|\mathcal{V}_\mathfrak{p}\rangle \in \mathcal{H}_b(\mathfrak{p}) \otimes \mathcal{H}_\partial(\mathfrak{p})$ **and exhibits certain entanglement between bulk and boundary DOF.**

Equivalently, $|\mathcal{V}_\mathfrak{p}\rangle : \mathcal{H}_b(\mathfrak{p}) \to \mathcal{H}_\partial(\mathfrak{p})$ **maps bulk states to boundary states.**

$$|T(\mathfrak{p})\rangle = \langle \phi_b(\mathfrak{p})|\mathcal{V}_\mathfrak{p}\rangle = \sum_{\mu_1,\cdots,\mu_M} T(\mathfrak{p})_{\mu_1,\cdots,\mu_M} |\mu_1\rangle \otimes \cdots \otimes |\mu_M\rangle$$ **tensor to build a CFT tensor network**

Gluing of Semiclassical Polyhedron and Large Semiclassical Geometry



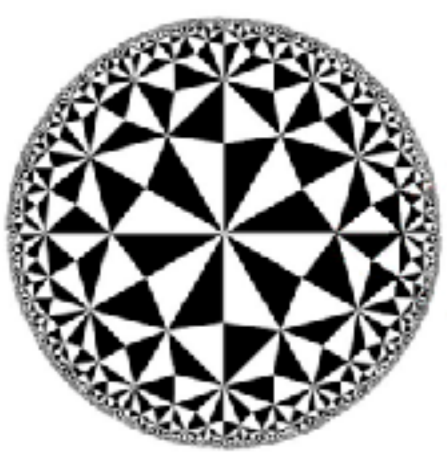
At each gluing interface f: $\quad |f\rangle = \sum_{\mu_f} |\mu_f\rangle_L \otimes |\mu_f\rangle_R$ **a maximally entangled state**

Polyhedra gluing = Projected Entangled Pair State (PEPS): $\quad \left|\mathfrak{p}\cup\mathfrak{p}'\right\rangle = \langle f\,|\,\mathcal{V}_\mathfrak{p}\otimes\mathcal{V}_{\mathfrak{p}'}\rangle$

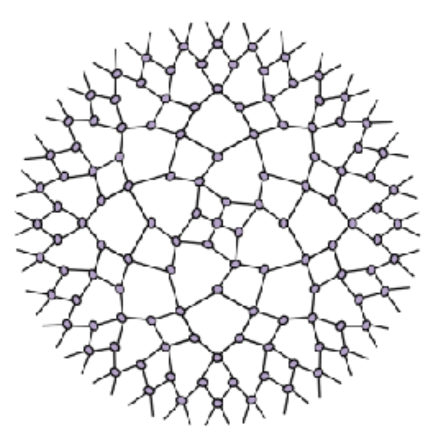
Large semiclassical geometry of the entire space Σ

$|\Sigma\rangle = \otimes_f\langle f|\ \otimes_\mathfrak{p}|\mathcal{V}_\mathfrak{p}\rangle$ **LQG state, Exact Holographic Mapping, and Random Tensor Network**

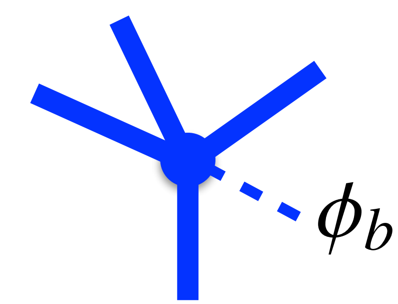
similar structure as proposed in [Hayden, Nezami, Qi, Thomas, Water, Yang 2016]. It's now derived from QG.



Σ



Recall $|T(\mathfrak{p})\rangle = \langle\phi_b(\mathfrak{p})|\mathcal{V}_\mathfrak{p}\rangle$ **is a tensor**



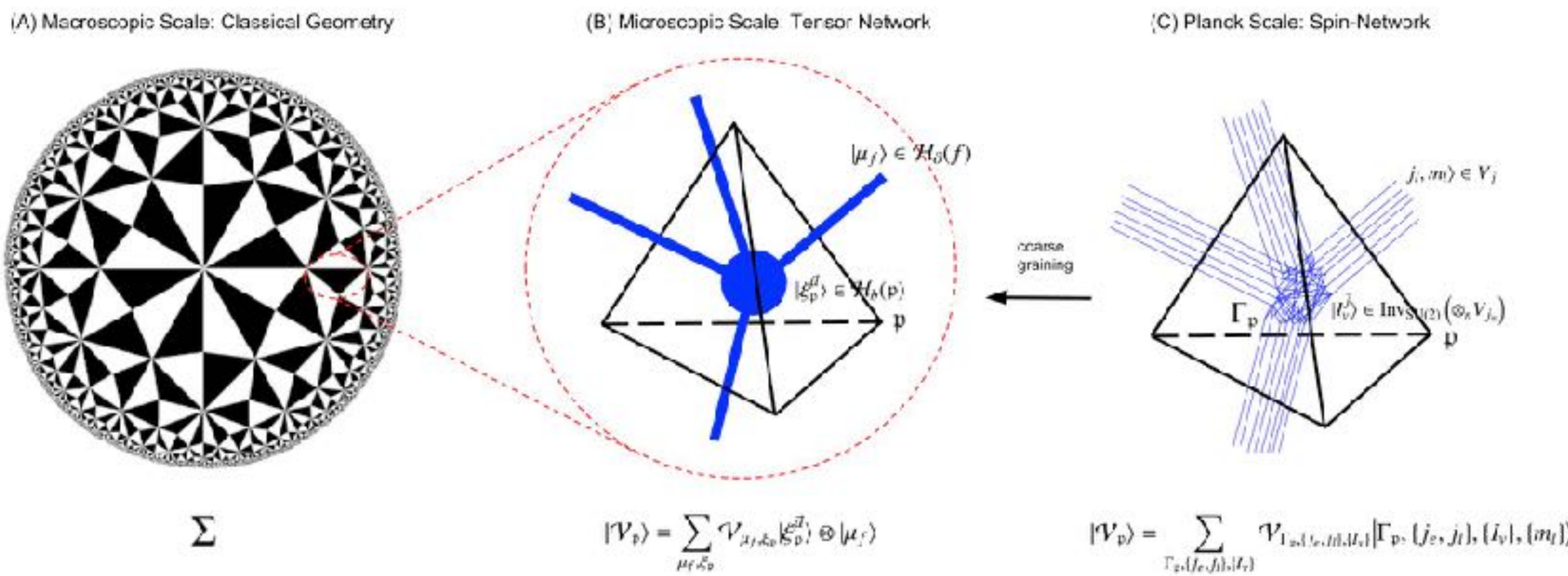
$\langle\otimes_\mathfrak{p}\phi_b(\mathfrak{p})\,|\,\Sigma\,\rangle$ **is a Tensor Network representing boundary CFT ground state**

The tensor network structure emerges from LQG and quantum geometry at Planck scale

Scales in Quantum Gravity and Semiclassical Approximation

Different physics at different scales:

IR ⟵ UV



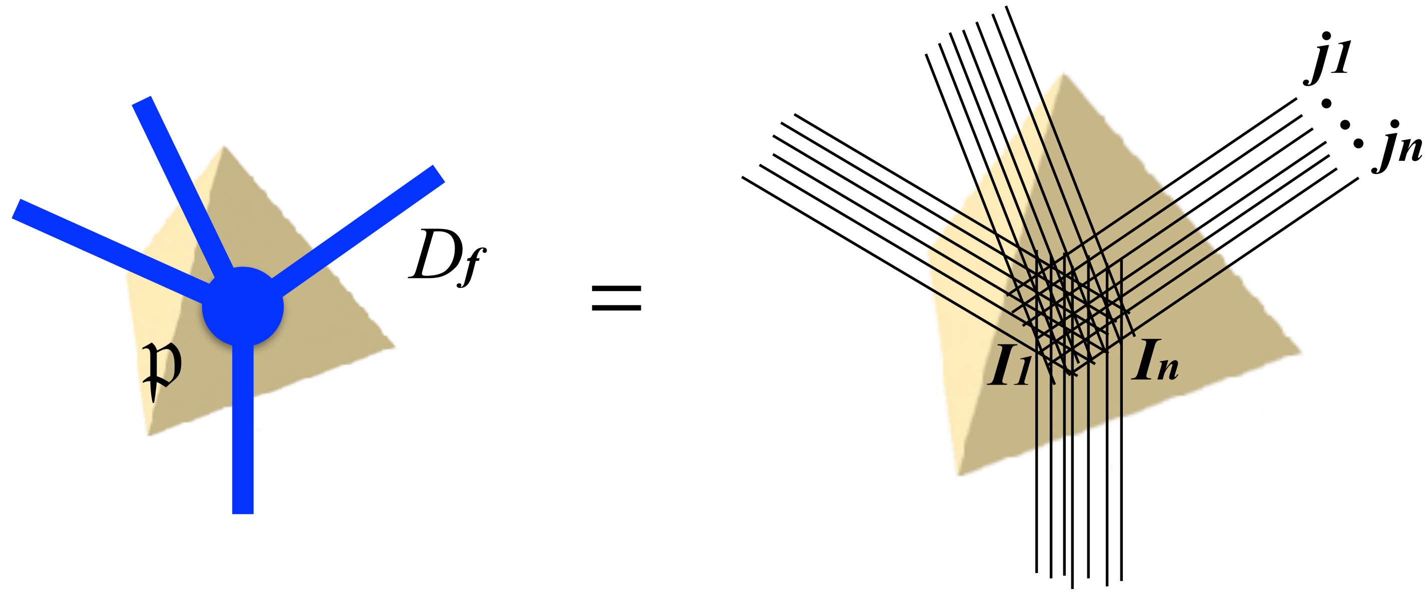
L **curvature radius** $\mathbf{Ar}_f$ **semiclassical face area** ℓ_P **Planck scale**

Semiclassical approximation: we zoom out such that the geometry is approximately smooth

$$L^2 \gg \mathbf{Ar}_f \gg \ell_P^2$$

The Ryu-Takayanagi formula of HEE will be reproduced in this regime

Area Law of Bond Dimension

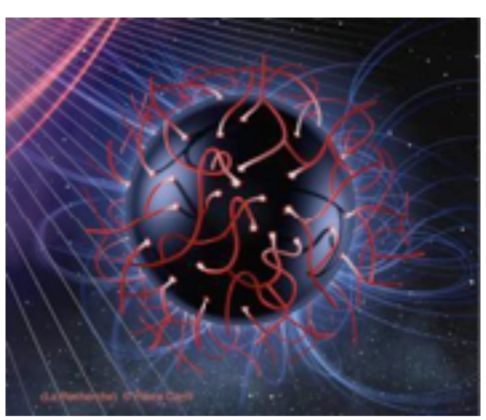


Each face area $\mathbf{Ar}_f = \ell_P^2 \gamma \sum_{l, l \cap f \neq \emptyset} \sqrt{j_l(j_l+1)}$ summing over all spin-network edges intersecting f

Fixing the face area $\mathbf{Ar}_f$ the bond dimension D_f at each tensor leg is the number of microstates on the spin-network edges.

The number of microstates is given by counting all possible spin configurations $\{j_l\}$ with the total face area being fixed

The same type of microstate counting has been well-studied in the context of LQG black hole entropy



quantum black hole horizon

Gosh, Perez 2011
Gosh, Perez 2012
Barbero, Perez 2015

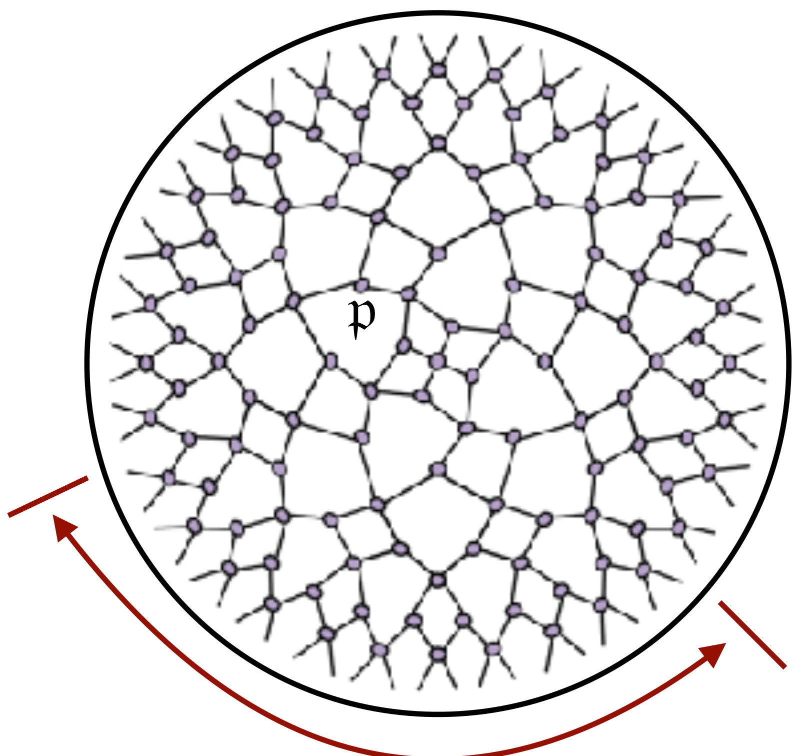
$$D_f \simeq \exp\left[\frac{\mathbf{Ar}_f}{4G_N}\right]$$

IR value $\frac{1}{4G_N} = \frac{\beta_0}{8\pi\gamma\ell_P^2}, \quad 2\pi\beta_0 \simeq 0.274...$

large bond dimension $D_f \gg 1$ in the semiclassical regime $\mathbf{Ar}_f \gg \ell_P^2$

Derivation of Ryu-Takayanagi Formula

Tensor network emerging from LQG $\langle \Phi_b | \Sigma \rangle = |\Psi_\partial\rangle$ $\rho = |\Psi_\partial\rangle\langle\Psi_\partial|$



swap trick: $S_2(A) = -\ln \frac{\mathrm{tr}\rho_A^2}{(\mathrm{tr}\rho_A)^2} = -\ln \frac{\mathrm{tr}\left[(\rho\otimes\rho)\mathcal{F}_A\right]}{\mathrm{tr}\left[\rho\otimes\rho\right]}$

swap operator:

$$\mathcal{F}_A\Big(|\mu_f\rangle_A|\mu_f\rangle_{\bar{A}}\Big)\otimes\Big(|\mu'_f\rangle_A|\mu'_f\rangle_{\bar{A}}\Big) = \Big(|\mu'_f\rangle_A|\mu_f\rangle_{\bar{A}}\Big)\otimes\Big(|\mu_f\rangle_A|\mu'_f\rangle_{\bar{A}}\Big)$$

Recall the exact holographic mapping:

$$|\Sigma\rangle = \otimes_f\langle f| \otimes_{\mathfrak{p}} |\mathcal{V}_{\mathfrak{p}}\rangle$$

$|\mathcal{V}_{\mathfrak{p}}\rangle$ **is a random state.**

Computing the random average of S_2 involves random averaging 4 copies of $|\mathcal{V}_{\mathfrak{p}}\rangle$ at each node $\mathfrak{p}$

Haar random average: $\int \mathrm{d}U\ \big(|\mathcal{V}_{\mathfrak{p}}\rangle\langle\mathcal{V}_{\mathfrak{p}}|\otimes|\mathcal{V}_{\mathfrak{p}}\rangle\langle\mathcal{V}_{\mathfrak{p}}|\big) \propto I_{\mathfrak{p}} + \mathcal{F}_{\mathfrak{p}}$ **Harrow 2011**

swap operator: $\mathcal{F}_{\mathfrak{p}} : \mathcal{H}_{\mathfrak{p}}\otimes\mathcal{H}_{\mathfrak{p}} \to \mathcal{H}_{\mathfrak{p}}\otimes\mathcal{H}_{\mathfrak{p}}$

When we insert the result into S_2 and expand, each term of the expansion corresponds to a choice of $I_{\mathfrak{p}}$ or $\mathcal{F}_{\mathfrak{p}}$ at each node.

Define Ising variable $s_{\mathfrak{p}} = \pm 1$ to denote the choice of $I_{\mathfrak{p}}$ or $\mathcal{F}_{\mathfrak{p}}$ at each node.

It relates S_2 to a partition function of Ising model. **Hayden, Nezami, Qi, Thomas, Water, Yang 2016**

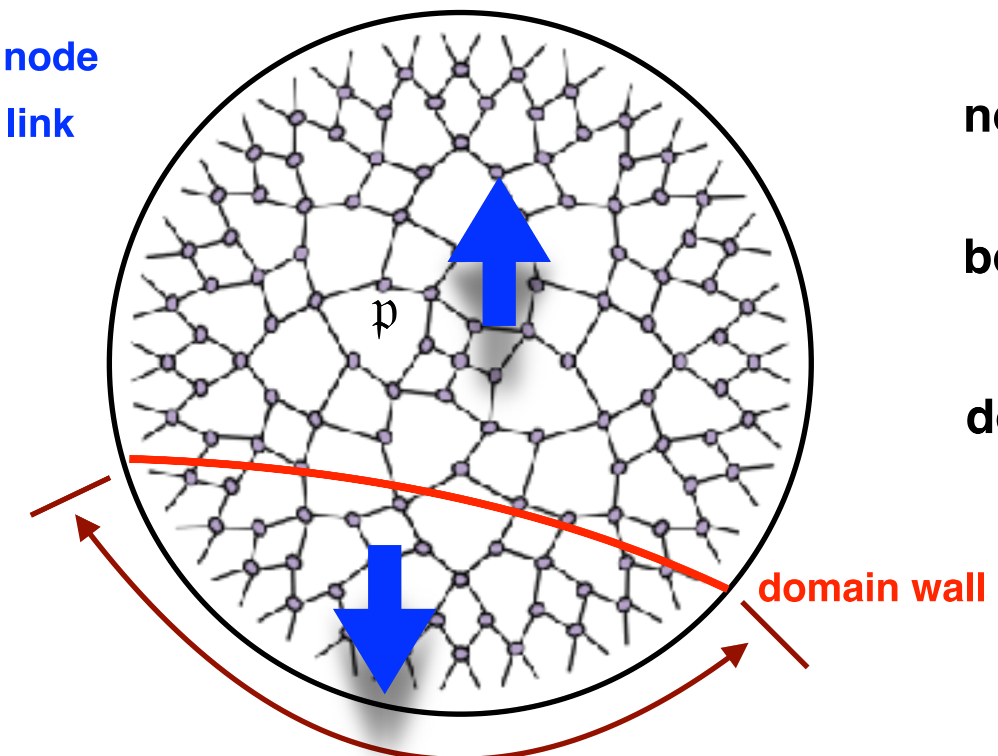
From Ising model to Nambu-Goto Action

S_2 **relates to a partition function of Ising model:** $e^{-S_2(A)} = \sum_{\{s_\mathfrak{p}=\pm 1\}} e^{-\mathscr{A}[s_\mathfrak{p}]}$

Ising action: $\mathscr{A}[s_\mathfrak{p}] = -\sum_{f \text{ bulk}} \frac{1}{2} \ln D_f (s_\mathfrak{p} s_{\mathfrak{p}'} - 1) - \sum_{f \text{ boundary}} \frac{1}{2} \ln D_f (h_\mathfrak{p} s_\mathfrak{p} - 1) + \text{const.}$

$\mathfrak{p}$ **node**

f **link**



nonconstant effective coupling $\ln D_f \simeq \frac{\mathbf{Ar}_f}{4G_N} \gg 1$

boundary condition: $h_\mathfrak{p} = 1$ $(h_\mathfrak{p} = -1)$ as $\mathfrak{p}$ close to $\bar{A}$ ($\mathfrak{p}$ close to A)

dominant Ising configurations:

A single domain wall $\mathcal{S}$ separating spin up and down.
The spin-down domain attaches to the boundary region A.

S_2 **reduces to a sum over the domain wall configurations (a functional integral of Nambu-Goto action)**

$$e^{-S_2(A)} \simeq \sum_{\mathcal{S}} e^{-\frac{1}{4G_N} \sum_{f \subset \mathcal{S}} \mathbf{Ar}_f} \simeq \int [D\mathcal{S}]\, e^{-\frac{1}{4G_N} \mathbf{Ar}_\mathcal{S}}$$

because of $L^2 \gg \mathbf{Ar}_f$

Because of $\mathbf{Ar}_f \gg \ell_P^2$ **the variational principle gives**

$$S_2(A) \simeq \frac{\mathbf{Ar}_{\min}}{4G_N}$$

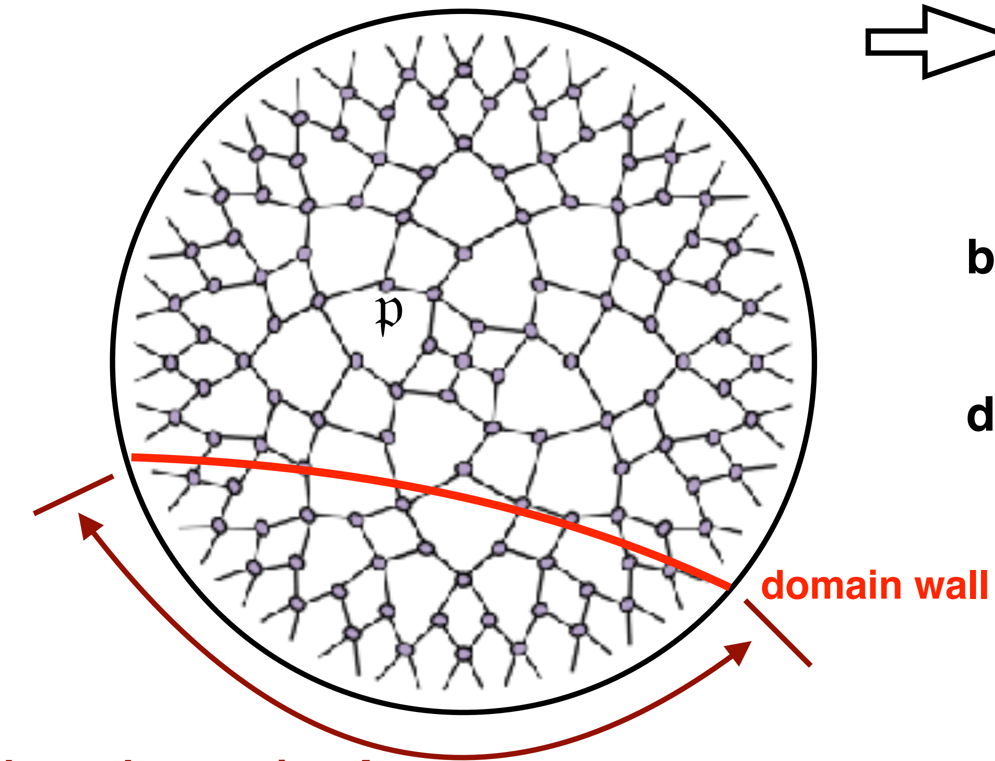
(RT formula of second Renyi entropy)

Higher Renyi Entropies

S_n can be computed in a similar manner:

Haar random average: $$\int \mathrm{d}U \, (|\mathcal{V}_\mathfrak{p}\rangle\langle\mathcal{V}_\mathfrak{p}|)^{\otimes n} \propto \sum_{g_\mathfrak{p} \in \mathrm{Sym}_n} g_\mathfrak{p}$$

permutation operator: $g_\mathfrak{p} : \mathcal{H}_\mathfrak{p}^{\otimes n} \to \mathcal{H}_\mathfrak{p}^{\otimes n}$ n! choices of permutations at each nodes



$\Rightarrow$ Sym_n-model over the tensor network lattice.

boundary condition: $h_\mathfrak{p} = 1$ ($h_\mathfrak{p}$ = cyclic) as $\mathfrak{p}$ close to $\bar{A}$ ($\mathfrak{p}$ close to A)

dominant Sym configurations:

A single domain wall $\mathcal{S}$ separating identity and cyclic permutation.
The cyclic permutation domain attaches to the boundary region A.

In the semiclassical regime: $$L^2 \gg \mathbf{Ar}_f \gg \ell_P^2$$

$$e^{-(n-1)S_n(A)} \simeq \int [D\mathcal{S}] \, e^{-(n-1)\frac{1}{4G_N}\mathbf{Ar}_\mathcal{S}} \qquad S_n(A) \simeq \frac{\mathbf{Ar}_{\min}}{4G_N}$$ (RT formula of higher Renyi entropy)

Von Neumann entropy: $$S_{EE}(A) \simeq \frac{\mathbf{Ar}_{\min}}{4G_N}$$

Conclusion and Outlook

We propose that the tensor network is an effective theory at larger scale emergent from LQG at Planck Scale.

The tensor network is obtained via coarse graining from Planck-scale quantum geometry (spin-networks).

The emergent tensor network is a LQG state, an Exact Holographic Mapping, and a Random Tensor Network.

The tensor network presents correctly Ryu-Takayanagi formula of Holographic Entanglement Entropy .

$$S_{EE}(A) \simeq \frac{\mathbf{Ar}_{\min}}{4G_N}$$

The result opens the window to understand holography from the 1st principle in the theory of quantum gravity.

- reproduce AdS/CFT dictionary from Exact Holographic Mapping
- input from LQG dynamics and the dynamics of Tensor Network
- holographic formulation of Black Holes and Information Paradox
- holographic scambling
- ……

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The end

Thanks for your attention !