Quantization of constantly curved tetrahedron

Chen-Hung Hsiao

Florida Atlantic University

In Collaboration with Muxin Han, Qiaoyin Pan
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Outline

- Motivation
- Strategies and results
- Quantization of constantly curved tetrahedron
  - Solution space of quantum curved tetrahedron
  - Area spectrum
  - Coherent state
- Conclusions and Outlooks

- **Motivation**
- **Strategies and results**
- Quantization of constantly curved tetrahedron
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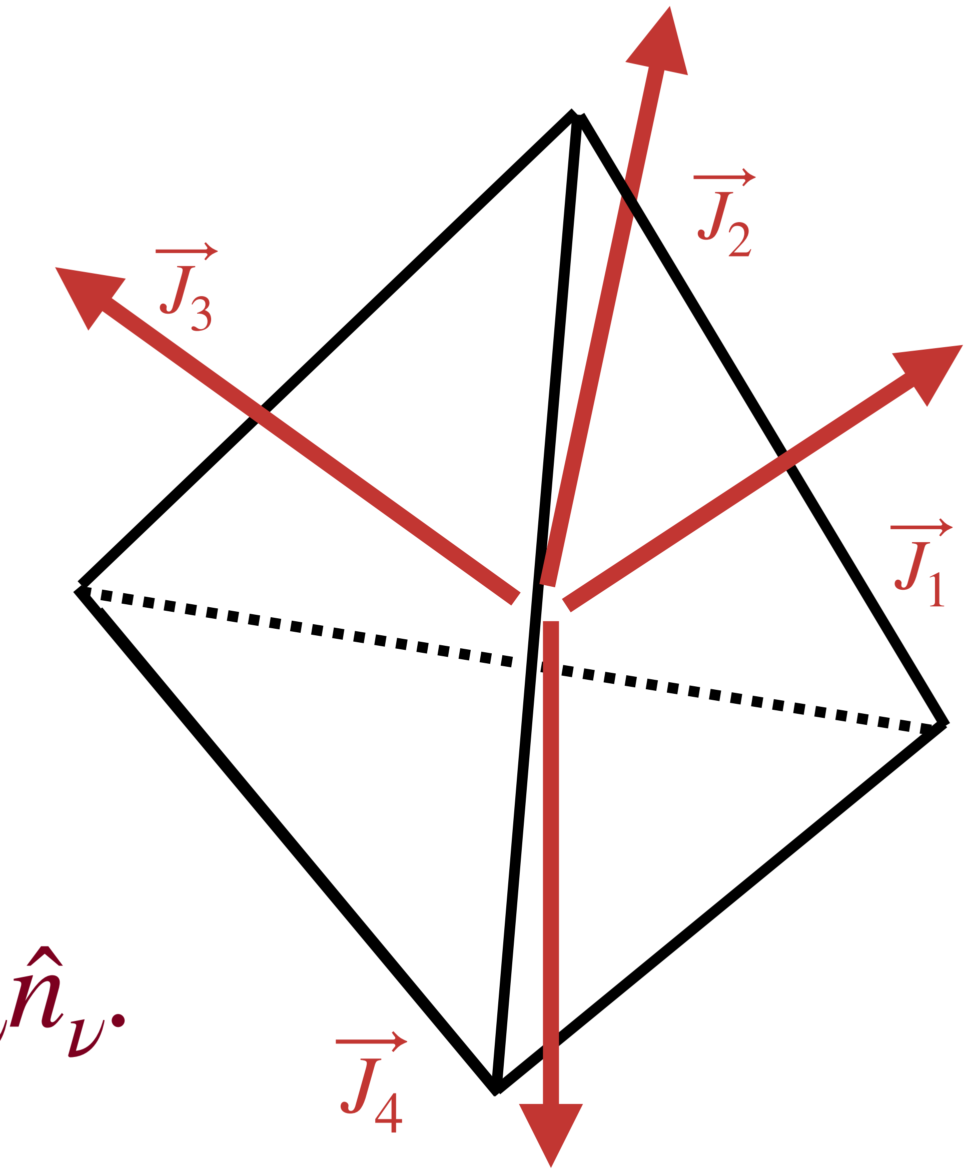
  - Area spectrum
  - Coherent state
- Conclusions and Outlooks

Motivation

- 3+1 D Lorentzian spinfoam models for $\Lambda = 0$: EPRL model .

- The building block of the boundary geometry is the flat tetrahedron; e.g., the boundary of a four-simplex consists of five flat tetrahedra.



- The closure condition of flat tetrahedron $\vec{J}_1 + \vec{J}_2 + \vec{J}_3 + \vec{J}_4 = \vec{0}, \quad \vec{J}_\nu = a_\nu \hat{n}_\nu$.

- When we quantize the tetrahedron, the Hilbert space is the 4-valent $SU(2)$ intertwiner space, the solution space of the quantum flat closure condition $\hat{\vec{J}}_1 + \hat{\vec{J}}_2 + \hat{\vec{J}}_3 + \hat{\vec{J}}_4 = 0$.

- For $\Lambda = 0$, the area spectrum of the tetrahedron's face is $\gamma \ell_p^2 \sqrt{J(J+1)}\ , \quad J \in \mathbb{N}_0/2$.

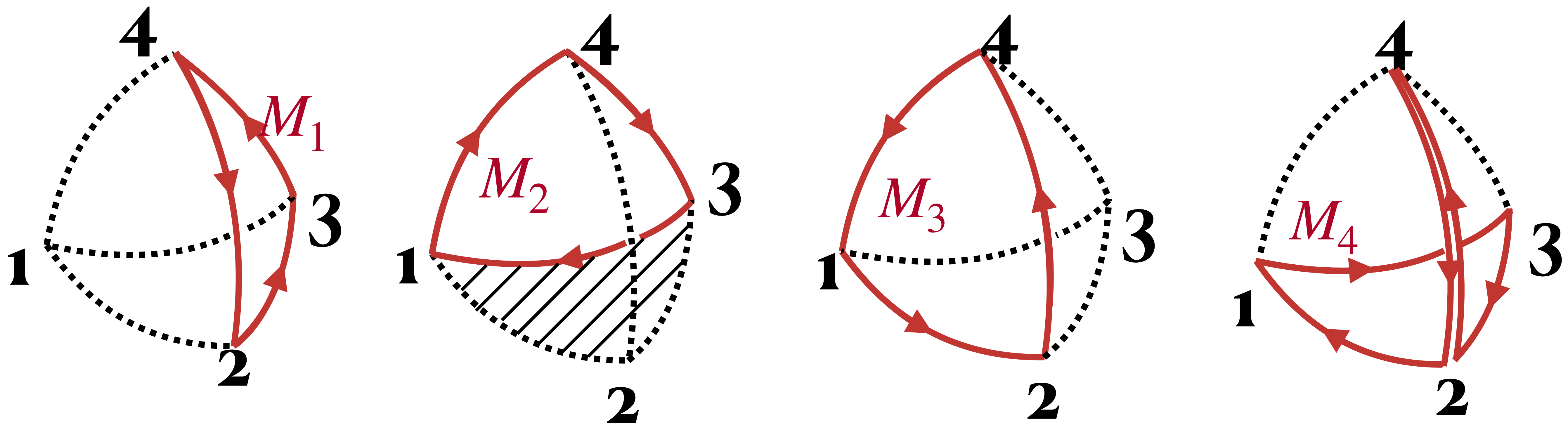
Motivation

- 3+1 D Lorentzian Spinfoam model with $\Lambda \neq 0$

  [H. Haggard, M. Han, W. Kamin´ski ,A. Riello '15; '19; M. Han '21]

- The building block of the boundary geometry is the curved tetrahedron. e.g. the boundary of the homogeneously curved four-simplex (involved in the vertex amplitude) consists of five constantly curved tetrahedra.

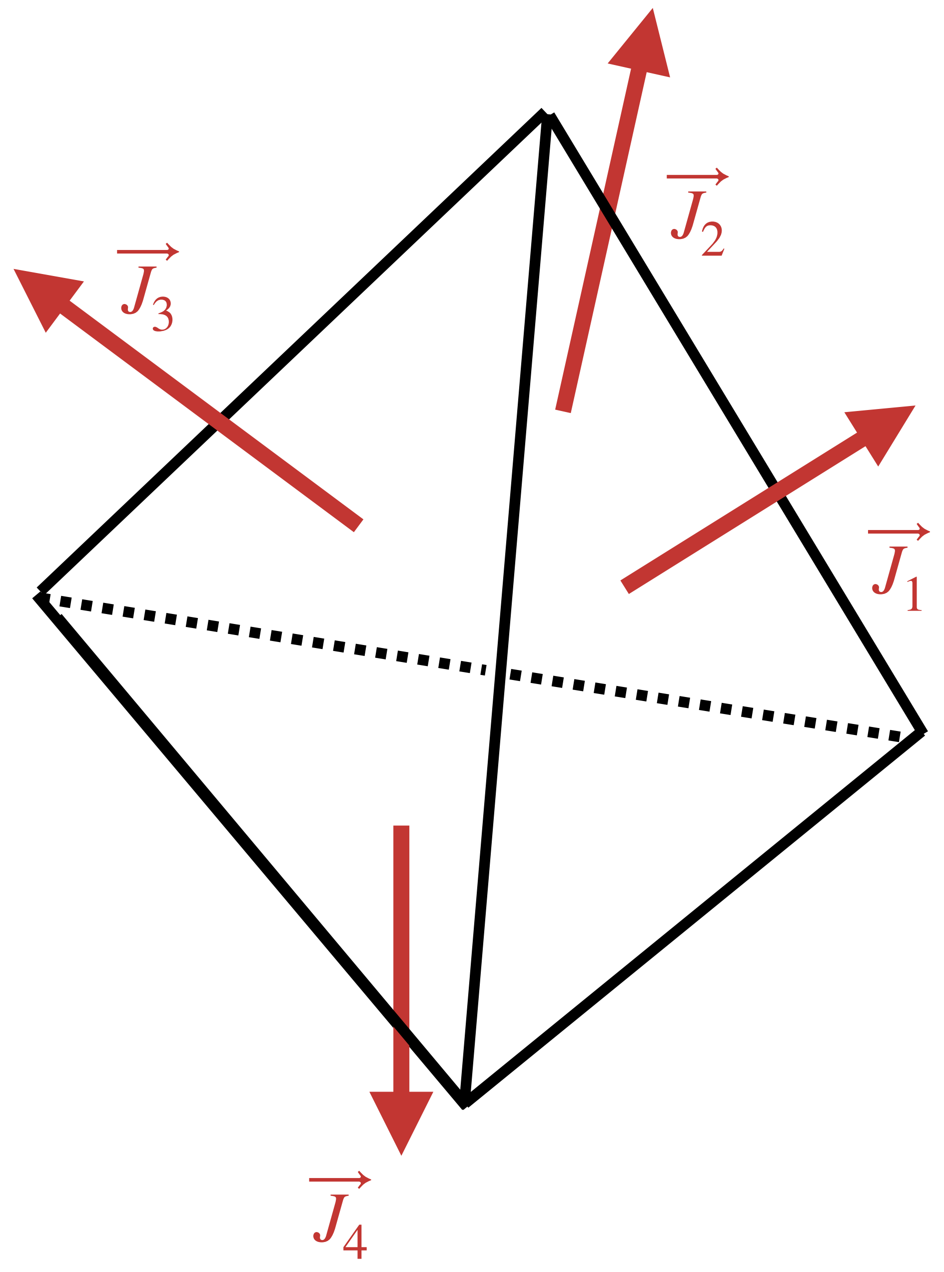
- The closure condition of constantly curved tetrahedron $M_4 M_3 M_2 M_1 = \mathrm{Id}$ [H. Haggard, M. Han, A. Riello '15]



$$M_\ell(p) = \exp\left[\frac{\Lambda}{3} a_\ell \hat{n}_\ell(p) \cdot \vec{\tau}\right] \in SU(2), \quad \tau = \frac{-i}{2}\vec{\sigma}$$

- **What is the quantization of the curved closure condition? What is the solution Hilbert space? What is the behavior of the area spectrum of the constantly curved tetrahedron?**

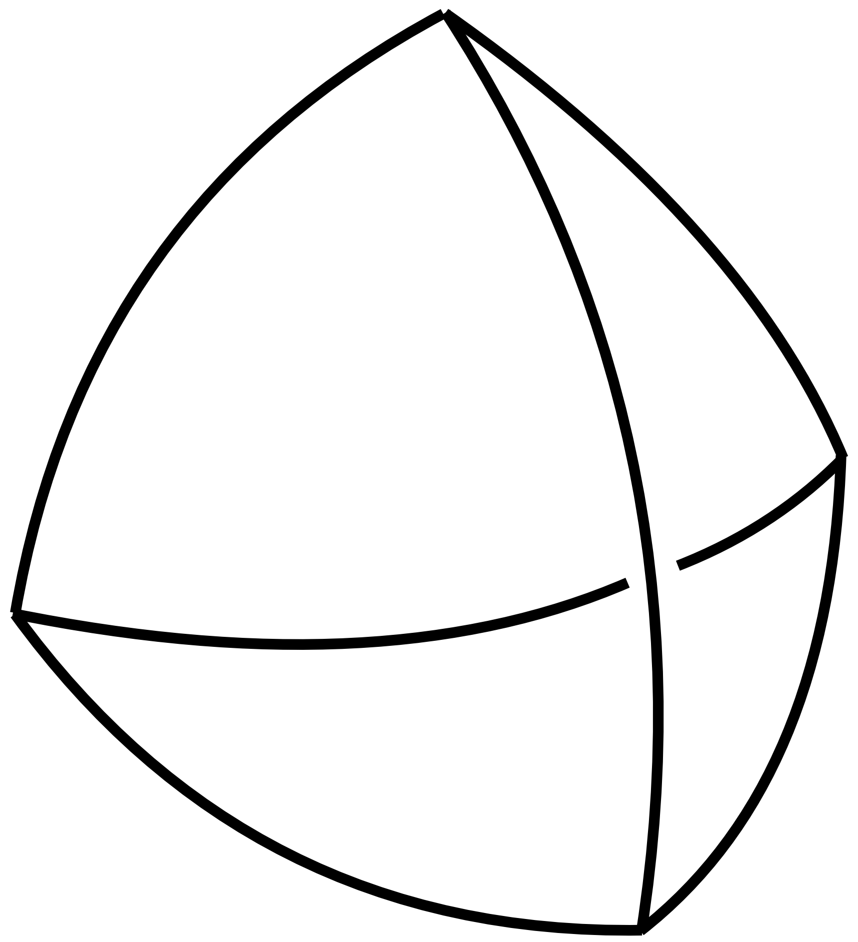
Motivation



Minkowski's theorem ⟷ $\vec{J}_1+\vec{J}_2+\vec{J}_3+\vec{J}_4=\vec{0}$ —**Quantization**→ $SU(2)$ intertwiner space

[H. Minkowski '97]

↓ **Generalized** ↓ ***q*-deformed ??**



Curved Minkowski's theorem ⟷ $M_4M_3M_2M_1 = \mathrm{Id}$ —**Quantization**→ $SU_q(2)$ **intertwiner space ??**

[H. Haggard, M. Han, A. Riello '15]

Strategies and Results

- All the results are valid for both $q \in U(1)$ (i.e. being a phase) and root of unity case (except for coherent state).

- The phase space of shapes of a homogeneously curved tetrahedron is the moduli space of $SU(2)$ flat connections on a four-holed sphere.

- Main strategy: Combinatorial description and quantization of the phase space.

[V. Fock, A. Rosly '92; A. Alekseev '93; A. Alekseev, H. Grosse, V. Schomerus '94; A. Alekseev, V. Schomerus '95]

- Quantum curved closure condition $\mathbf{M}_4\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 = \zeta \, \mathrm{Id}$, and ζ has the classical limit $\zeta \xrightarrow{q \to 1} 1$, where $q = e^{i\theta}$, $\theta \in (0,2\pi)$ including root of unity case.

- The solution space of the quantum curved closure condition is the intertwiner space, $\mathrm{Inv}_q(J_1, J_2, J_3, J_4)$, of the quantum group $\mathcal{U}_q(su(2))$.

This approach aligns with that used for the flat tetrahedron: Quantizing the closure condition reveals the solution space, which corresponds to the physical Hilbert space of the quantum tetrahedron.

Strategies and Results

- Area spectrum is bounded from above, i.e. $a_\nu < \frac{6\pi}{|\Lambda|}$

For q generic phase

$$a = \begin{cases} \gamma \ell_p^2 \left(J + \frac{1}{2}\right), & 0 \leq J < \frac{1}{2}B \\ \frac{12\pi}{|\Lambda|} - \gamma \ell_p^2 \left(J + \frac{1}{2}\right), & \frac{1}{2}B < J \leq B \end{cases} \qquad B = \frac{12\pi}{\ell_p^2 \gamma |\Lambda|} - 1$$

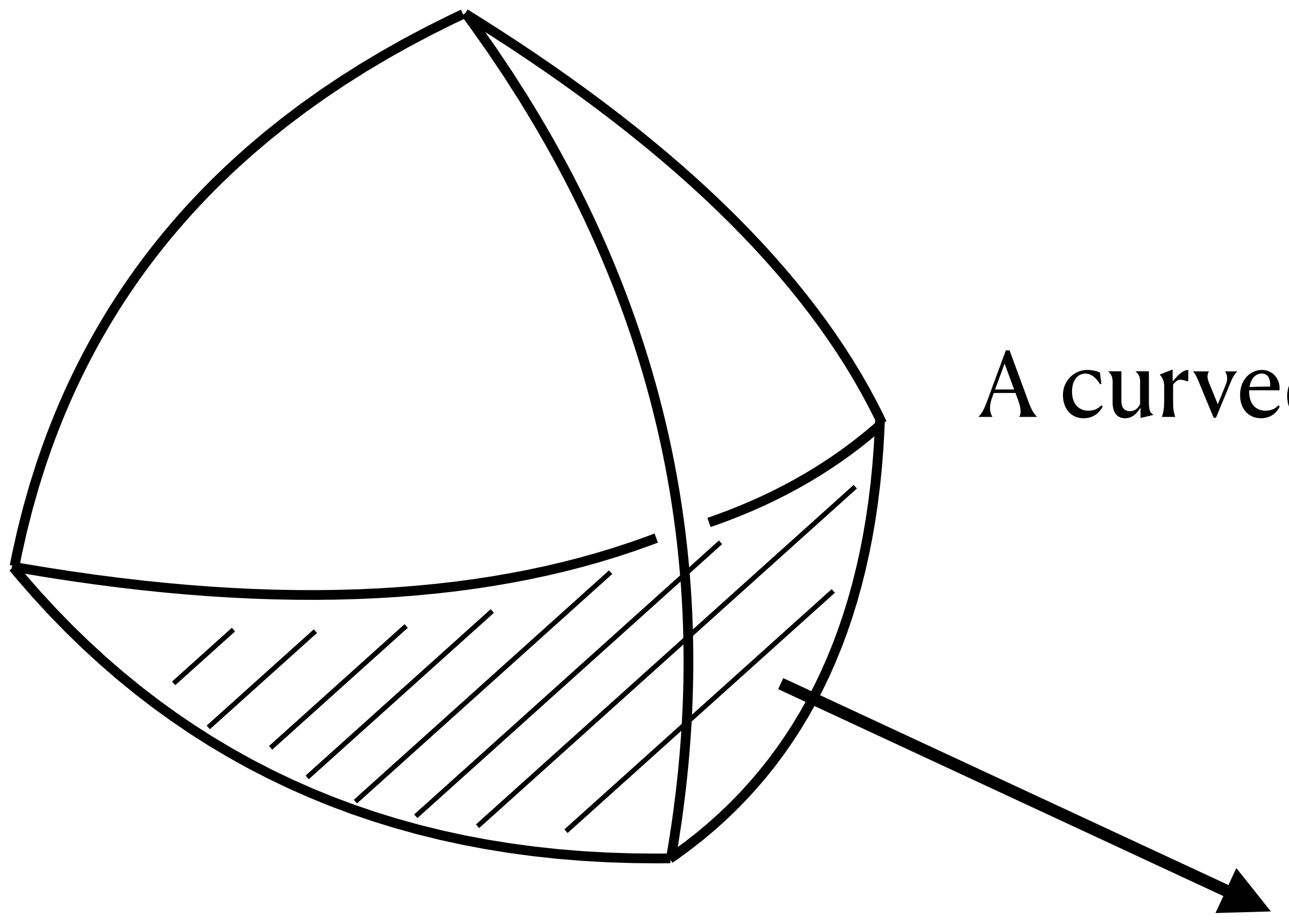
For q root of unity,

$$a = \gamma \ell_p^2 \left(J + \frac{1}{2}\right), \quad 0 \leq J \leq A \qquad A = \frac{6\pi}{\ell_p^2 \gamma |\Lambda|} - 1$$

  - In the limit $|\Lambda| \to 0$ and large J, the area grows linear in J, the area becomes $\gamma \ell_p^2 J$, which is consistent with standard LQG
  - The small J behavior is different from the standard LQG, particularly when $J = 0$, the minimal area is $\frac{1}{2}\gamma \ell_p^2$ instead of being trivial.

- Coherent state (Analog of coherent intertwiner of flat tetrahedron). [E.Livine, S. Speziale '07]
  - In the semiclassical limit, their expectation values relate to the point of phase space, which describes the shapes of a constantly curved tetrahedron.

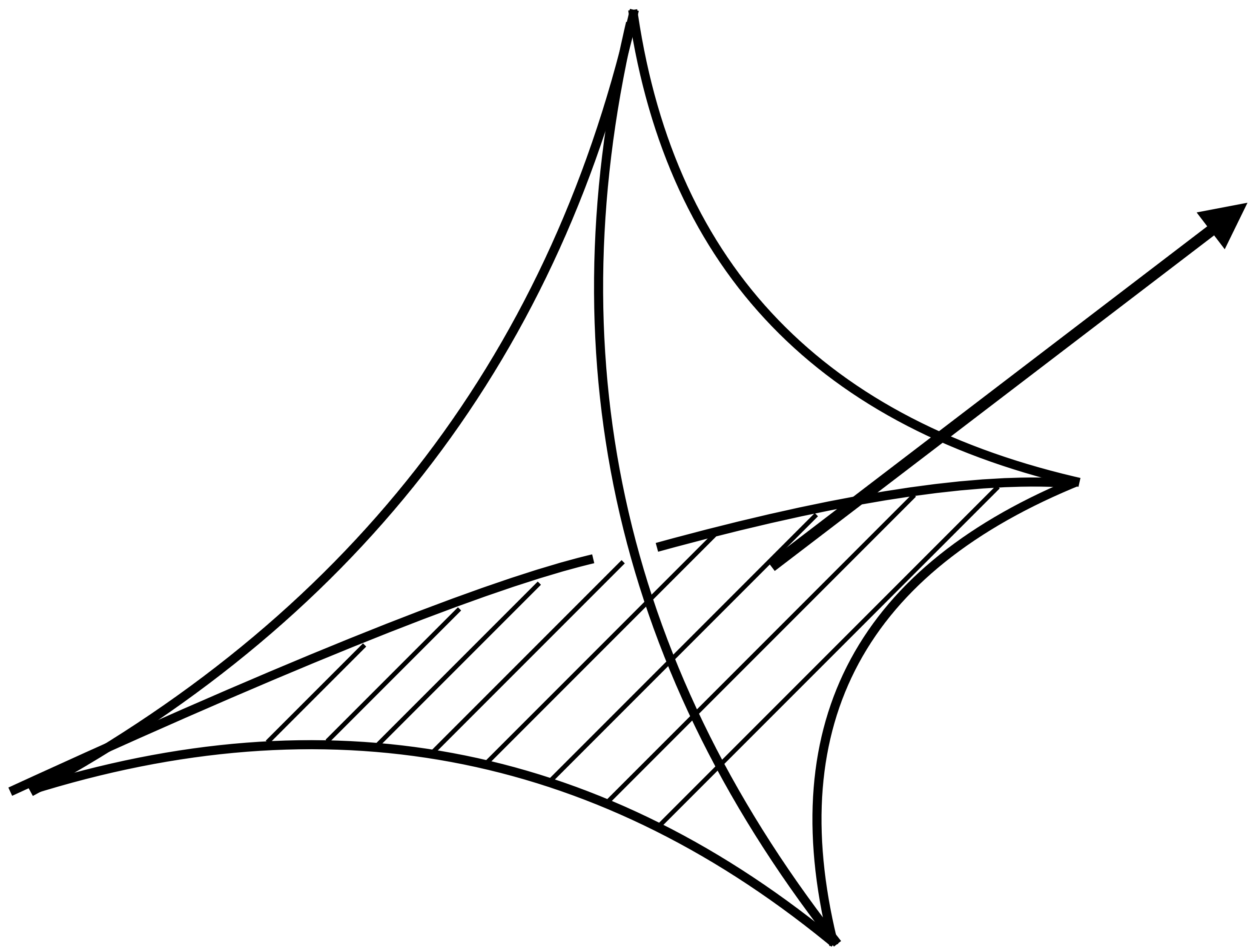
What is a constantly curved tetrahedron?



[H. Haggard, M. Han, A. Riello '15]

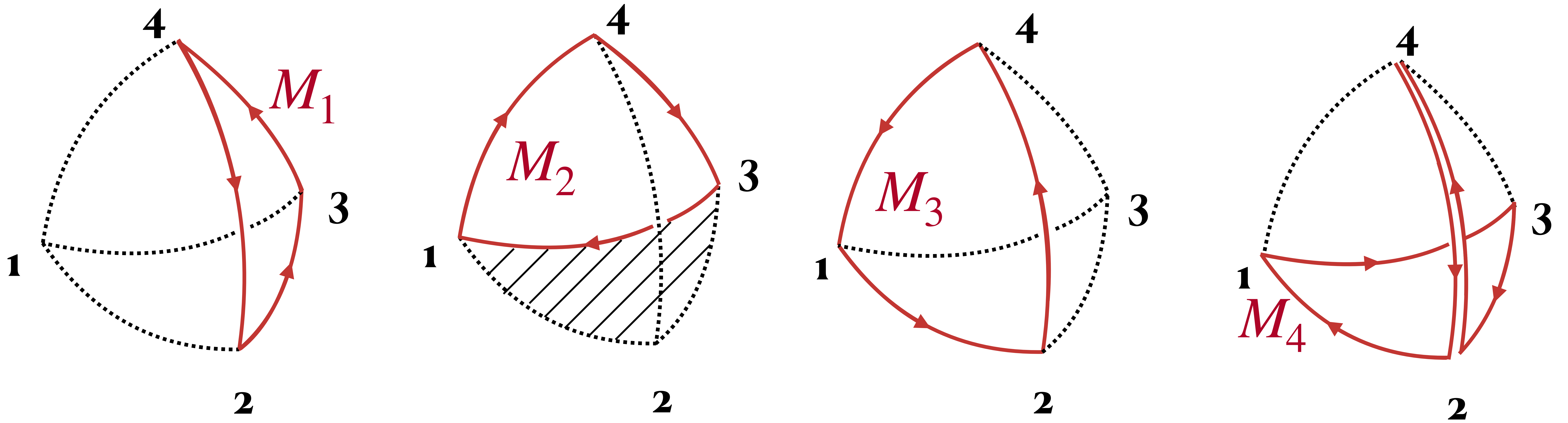
A curved tetrahedron is a tetrahedron in a three-sphere S^3 or a hyperbolic three-space H^3 where each face is flatly embedded.

A flatly embedded surface is a surface without extrinsic curvature.



In the LQG context, the curvature is identified as Λ.

How to describe a curved tetrahedron?



The holonomy of the Levi-Civita connection along ℓ at the base point p given by

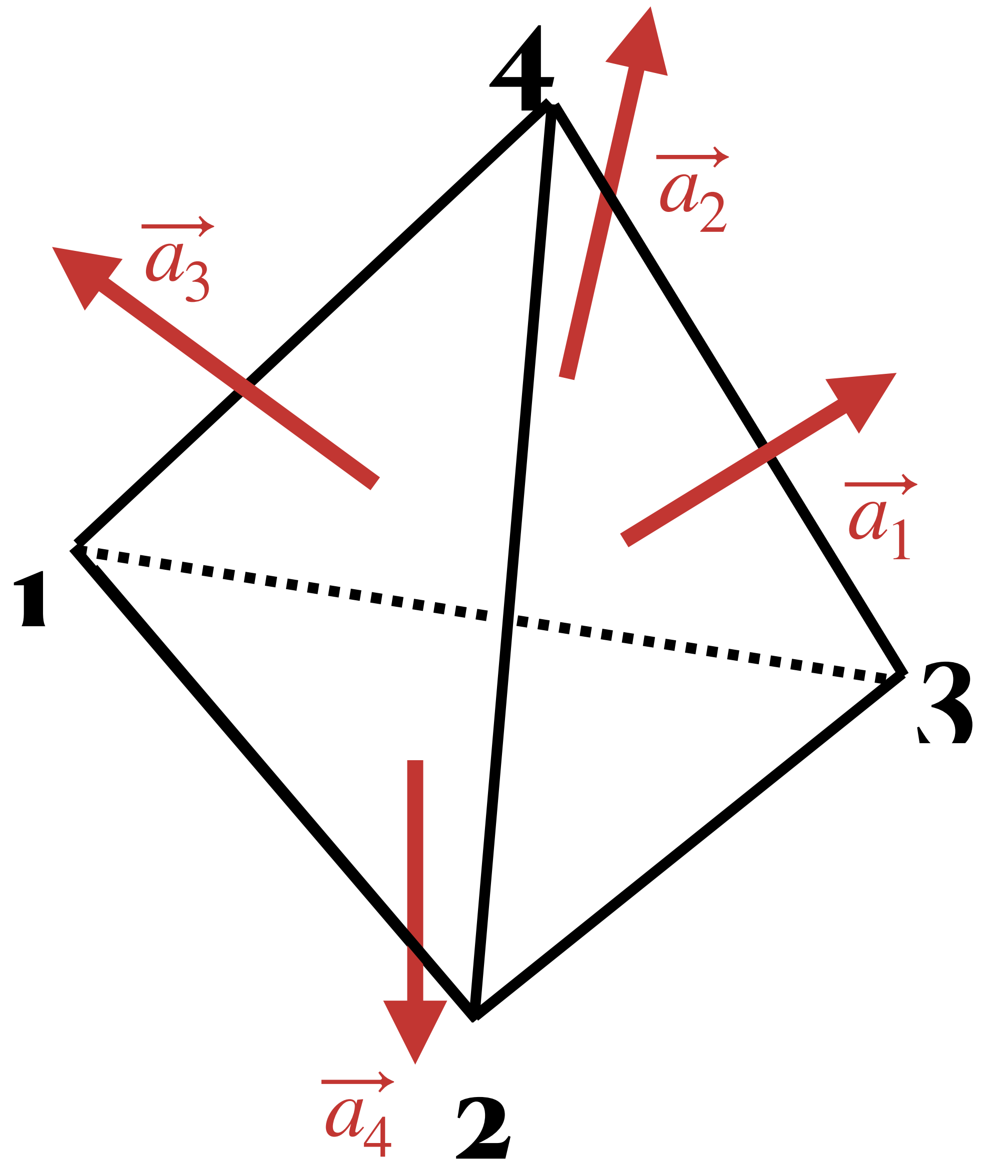
$$M_\ell(p) = \exp\left[\frac{\Lambda}{3} a_\ell \hat{n}_\ell(p) \cdot \vec{\tau}\right] \in SU(2), \quad \tau = \frac{-i}{2}\vec{\sigma}$$

a_ℓ is the area of the face enclosed by the loop ℓ, and $n_\ell(p)$ is the normal of the face at the base point p.

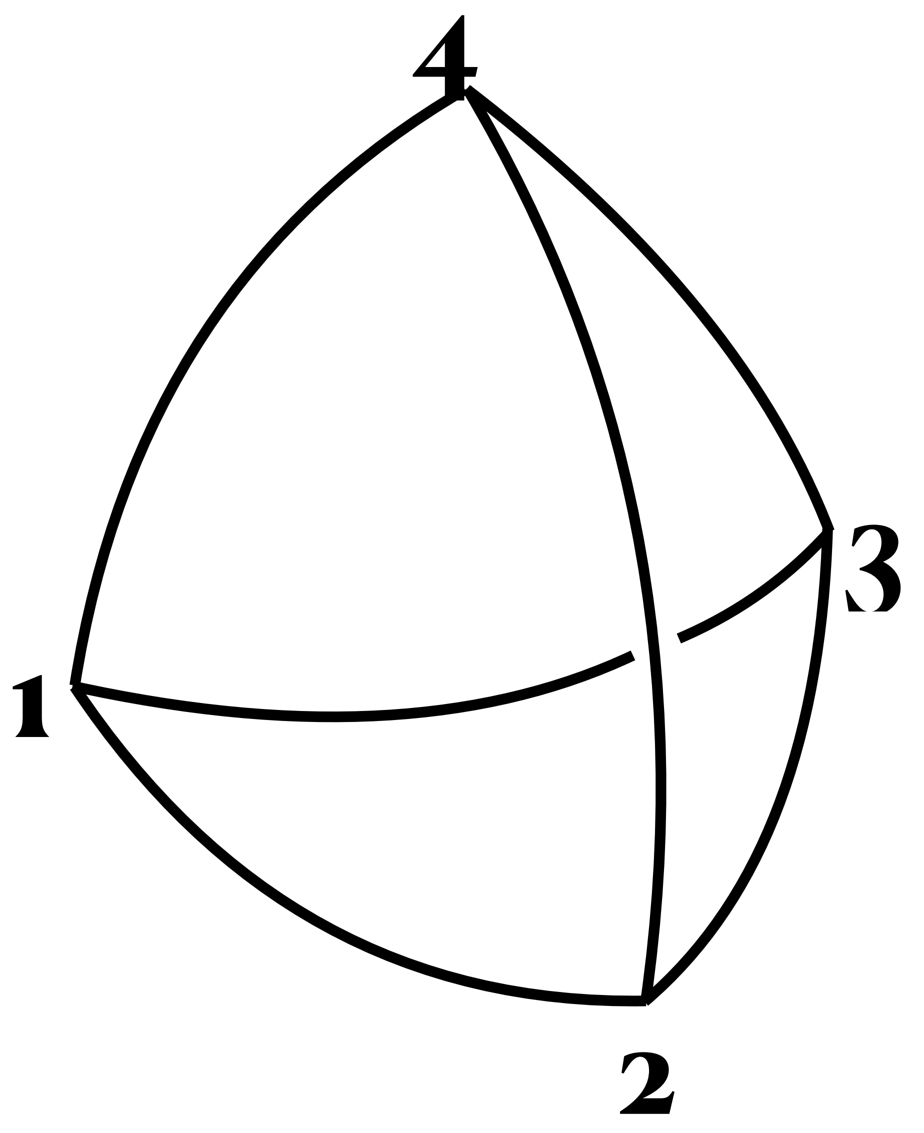
Curved closure condition: $M_4 M_3 M_2 M_1 = \mathrm{Id}$

[H. Haggard, M. Han, A. Riello '15]

Curved closure condition of a curved tetrahedron



Generalized



Flat Minkowski's theorem:

A flat tetrahedron is uniquely determined by a set of four vectors, satisfying the closure condition.

$$\vec{a_1} + \vec{a_2} + \vec{a_3} + \vec{a_4} = \vec{0}$$

[H. Minkowski '97]

$$M_\ell = e^{\Lambda \vec{a}_\ell \cdot \vec{\tau}}$$

Curved Minkowski's theorem:

Given four $SU(2)$ matrices satisfying curved closure condition, the curved tetrahedron is uniquely determined.

$$M_4 M_3 M_2 M_1 = 1$$

[H. Haggard, M. Han, A. Riello '15]

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$M_4 M_3 M_2 M_1 = \mathrm{Id}$

Classical closure condition → (Solution space) → Moduli space of SU(2) flat connections

$\mathscr{M}_{flat}(\Sigma_{0,4}, SU(2)) := Hom(\pi_1(\Sigma_{0,4}), SU(2))/SU(2)$

Classical closure condition ↓ (Quantization) ↓ Quantum closure condition

Moduli space of SU(2) flat connections ↓ (Quantization) ↓ Intertwiner space of quantum group

Quantum closure condition → (Solution space) → Intertwiner space of quantum group

$\mathbf{M}_4 \mathbf{M}_3 \mathbf{M}_2 \mathbf{M}_1 = \zeta \, \mathrm{Id}$

$\mathrm{Inv}_q(J_1, J_2, J_3, J_4)$

Moduli Space of SU(2) Flat Connections

- The solution space of the curved closure condition is the moduli space of SU(2) flat connections on the four-punctured sphere $\Sigma_{0,4}$. Representations of the fundamental group $\pi_1(\Sigma_{0,4})$ define the solution space as follows

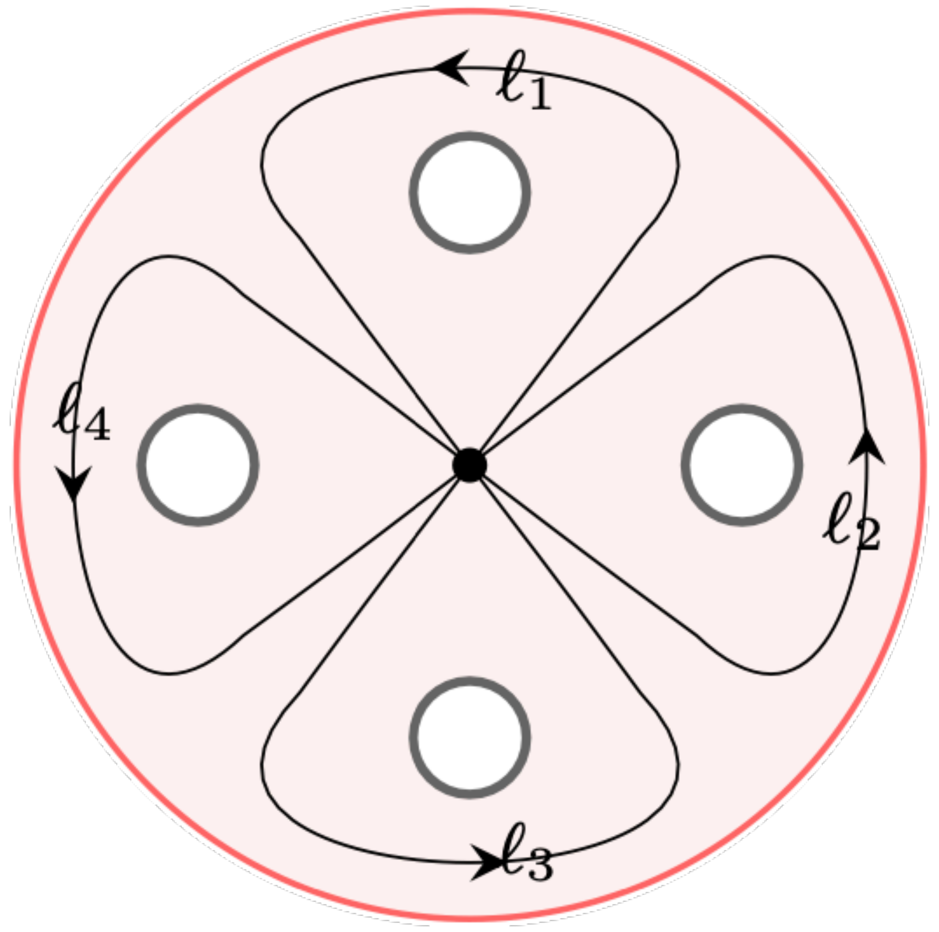
$$\mathscr{M}_{flat}(\Sigma_{0,4}, SU(2)) := Hom(\pi_1(\Sigma_{0,4}), SU(2))/SU(2)$$

$$\pi_1(\Sigma_{0,4}) = \{\ell_1, \ell_2, \ell_3, \ell_4 : \ell_4 \circ \ell_3 \circ \ell_2 \circ \ell_1 = 1\}$$

$Hom(\pi_1(\Sigma_{0,4}), SU(2))$ is the set of the maps that map ℓ to $SU(2)$ holonomies

$$M_4 M_3 M_2 M_1 = id$$

- The simple(standard) graph (in black) on the four-punctured sphere (in red) is presented graphically:
  - It contains one vertex and four loops $\ell_1, \ell_2, \ell_3, \ell_4$.
  - $\ell_1, \ell_2, \ell_3, \ell_4$ are generators of $\pi_1(\Sigma_{0,4})$.



- The moduli space of $SU(2)$ flat connections is naturally equipped with the symplectic form:

$$\Omega = \int_\Sigma Tr(\delta A \wedge \delta A) \qquad \{A_i^a(x_1), A_j^b(x_2)\} = \delta^{ab}\epsilon_{ij}\delta^{(2)}(x_1 - x_2)$$

- The Poisson bracket of monodromies (holonomies along the closed loop $\ell_\nu, \nu = 1,2,3,4$) is defined as

$$\overset{1}{M}{}^I_\ell = M^I_\ell \otimes id^J, \quad \overset{2}{M}{}^J_\ell = id^I \otimes M^J_\ell$$

For each ℓ:
$$\{\overset{1}{M}{}^I_\ell, \overset{2}{M}{}^J_\ell\} = r^{IJ}\, \overset{1}{M}{}^I_\ell \overset{2}{M}{}^J_\ell + \overset{2}{M}{}^J_\ell\, (r')^{IJ}\, \overset{1}{M}{}^I_\ell - \overset{1}{M}{}^I_\ell\, r^{IJ}\, \overset{2}{M}{}^J_\ell - \overset{1}{M}{}^I_\ell \overset{2}{M}{}^J_\ell\, (r')^{IJ}$$

For $\ell \prec \ell'$:
$$\{\overset{1}{M}{}^I_\ell, \overset{2}{M}{}^J_{\ell'}\} = r^{IJ}\, \overset{1}{M}{}^I_\ell \overset{2}{M}{}^J_{\ell'} + \overset{1}{M}{}^I_\ell \overset{2}{M}{}^J_{\ell'}\, r^{IJ} - \overset{2}{M}{}^J_{\ell'}\, r^{IJ}\, \overset{1}{M}{}^I_\ell - \overset{1}{M}{}^I_\ell\, r^{IJ}\, \overset{2}{M}{}^I_{\ell'}$$

- A classical r-matrix determines the Poisson Structure of the phase space, $r \in \mathfrak{su}(2) \otimes \mathfrak{su}(2)$.

[V. Fock, A. Rosly '92; A. Alekseev, H. Grosse, V. Schomerus '94; A. Alekseev, V. Schomerus '95]

Quantization of moduli space

Classical closure condition ✓ → (Solution space) → Moduli space of SU(2) flat connections ✓

↓ Quantization

Representation space of quantum algebra

Graph Algebra(Multi-loops Algebra) $\mathscr{L}_{0,4}$

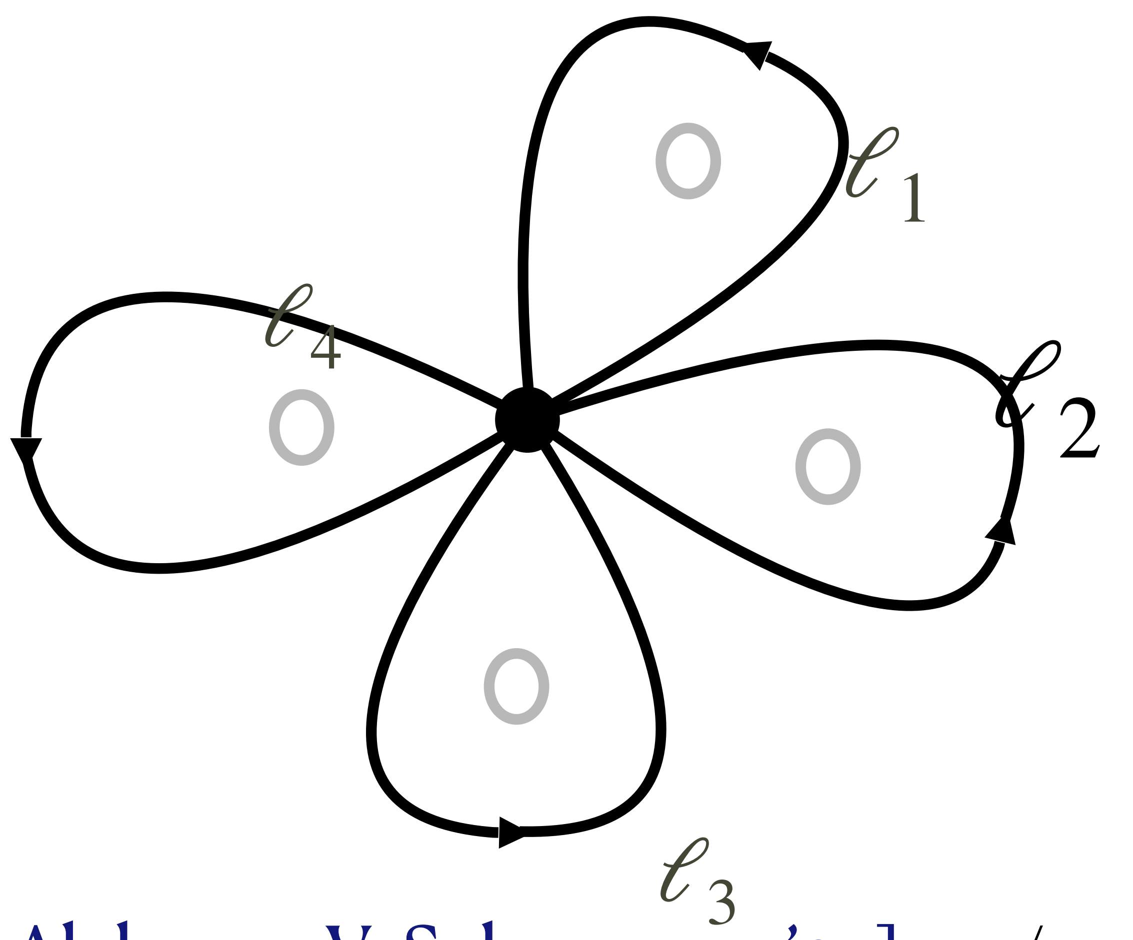
- It is a *-algebra and is generated by the matrix elements of quantum monodromies $\mathbf{M}^I_{\ell_\nu}$, which I runs through all irreducible representations of $\mathscr{U}_q(su(2))$ and $\nu = 1,2,3,4$

- Quantize the Poisson bracket into commutation relations, the Poisson bracket is recovered when one expands the quantum R-matrix to the first-order of $\hbar$: $R = 1 + \hbar r + O(\hbar^2)$

- The $\mathbf{M}^I_\ell$ is the quantization of the holonomies matrices in the curved closure condition.

For each ℓ: $$(R^{-1})^{IJ} \overset{1}{\mathbf{M}}{}^I_\ell \, R^{IJ} \, \overset{2}{\mathbf{M}}{}^J_\ell = \overset{2}{\mathbf{M}}{}^J_\ell \, (R')^{IJ} \, \overset{1}{\mathbf{M}}{}^I_\ell \, (R'^{-1})^{IJ}$$

For $\ell \prec \ell'$: $$(R^{-1})^{IJ} \overset{1}{\mathbf{M}}{}^I_\ell \, R^{IJ} \, \overset{2}{\mathbf{M}}{}^J_{\ell'} = \overset{2}{\mathbf{M}}{}^J_{\ell'} \, (R^{-1})^{IJ} \, \overset{1}{\mathbf{M}}{}^I_\ell \, R^{IJ}$$

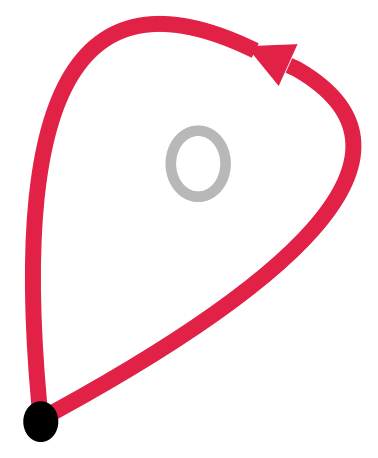


[A. Alekseev, H. Grosse, V. Schomerus '94; A. Alekseev, V. Schomerus '95]

Loop Algebra $\mathscr{L}_{0,1}$

- The loop algebra is a *-algebra and is generated by the matrix elements of quantum monodromies (holonomies along the closed loop ℓ) $\mathbf{M}^I_\ell$ [A. Alekseev, H. Grosse, V. Schomerus '94; A. Alekseev, V. Schomerus '95]

For each ℓ: $$(R^{-1})^{IJ}\,\overset{1}{\mathbf{M}}{}^I_\ell\,R^{IJ}\,\overset{2}{\mathbf{M}}{}^J_\ell=\overset{2}{\mathbf{M}}{}^J_\ell\,(R')^{IJ}\,\overset{1}{\mathbf{M}}{}^I_\ell\,(R'^{-1})^{IJ}$$



- The loop algebra is isomorphic to $\mathscr{U}_q(su(2))$, the isomorphism is defined as:

[A. Alekseev '93 ;A. Alekseev, V. Schomerus '95]

$$\mathbf{M}^I_\ell=\kappa_I^{-1}\mathbf{X}^I_+(\mathbf{X}^{-1}_-)^I,\quad \mathbf{X}^I_+\equiv(R')^I,\mathbf{X}^I_-\equiv(R^{-1})^I,\quad \kappa_I=q^{-\frac{1}{2}I(I+1)}\,.$$

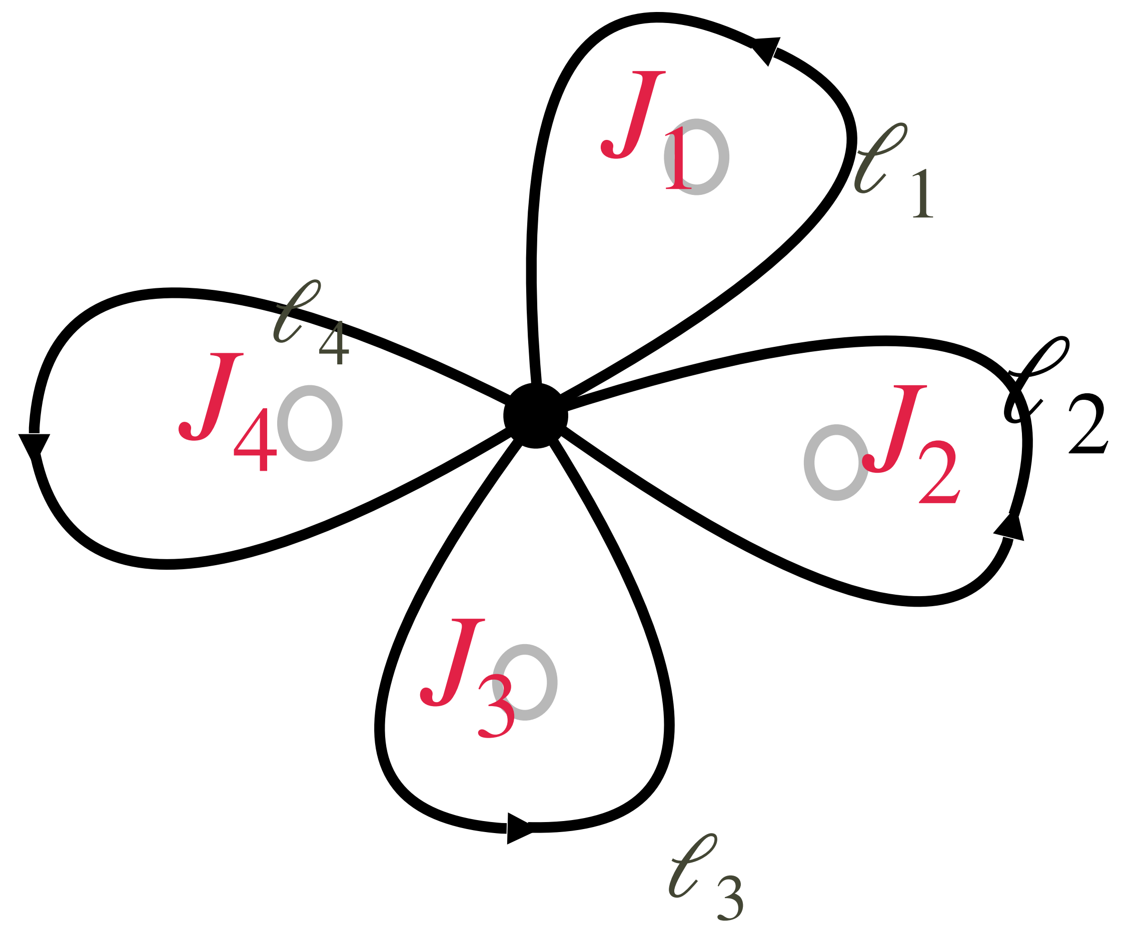
- The representation of the loop algebra is realized in the carrier space V^J of $\mathscr{U}_q(su(2))$.

	Flat tetrahedron	Curved tetrahedron
Single face	The flux operators form the $su(2)$ Lie algebra.	Loop algebra $\mathscr{L}_{0,1}$ generated by $\mathbf{M}^I_\ell$, $\mathscr{L}_{0,1} \cong \mathscr{U}_q(su(2))$
Four faces	Four copies of $su(2)$ Lie algebra.	Four copies of $\mathscr{L}_{0,1}$ modulo mutual commutation relation (Graph algebra, $\mathscr{L}_{0,4}$).
Representation space	$V^{J_1} \otimes V^{J_2} \otimes V^{J_3} \otimes V^{J_4}$	$V^{J_1} \otimes V^{J_2} \otimes V^{J_3} \otimes V^{J_4}$ of quantum group
Closure condition	$\hat{\vec{J}}_4 + \hat{\vec{J}}_3 + \hat{\vec{J}}_2 + \hat{\vec{J}}_1 = \hat{\vec{0}}$	$\mathbf{M}_4\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 = \zeta\, id,\ \zeta = \kappa_I^{-4}, \kappa_I = q^{-\frac{1}{2}I(I+1)}$
Solution space	Intertwiner space of SU(2)	Intertwiner space of $\mathscr{U}_q(su(2))$

Solution space of quantum closure condition

THEOREM:

For any set J_1, J_2, J_3, J_4 labeling four punctures, the intertwiner space $W^0(J_1, J_2, J_3, J_4) \equiv Inv_q(V^{J_1} \otimes V^{J_2} \otimes V^{J_3} \otimes V^{J_4})$ is the ***only*** solution space to the quantum closure condition.



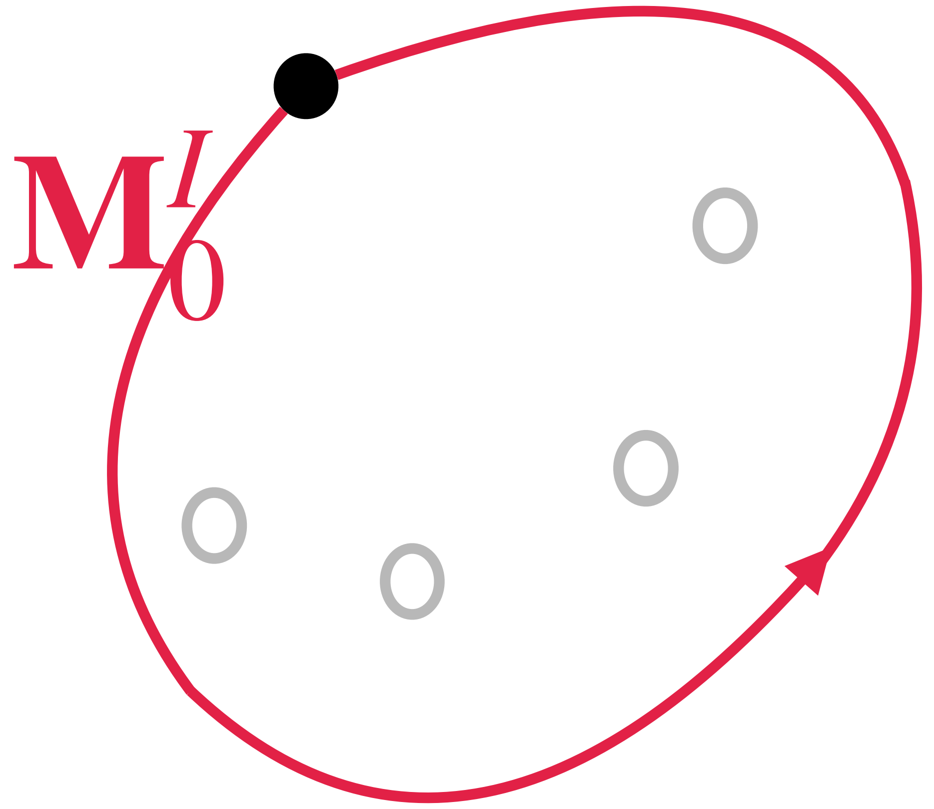
Solution space of quantum closure condition

Sketch of proof:

$$\mathcal{T}(J_1, J_2, J_3, J_4) := V^{J_1} \otimes V^{J_2} \otimes V^{J_3} \otimes V^{J_4} = \sum_L V^L \otimes W^L(J_1, J_2, J_3, J_4)$$

- The proof is obtained by the representation theory constructed by A. Alekseev et. al., and the algebraic relation: $M_0^I \chi_0^0 = \kappa_I^{-1} id^I$, where χ_0^0 is the central element and projects the multiplicity spaces $W^L(J_1, J_2, J_3, J_4)$ into the intertwiner space $W^0(J_1, J_2, J_3, J_4)$ when we evaluate it in the representation, the representation theory is faithful by construction.

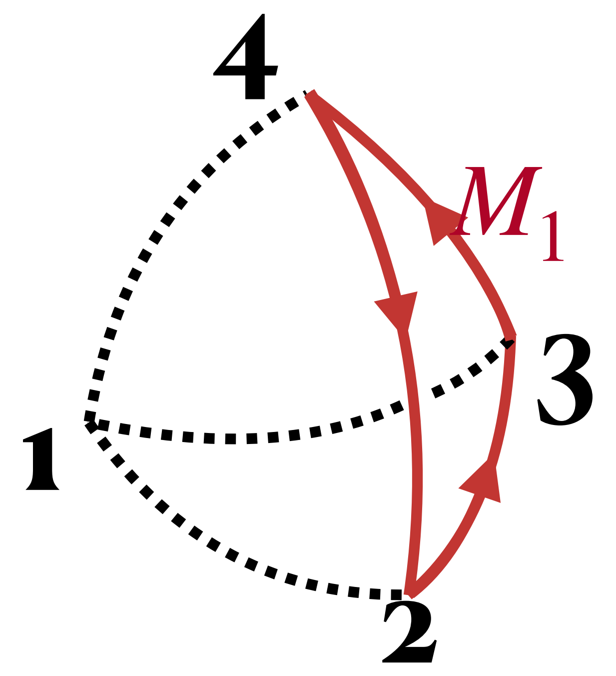
$$\rho^{J_1,J_2,J_3,J_4}(\chi_0^0)\,|_{W^L} = \delta_{L,o} id_{W^0}$$



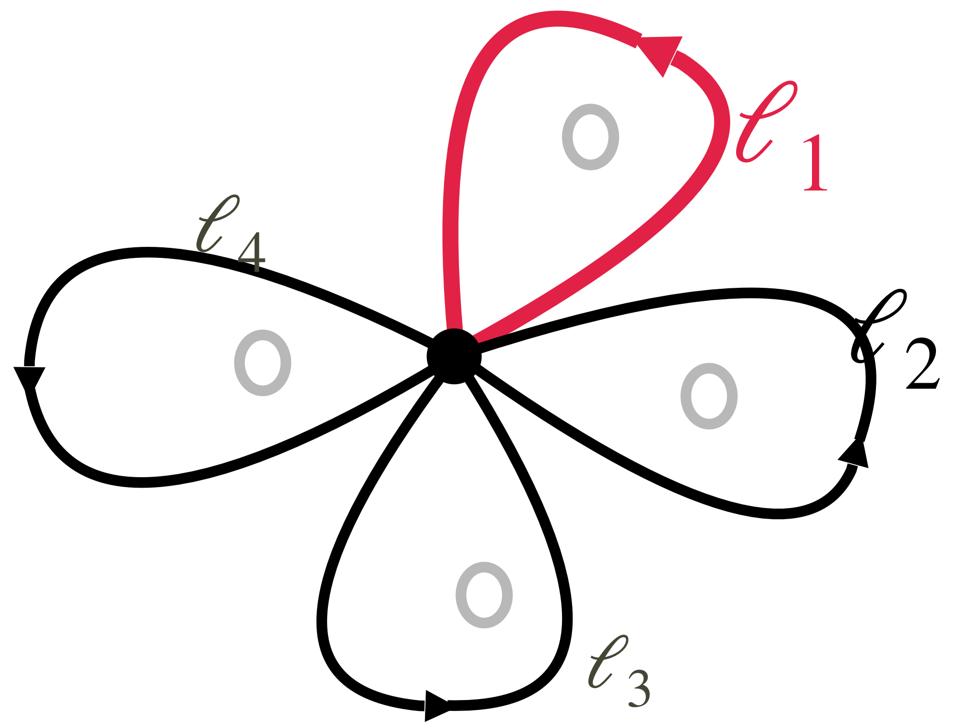
$$\mathbf{M}_0^I W^0 = \kappa_I^{-1} W^0$$

$$\mathbf{M}_0^I W^0 \equiv \kappa_I^3 \mathbf{M}_4^I \mathbf{M}_3^I \mathbf{M}_2^I \mathbf{M}_1^I W^0 = \kappa_I^{-1} W^0$$

Spectrum of Area operator



- Classically, the information of area is obtained from: $\frac{1}{2}Tr\left(M_\nu\right) = \cos\left(\frac{|\Lambda|}{6}a_\nu\right).$



- A proposed quantization will be: $\cos\left(\frac{|\Lambda|}{6}\mathbf{a}_\nu\right) := \frac{1}{2}\rho^{J_\nu}\left[Tr_q^{\frac{1}{2}}\left(\mathbf{M}_\nu^{\frac{1}{2}}\right)\right] = \cos\left[\frac{\theta}{2}(2J_\nu + 1)\right]$

- The spectrum of the operator $a_\nu \in [0,\pi)$ is bounded from above, i.e. $a_\nu < \frac{6\pi}{|\Lambda|}$, and is given by

For generic $q = e^{i\theta},\ \theta = \frac{1}{6}\ell_p^2\gamma|\Lambda|$

$$a = \begin{cases} \gamma\ell_p^2\left(J + \frac{1}{2}\right), & 0 \le J < \frac{1}{2}B \\ \frac{12\pi}{|\Lambda|} - \gamma\ell_p^2\left(J + \frac{1}{2}\right), & \frac{1}{2}B < J \le B \end{cases} \qquad B = \frac{12\pi}{\ell_p^2\gamma|\Lambda|} - 1$$

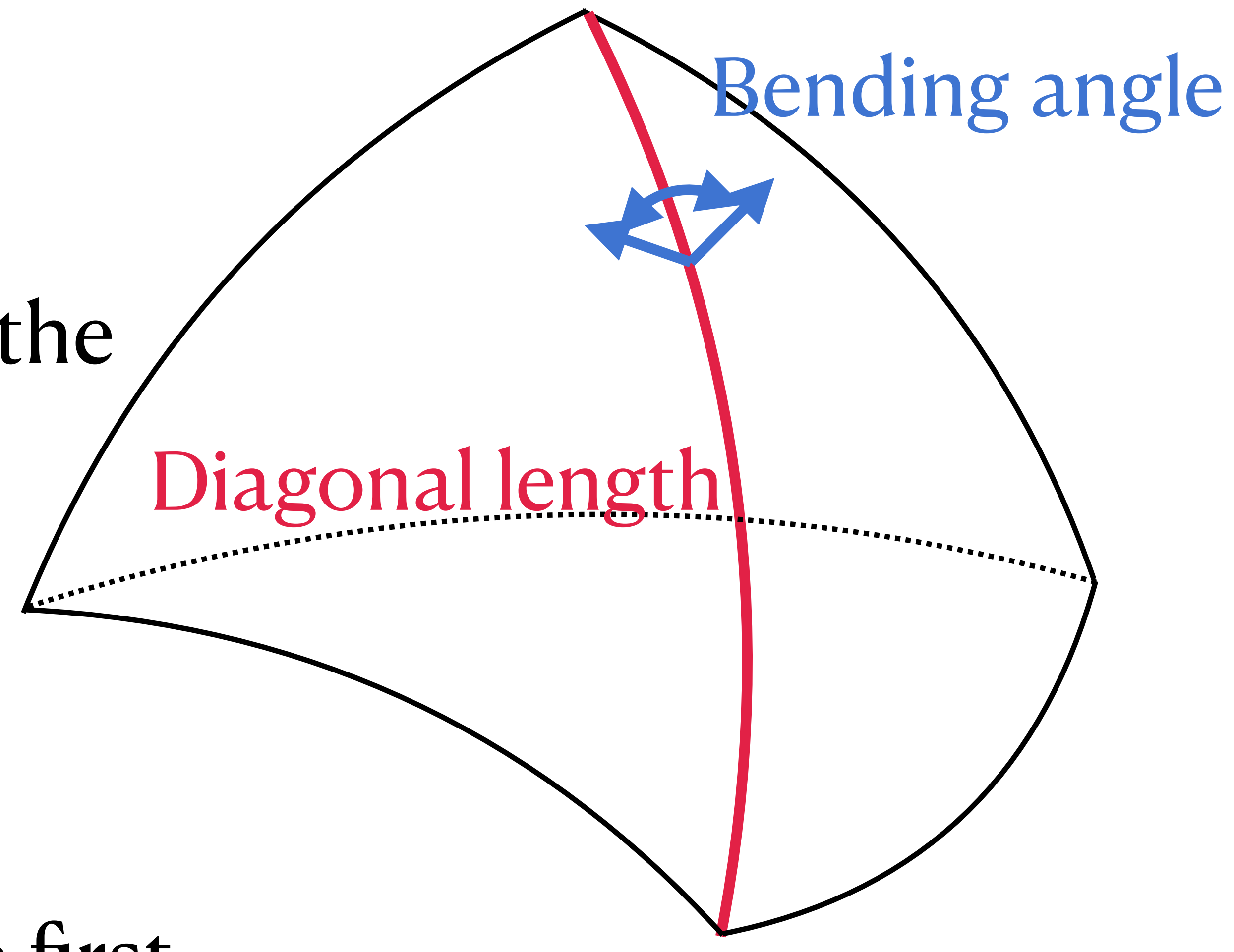
For $q = \exp\left(\frac{2\pi i}{k+2}\right), k + 2 = \frac{12\pi}{\ell_p^2\gamma|\Lambda|}$

$$a = \gamma\ell_p^2(J + \frac{1}{2}), \quad 0 \le J \le A \qquad A = \frac{6\pi}{\ell_p^2\gamma|\Lambda|} - 1$$

Coherent State

- Classically, the phase space of shapes of a curved tetrahedron is the moduli space of $SU(2)$ flat connections and can be described by a 4-gon on S^3, which is the generalization of Kapovich & Millson description.

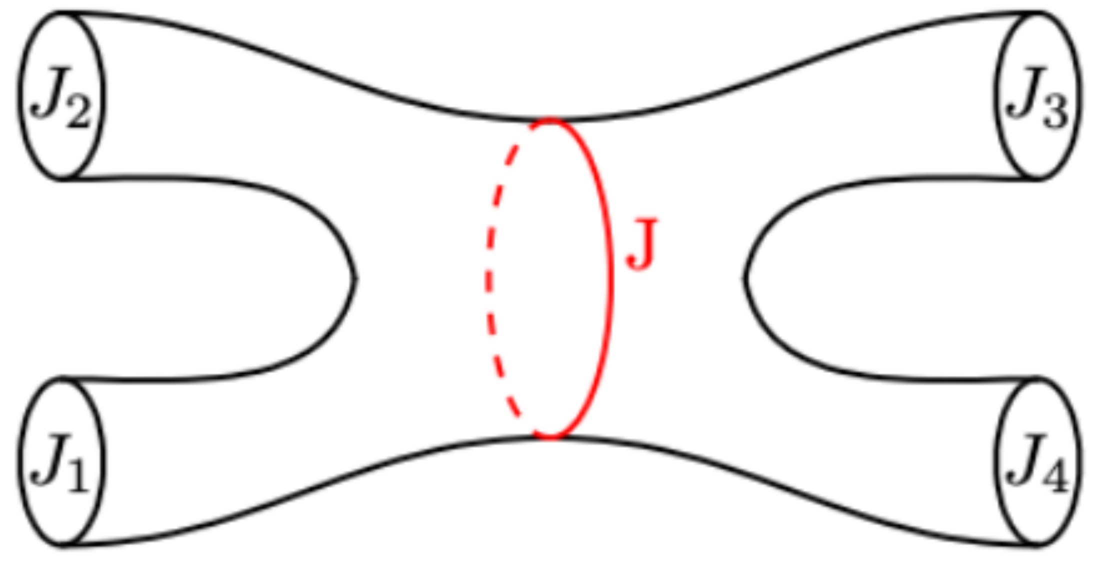
Phase space of shapes of a curved tetrahedron	Moduli space of flat connections
The diagonal length and bending angle	Complex Fenchel-Nielson coordinates



- The 4-gon on S^3 is parametrized by the diagonal length, $\ln(x)$, and bending angle, $\ln(y^2)$, which is the analogy of the complex Fenchel-Nielson coordinate for the flat connections and has the following Poisson bracket:

$$\{\ln(x), \ln(y^2)\} = 1$$

- The quantum diagonal operator, $\mathbf{x}$, is obtained from the q-deformed Wilson loop that encloses the first and the second punctures. The eigenvalue of the $\mathbf{x}$ in the intertwiner space, $\mathrm{Inv}_q(J_1, J_2, J_3, J_4)$, is given by:



$$\mathbf{x}\,|\psi^J\rangle = e^{\frac{i\pi}{k+2}(2J+1)}\,|\psi^J\rangle \qquad \mathbf{y^2}\,|\psi^J\rangle = |\psi^{J+1}\rangle$$

$$\mathbf{x}\mathbf{y^2} = q\mathbf{y^2}\mathbf{x}$$

  - J satisfies the triangle inequality: $\max\left(|J_1 - J_2|, |J_3 - J_4|\right) \le J \le \min\left(u(J_1, J_2), u(J_3, J_4)\right)$ $u(I, J) = \min\left(I + J, k - I - J\right)$

Coherent State

- We define the auxiliary space, $\mathscr{H}_{\text{aux}} = \mathbb{C}^{2k+4}$, as the irreducible representations of Weyl algebra (i.e. $\mathbf{xy} = q^{\frac{1}{2}}\mathbf{yx}$). Quantization of the torus as the phase space, there is a set of well-defined coherent states $|\psi_{(x_0,y_0)}\rangle$, obtained by averaging harmonic oscillator coherent states over the periodicity.

[J. Gazeau, 2009]

$$\frac{k+2}{2\pi^2}\int_{\mathbb{T}_2} dx_0 dy_0 |\psi_{(x_0,y_0)}\rangle\langle\psi_{(x_0,y_0)}| = Id_{\mathscr{H}_{\text{aux}}}$$

- We define a projector $P = \sum_{2J} |2J\rangle\langle 2J| : \mathscr{H}_{\text{aux}} \to W^0(J_1, J_2, J_3, J_4)$. The coherent state of $W^0(J_1, J_2, J_3, J_4)$ is given by $|\tilde{\psi}_{(x_0,y_0)}\rangle = P|\psi_{(x_0,y_0)}\rangle$.

$$\frac{k+2}{2\pi^2}\int_{\mathbb{T}_2} dx_0 dy_0 |\tilde{\psi}_{(x_0,y_0)}\rangle\langle\tilde{\psi}_{(x_0,y_0)}| = Id_{W^0}$$

- In the semi-classical limit (i.e., $k = \lambda k$, $J_\nu = \lambda J_\nu$, and $\lambda \to \infty$), the expectation value of $\mathbf{x}$ and $\mathbf{y}^2$ in the projected coherent state is given:

$$\langle\mathbf{x}\rangle = e^{ix_0} + O\left(e^{-\lambda}/\sqrt{\lambda}\right), \quad \langle\mathbf{y^2}\rangle = e^{2iy_0} + O\left(e^{-\lambda}/\sqrt{\lambda}\right).$$

  - x_0 needs to satisfy the triangle inequality
  - Otherwise, they exhibit exponential decay: $\langle\mathbf{x}\rangle = O\left(e^{-\lambda}/\sqrt{\lambda}\right), \quad \langle\mathbf{y^2}\rangle = O\left(e^{-\lambda}/\sqrt{\lambda}\right).$

Conclusion and outlook

Conclusion

	Curved tetrahedron
Single face	Loop algebra $\mathscr{L}_{0,1}$ generated by $\mathbf{M}^I_\ell$, $\mathscr{L}_{0,1} \cong \mathscr{U}_q(su(2))$
Four faces	Four copies of $\mathscr{L}_{0,1}$ modulo mutual commutation relation (Graph algebra, $\mathscr{L}_{0,4}$).
Representation space	$V^{J_1} \otimes V^{J_2} \otimes V^{J_3} \otimes V^{J_4}$ of quantum group
Closure condition	$\mathbf{M}_4\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 = \zeta\, id,\ \zeta = \kappa_I^{-4}\ , \kappa_I = q^{-\frac{1}{2}I(I+1)}$
Solution space	Intertwiner space of $\mathscr{U}_q(su(2))$
Area spectrum	Area spectrum in both q generic phase and root of unity cases
Coherent states	$\lvert\tilde{\psi}_{(x_0,y_0)}\rangle = P\lvert\psi_{(x_0,y_0)}\rangle$

Conclusion and outlook

Outlook

- Generalize the quantization to arbitrary 3D cellular complex. The moduli space of SU(2) flat connections on higher-genus surfaces is closely related to the new LQG phase space on cellular complexes.

  [L.Smolin'96; J. Lewandowski, A. Okolow '08; M.Han, Z.Huang '17]

- Relate the coherent state and Intertwiner space of the quantum group to spin foam with the cosmological constant.

- Volume operator.

Thank you for your listening!