Homogeneous-isotropic sector of loop quantum gravity: new approach

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Section 1

Motivation

Motivation

- Promising candidates for the models of Quantum Gravity have been proposed in the canonical Loop Quantum Gravity:
  - Giesel K and Thiemann T 2010 Class. Quant. Grav. 27 175009,
  - Domagala M, Giesel K, Kaminski W and Lewandowski J 2010 Phys. Rev. D82 104038.
- M. Domagala, K. Giesel, W. Kaminski and J. Lewandowski, *Gravity quantized: Loop Quantum Gravity with a Scalar Field*: "In the LQC models of the homogeneous massless scalar field coupled to gravity, big bang turns out to be replaced by big bounce, as the result of the quantum gravity effects. Now, with our model, we can consider the same system of fields from the point of view of the full theory, without the symmetry reduction. Similarly, we can also consider the quantum gravitational collapse, quantum black holes, and theory entropy. All those cases are manageable within our model, and the only difficulty is of a technical nature."
- We focus on a class of physical Hamiltonians which seem to be technically simpler: Lewandowski J and Sahlmann H 2015 Phys. Rev. D91 044022; Alesci E, Assanioussi M, Lewandowski J and Mäkinen I 2015 Phys. Rev. D91 124067; Assanioussi M, Lewandowski J and Mäkinen I 2015 Phys. Rev. D92 044042; Assanioussi M, Lewandowski J and Mäkinen I 2017 Phys. Rev. D96 024043.

Outline

Section 2

Symmetric states on a lattice

The lattice

Our spatial manifold will be the 3-torus:

$$\Sigma = \mathbb{T}^3.$$

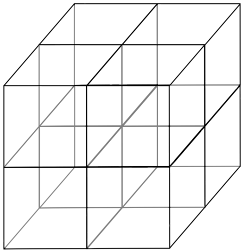
It will be convenient to consider it as a quotient space. Given a real number $a \in \mathbb{R}$, let us consider a subgroup of the group of translations $\mathcal{T}_a = \{T_{(pa,qa,ra)} : p, q, r \in \mathbb{Z}\}$, where T_v denotes a translation by a vector v. The quotient space $\mathbb{R}^3/\mathcal{T}_a$ is diffeomorphic to $\mathbb{T}^3$:

$$\mathbb{T}^3 = \mathbb{R}^3/\mathcal{T}_a.$$

Let us fix a number $\epsilon \in \mathbb{R}$ such that $\frac{1}{\epsilon} \in \mathbb{N}^+$. Let us consider an infinite cubical lattice $\widetilde{\Gamma}$ in $\mathbb{R}^3$ such that the coordinates of the nodes are

$$(p\,\epsilon a, q\,\epsilon a, r\,\epsilon a),$$

where $p, q, r \in \mathbb{Z}$, This lattice naturally descends to a lattice on $\mathbb{T}^3 = \mathbb{R}^3/\mathcal{T}_a$. We will denote the corresponding lattice on $\mathbb{T}^3$ by Γ.



The symmetry group

We consider a group O_{cube} of orientation preserving symmetries of a cube. This group is a subgroup of a group generated by matrices

$$R_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

formed by matrices of determinant 1. The subgroup of the Euclidean group $\mathcal{T}_{\epsilon a} \rtimes O_{\mathrm{cube}}$ acting in $\mathbb{R}^3$ preserves the infinite lattice $\tilde{\Gamma}$. Let us denote by $\mathcal{T}_\epsilon$ the quotient group:

$$\mathcal{T}_\epsilon := \mathcal{T}_{\epsilon a}/\mathcal{T}_a.$$

The group

$$\mathcal{T}_\epsilon \rtimes O_{\mathrm{cube}}$$

is a subgroup of isometries of $\mathbb{T}^3 = \mathbb{R}^3/\mathcal{T}_a$ equipped with the canonical flat metric (the metric for which $\pi : \mathbb{R}^3 \to \mathbb{R}^3/\mathcal{T}_a$ is the Riemannian covering map). The group $\mathcal{T}_\epsilon \rtimes O_{\mathrm{cube}}$ preserves the lattice Γ.

Homogeneous-isotropic states

We will say that a spin network s defined on a cubical lattice is homogeneous-isotropic if it is invariant under $\mathcal{T}_\epsilon \rtimes O_{\rm cube}$ up to a phase, i.e. for any diffeomorphism $g \in \mathcal{T}_\epsilon \rtimes O_{\rm cube}$

$$g^*|s >= e^{i\Phi_s(g)}|s >,$$

where $e^{i\Phi_s(g)}$ is 1-dimensional unitary representation of the symmetry group $\mathcal{T}_\epsilon \rtimes O_{\rm cube}$. For homogeneous isotropic spin networks it follows in particular that

$$\forall_{\ell \in {\rm Links}(\Gamma)} \dim \rho_\ell = 2j+1$$

for a given fixed spin j. The phase, which we will choose, depends only on this spin j. Therefore, instead of writing $\Phi_s(g)$ we will write $\Phi_j(g)$.

Symmetric intertwiners

Let us focus on the intertwiner space corresponding to a single node n. Let us denote the links incident at n by $\ell_1, \ell_2, \ldots, \ell_6$. Without loss of generality, we can assume that the coordinates of the node are $(0,0,0)$ and that all links are outgoing from the node. To each group element $g \in O_{\mathrm{cube}}$ there corresponds a permutation of the links incident at the node n. We will denote the permutation corresponding to g by σ_g. We consider a projection operator $P_j : \mathrm{Inv}\left((\mathcal{H}_j)^{\otimes 6}\right) \to \mathrm{Inv}\left((\mathcal{H}_j)^{\otimes 6}\right)$ acting on an intertwiner ι in the following way:

$$(P_j \iota)^{A_1 \ldots A_6} = \frac{1}{24} \sum_{g \in O_{\mathrm{cube}}} e^{-i\Phi_j(g)} \iota^{A_{\sigma_g^{-1}(1)} \ldots A_{\sigma_g^{-1}(6)}}.$$

The image of $\mathrm{Inv}\left((\mathcal{H}_j)^{\otimes 6}\right)$ with the projection operator P_j will be denoted by $\mathcal{H}_{j,0}^{\mathrm{cube}}$:

$$\mathcal{H}_{j,0}^{\mathrm{cube}} := P_j(\mathrm{Inv}\left((\mathcal{H}_j)^{\otimes 6}\right)).$$

For homogeneous-isotropic states all nodes are labelled with the same symmetric intertwiner, an element of $\mathcal{H}_{j,0}^{\mathrm{cube}}$.

Fixing the phase

Let us notice that the group O_{cube} is isomorphic to the group S_4 of permutations of the 4 diagonals of the cube. To each element $g \in O_{\mathrm{cube}}$ we will assign a number $\mathrm{sgn}(g) = \pm 1$ equal to the sign of the corresponding permutation ν_g in S_4:

$$\mathrm{sgn}(g) := \mathrm{sgn}(\nu_g).$$

We choose the phase to be equal to

$$e^{i\Phi_j(g)} = (\mathrm{sgn}(g))^{2j},$$

where j is the spin assigned to each link. With this choice of phase the space $\mathcal{H}^{\mathrm{cube}}_{j,0}$ has at least two non-trivial elements. Both are the Livine-Speziale coherent intertwiners:

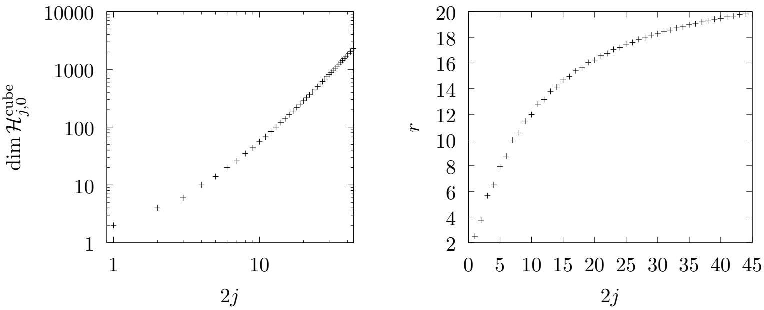
$$|j, \vec{n}_1 \ldots \vec{n}_6 > := \int_{\mathrm{SU}(2)} du \bigotimes_{i=1}^{6} \rho_j(u) |j, \vec{n}_i > .$$

First corresponds to a cube oriented in accordance with standard orientation of $\mathbb{R}^3$:

$$\vec{n}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{n}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{n}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \vec{n}_4 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \ \vec{n}_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \vec{n}_6 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Second corresponds to a cube with opposite orientation.

Symmetry reduction in the space of invariants



(a) (b)

Let us introduce a ratio

$$r(j) = \frac{\dim \mathrm{Inv}\left((\mathcal{H}_j)^{\otimes 6}\right)}{\dim \mathcal{H}^{\mathrm{cube}}_{j,0}}.$$

Section 3

Vertex Hilbert space

The kinematical Hilbert space

The kinematical Hilbert space is the Cauchy completion of the space of cylindrical functions $\mathcal{H}_{\rm kin} := \overline{\rm Cyl}$.The space of functions cylindrical with respect to a graph γ will be denoted by ${\rm Cyl}_\gamma$. The Cauchy completion of ${\rm Cyl}_\gamma$ will be denoted by $\tilde{\mathcal{H}}_\gamma$:
$\tilde{\mathcal{H}}_\gamma := \overline{{\rm Cyl}_\gamma}$.
Following we will say that $\Psi \in \tilde{\mathcal{H}}_\gamma$ is a proper element of $\tilde{\mathcal{H}}_\gamma$ if the following holds:

$$\tilde{\mathcal{H}}_{\gamma'} < \tilde{\mathcal{H}}_\gamma \Rightarrow \Psi \perp \tilde{\mathcal{H}}_{\gamma'}.$$

The subspace of $\tilde{\mathcal{H}}_\gamma$ formed by proper states will be denoted by $\mathcal{H}_\gamma$. The kinematical Hilbert space is a direct sum (of orthogonal Hilbert spaces):

$$\mathcal{H}_{\rm kin} = \bigoplus_\gamma \mathcal{H}_\gamma.$$

Vertex Hilbert space

The vertex Hilbert space is obtained by averaging the states in each space $\mathcal{H}_\gamma$ with respect to all diffeomorphisms $\mathrm{Diff}_{\mathrm{Nodes}(\gamma)}$ that act trivially on the set $\mathrm{Nodes}(\gamma)$ of nodes of the graph γ. The averaging is done by considering a map η

$$\mathcal{H}_\gamma \ni \Psi \mapsto \eta(\Psi) \in \mathrm{Cyl}^*$$

defined by

$$\eta(\Psi) := \frac{1}{N_\gamma} \sum_{[f]\in \mathrm{Diff}_{\mathrm{Nodes}(\gamma)}/\mathrm{TDiff}_\gamma} < U_f \Psi|.$$

In the formula above TDiff_γ is formed by diffeomorphisms acting trivially on $\mathcal{H}_\gamma$. The operator $U_f : \mathcal{H}_{\mathrm{kin}} \to \mathcal{H}_{\mathrm{kin}}$ is the unitary operator defined by the (analytic) diffeomorphism $f \in \mathrm{Diff}(\Sigma)$: $U_f \Psi(A) = \Psi(f^* A)$.
The vertex Hilbert space was originally defined as a completion of $\eta(\mathrm{Cyl})$ under the norm defined by the scalar product

$$\langle \eta(\Psi) | \eta(\Psi') \rangle := \eta(\Psi)(\Psi').$$

In the following we will restrict to SU(2) gauge invariant cylindrical functions $\mathrm{Cyl}^{\mathrm{Gauss}}$ and we will call the vertex Hilbert space the following space:

$$\mathcal{H}_{\mathrm{vtx}} := \overline{\eta(\mathrm{Cyl}^{\mathrm{Gauss}})}.$$

Decomposition of the vertex Hilbert space

If two graphs γ and γ' can be related by an action of a diffeomorphism fixing each element in the set $\mathrm{Nodes}(\gamma) = \mathrm{Nodes}(\gamma')$, then the spaces $\eta(\mathcal{H}_\gamma)$ and $\eta(\mathcal{H}_{\gamma'})$ are the same: $\eta(\mathcal{H}_\gamma) = \eta(\mathcal{H}_{\gamma'})$. It is therefore justified to introduce a notation:

$$\mathcal{H}_{[\gamma]} = \eta(\mathcal{H}_\gamma),$$

where $[\gamma]$ is $\mathrm{Diff}_{\mathrm{Nodes}(\gamma)}$-equivalence class of graphs with representative γ.
The space $\mathcal{H}_{\mathrm{vtx}}$ has an orthogonal decomposition:

$$\mathcal{H}_{\mathrm{vtx}} = \bigoplus_{V \in \mathrm{FS}(\Sigma)} \mathcal{H}_V,$$

where $\mathrm{FS}(\Sigma)$ is the set of finite subsets of Σ. Each state in $\mathcal{H}_V$ is invariant under the action of the group Diff_V of diffeomorphism that act trivially on the set $V \in \mathrm{FS}(\Sigma)$.
Each space $\mathcal{H}_V$ has an orthogonal decomposition:

$$\mathcal{H}_V = \bigoplus_{[\gamma] \in [\gamma(V)]} \mathcal{H}_{[\gamma]}^{\mathrm{Gauss}},$$

where

$$[\gamma(V)] := \{[\gamma] : \mathrm{Nodes}(\gamma) = V\}.$$

Section 4

Scalar constraint operator

General properties of the scalar constraint operator

- It is local:

$$\hat{C}(N) = \sum_{x\in\Sigma} N(x)\hat{C}_x, \tag{1}$$

$$\hat{C}_x\mathcal{H}_V \subset \mathcal{H}_V, \tag{2}$$

$$\hat{C}_x|_{\mathcal{H}_V} = 0, \ \text{unless } x \in V \tag{3}$$

- Each operator $\hat{C}_x$ does not change the intertwiners associated to nodes different from x.
- It is covariant:

$$U_f\hat{C}_x U_f^{-1} = \hat{C}_{f(x)} \ \text{for all } f \in \mathrm{Diff}(\Sigma).$$

Decomposition of the vertex Hilbert space

- Each operator $\hat{C}_x$ has the following splitting

$$\hat{C}_x = \sum_{\ell,\ell'} \epsilon(\dot{\ell},\dot{\ell}')\hat{C}_{x\,\ell,\ell'},$$

  where the sum is over pairs of links at the node x. The coefficient $\epsilon(\dot{\ell},\dot{\ell}')$ is zero if the vectors $\dot{\ell},\dot{\ell}'$ are colinear and 1 otherwise. Each of the operators $\hat{C}_{x\,\ell,\ell'}$ satisfies:

$$\hat{C}_{x\,\ell,\ell'}\mathcal{H}_{[\gamma]} \subset \mathcal{H}_{[\gamma_-]} \oplus \mathcal{H}_{[\gamma]} \oplus \mathcal{H}_{[\gamma_+]},$$

  where γ_+ is obtained from γ by adding a loop tangential to links ℓ,ℓ' according to the prescription which we will shortly describe and γ_- is obtained by removing such loop.

- The operator does not change representation labels of the links. The new loop or removed loop is labelled with fixed unitary irreducible representation $\rho_{(l)}$ of dimension $2l+1$. The corresponding representation space will be denoted by $\mathcal{H}_{(l)}$.

Regularization of the constraint operator

The regularization proposed by Alesci, Assanioussi, Lewandowski, Makinen:

- Consider a node n and a link ℓ_I. We will denote by $T(\ell_I, \ell_J) \geq 0$ the order of tangentiality of ℓ_I with another link ℓ_J incident at n. We will use a shorthand notation $k_{n,IJ} = T(\ell_I, \ell_J)$. The order of tangentiality $k_{n,I}$ of the link ℓ_I at the node n is

$$k_{n,I} = \max_{J \neq I} k_{n,IJ}.$$

- The special loop $\alpha_{n,IJ}$ is tangent to the two links ℓ_I and ℓ_J at the node n up to orders $k_{n,I} + 1$ and $k_{n,J} + 1$ respectively.

Regularization of the constraint operator

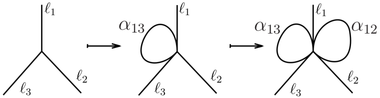
Our regularization:

- Denote by $\mathring{\gamma}$ the subgraph of γ formed by links $\mathring{\ell}_{\mathring{I}}$ which do not form loops. We will call them regular links.
- For simplicity we will assume that the regular links are pairwise non-tangential $\forall_{\mathring{I},\mathring{J}} \mathring{k}_{n,\mathring{I}\mathring{J}} = 0$ and any loop in γ is tangential to precisely 2 different regular links.
- Denote by $\kappa_{n,\mathring{I}\mathring{J}}$ the order of the wedge $(\mathring{\ell}_{\mathring{I}}, \mathring{\ell}_{\mathring{J}})$, which we will define as

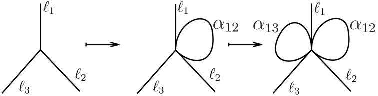
$$\kappa_{n,\mathring{I}\mathring{J}} = \max\{T(\mathring{\ell}_{\mathring{I}}, \alpha), T(\mathring{\ell}_{\mathring{J}}, \alpha) : \alpha \text{ is tangent to links } \mathring{I} \text{ and } \mathring{J} \text{ at } n\}.$$

- The special loop $\alpha_{n,\mathring{I}\mathring{J}}$ added by the constraint operator is tangent to the two links $\mathring{\ell}_{\mathring{I}}$ and $\mathring{\ell}_{\mathring{J}}$ at the node n up to orders $\kappa_{n,\mathring{I}\mathring{J}} + 1$. We will call $\kappa_{n,\mathring{I}\mathring{J}} + 1$ the order of the special loop $\alpha_{n,\mathring{I}\mathring{J}}$.
- A loop α in the graph γ has the property that there is a unique pair of links $\mathring{\ell}_{\mathring{I}}, \mathring{\ell}_{\mathring{J}}$ to which it is tangent. The special loop tangent to α coincides with the special loop $\alpha_{n,\mathring{I}\mathring{J}}$ described in the previous point.

Differences between regularizations



(a)



(b)

In our regularization:

$$T(\alpha_{13}, \ell_1) = T(\alpha_{13}, \ell_3) = T(\alpha_{12}, \ell_1) = T(\alpha_{12}, \ell_2) = 1.$$

In the regularization proposed by Alesci, Assanioussi, Lewandowski, Makinen the tangentiality orders are different for the two sequences. For sequence (a) they are:

$$T(\alpha_{13}, \ell_1) = T(\alpha_{13}, \ell_3) = 1, \quad T(\alpha_{12}, \ell_1) = 2, \quad T(\alpha_{12}, \ell_2) = 1.$$

For sequence (b) they are:

$$T(\alpha_{13}, \ell_1) = 2, \quad T(\alpha_{13}, \ell_3) = 1, \quad T(\alpha_{12}, \ell_1) = T(\alpha_{12}, \ell_2) = 1.$$

Invariant space

Let Γ be the cubical lattice. At each node n we can choose ordering of the links $\ell_1, \dots, \ell_{N_n}$. Let $\{\mathfrak{l}_n\}_{n \in \mathrm{Nodes}(\Gamma)}$ be a family of maps

$$\mathfrak{l}_n : \{(I,J) : I < J, \epsilon(\dot{\ell}_I, \dot{\ell}_J) \neq 0\} \rightarrow \mathbb{N}_n.$$

We will denote by $\mathfrak{L}(\mathfrak{l}_n)$ the total number of loops in $\mathfrak{l}_n$: $\mathfrak{L}(\mathfrak{l}_n) = \sum_{(I,J)} \mathfrak{l}_n(I,J)$. Let $\Gamma_{\mathfrak{l}}$ be a graph obtained from Γ by adding, at each node n, $\mathfrak{l}_n(I,J)$ special loops tangent at n to the links ℓ_I and ℓ_J such that the beginning of each loop is tangent to ℓ_I, the end is tangent to ℓ_J, and the orders of the loops are $1, \dots, \mathfrak{l}_n(I,J)$. The space

$$\mathcal{H}_{\Gamma}^{\mathrm{loop}} := \bigoplus_{\mathfrak{l}} \mathcal{H}_{[\Gamma_{\mathfrak{l}}]}^{\mathrm{Gauss}} \subset \mathcal{H}_V, \quad V = \mathrm{Nodes}(\Gamma)$$

is invariant.

Invariant space

Let us notice that the group of symmetries of Γ fixing each node of Γ is trivial $\#\mathrm{Sym}_\Gamma = 1$. In this case the averaging map

$$\eta : \mathcal{H}^{\mathrm{Gauss}}_{\Gamma_{\mathfrak{l}}} \to \mathcal{H}^{\mathrm{Gauss}}_{[\Gamma_{\mathfrak{l}}]}$$

is an isometry. Each of the spaces $\mathcal{H}^{\mathrm{Gauss}}_{\Gamma_{\mathfrak{l}}}$ can be further decomposed using Peter-Weyl theorem

$$\mathcal{H}^{\mathrm{Gauss}}_{\Gamma_{\mathfrak{l}}} \cong \bigoplus_{\mathcal{J}} \bigotimes_{n \in \mathrm{Nodes}(\Gamma_{\mathfrak{l}})} \mathrm{Inv}\left(\mathcal{H}_n\right).$$

Let us notice that for different total numbers of loops $\mathfrak{L}(\mathfrak{l}_n)$ in $\mathfrak{l}_n$ the spaces $\mathrm{Inv}\left(\mathcal{H}_n\right)$ are different. To make this dependence more explicit we introduce the following notation:

$$\mathcal{H}_{\mathfrak{L}(\mathfrak{l}_n)} = \mathrm{Inv}\left(\underbrace{\mathcal{H}_{(l)} \otimes \mathcal{H}^*_{(l)} \otimes \ldots \otimes \mathcal{H}_{(l)} \otimes \mathcal{H}^*_{(l)}}_{2\mathfrak{L}(\mathfrak{l}_n)} \otimes \mathcal{H}_{\mathcal{J}1} \otimes \ldots \otimes \mathcal{H}_{\mathcal{J}N}\right),$$

where $\mathcal{H}_{(l)}$ denotes the spin l representation space of SU(2): $\dim \mathcal{H}_{(l)} = 2l + 1$.

Invariant space

In summary, the invariant space $\mathcal{H}_{\Gamma}^{\rm loop}$ has a decomposition

$$\mathcal{H}_{\Gamma}^{\rm loop} = \bigoplus_{\jmath} \mathcal{H}_{\Gamma,\jmath}^{\rm loop},$$

where

$$\mathcal{H}_{\Gamma,\jmath}^{\rm loop} \cong \bigoplus_{\mathfrak{l}} \bigotimes_{n \in {\rm Nodes}(\Gamma_{\mathfrak{l}})} \mathcal{H}_{\mathfrak{L}(\mathfrak{l}_n)}.$$

Let us notice that also $\mathcal{H}_{\Gamma,\jmath}^{\rm loop}$ is invariant and that it is isomorphic to:

$$\mathcal{H}_{\Gamma,\jmath}^{\rm loop} \cong \bigotimes_{n \in {\rm Nodes}(\Gamma)} \bigoplus_{\mathfrak{l}_n} \mathcal{H}_{\mathfrak{L}(\mathfrak{l}_n)}.$$

The operator $\hat{C}_n$ acting in $\mathcal{H}_{\Gamma,\jmath}^{\rm loop}$ can be viewed as an operator in:

$$\mathcal{H}_{\Gamma,\jmath,n}^{\rm loop} := \bigoplus_{\mathfrak{l}_n} \mathcal{H}_{\mathfrak{L}(\mathfrak{l}_n)}.$$

In the diagonalization problem it is enough to find the eigenvalues of the operators $\hat{C}_n$ separately. By taking tensor product of eigenstates corresponding to different n we obtain obtain simultaneous eigenstates of the operators $\hat{C}_n$, where $n \in {\rm Nodes}(\Gamma)$.

Section 5

Symmetric states on a lattice with loops

Symmetric states on lattice with loops

The symmetric states on a lattice with loops $\Gamma_{\mathfrak{l}}$ satisfy

$$\forall_{\ell \in \mathrm{Links}(\Gamma)}\ \jmath_\ell = j.$$

We will denote by $\mathcal{H}^{\mathrm{loop}}_{\Gamma,j}, \mathcal{H}^{\mathrm{loop}}_{\Gamma,j,n}$ the spaces $\mathcal{H}^{\mathrm{loop}}_{\Gamma,\jmath}, \mathcal{H}^{\mathrm{loop}}_{\Gamma,\jmath,n}$ such that $\forall_{\ell \in \mathrm{Links}(\Gamma)}\ \jmath_\ell = j$. Let us focus on the space $\mathcal{H}^{\mathrm{loop}}_{\Gamma,j,n_0}$, $n_0 = (0,0,0)$. The diffeomorphisms $g \in O_{\mathrm{cube}}$ leave the space $\mathcal{H}^{\mathrm{loop}}_{\Gamma,j,n_0}$ invariant. Since we assumed that $\hat{C}_{n_0}$ is covariant with respect to diffeomorphisms, the operator

$$P^{\mathrm{cube}}_j = \frac{1}{24} \sum_{g \in O_{\mathrm{cube}}} e^{-i\Phi_j(g)} U_g$$

commutes with the operator $\hat{C}_{n_0}$:

$$P^{\mathrm{cube}}_j \hat{C}_{n_0} = \hat{C}_{n_0} P^{\mathrm{cube}}_j.$$

As a result the space $\mathcal{H}^{\mathrm{cube}}_j = \mathrm{Im}(P^{\mathrm{cube}}_j)$ is also invariant under the action of $\hat{C}_{n_0}$. The space of symmetric states $\mathcal{H}^{\mathrm{sym}}_j$ is the diagonal subspace of $\mathcal{H}^{\mathrm{loop}}_{\Gamma,j} \cong \bigotimes_{n \in \mathrm{Nodes}(\Gamma)} \bigoplus_{\mathfrak{l}_n} \mathcal{H}_{\mathfrak{L}(\mathfrak{l}_n)}$ corresponding to $\mathcal{H}^{\mathrm{cube}}_j$, i.e. a subspace spanned by vectors

$$\bigotimes_{n \in \mathrm{Nodes}(\Gamma)} v_n, \ \text{where } v_n = v \in \mathcal{H}^{\mathrm{cube}}_j.$$

Truncation

Our goal will be to look for the eigenstates numerically. To this end we will truncate the invariant Hilbert space $\mathcal{H}^{\mathrm{loop}}_{\Gamma,j,n}$ to loop configurations $\mathfrak{l}_n$ with at most L loops ($\mathfrak{L}(\mathfrak{l}_n) \leq L$), i.e. we will consider a space

$$\mathcal{H}^{\mathrm{loop}}_{\Gamma,j,n,L} = \bigoplus_{\mathfrak{l}_n : \mathfrak{L}(\mathfrak{l}_n) \leq L} \mathcal{H}_{\mathfrak{L}(\mathfrak{l}_n)}.$$

Let us notice that this truncation is compatible with the symmetry reduction. This means that the projection operator P^{cube}_j leaves the subspace $\mathcal{H}^{\mathrm{loop}}_{\Gamma,j,n,L}$ invariant. As a result we introduce symmetry reduced truncated space:

$$\mathcal{H}^{\mathrm{cube}}_{j,L} = P^{\mathrm{cube}}_j(\mathcal{H}^{\mathrm{loop}}_{\Gamma,j,n,L}).$$

For $j = \frac{1}{2}$ this truncation can be viewed as a truncation in the volume (see M.K. Class.Quant.Grav. 38 (2021) 19, 195023).

The space $\mathcal{H}^{\mathrm{cube}}_{j,L}$

The action of the diffeomorphisms $g \in O_{\mathrm{cube}}$ on the spin-network states induces an action in $\mathcal{H}^{\mathrm{loop}}_{\Gamma,j,n_0} = \bigoplus_{\mathfrak{l}_{n_0}} \mathcal{H}_{\mathfrak{L}(\mathfrak{l}_{n_0})}$:

$$U_g(\mathfrak{l}_{n_0}, \iota_{n_0}) = (g \cdot \mathfrak{l}_{n_0}, R(g)\iota_{n_0}).$$

The action of g on $\mathfrak{l}$ is the following:

$$(g \cdot \mathfrak{l})(I, J) = \begin{cases} \mathfrak{l}(g^{-1}(I), g^{-1}(J)), & \text{if } g^{-1}(I) < g^{-1}(J) \\ \mathfrak{l}(g^{-1}(J), g^{-1}(I)) & \text{otherwise.} \end{cases}$$

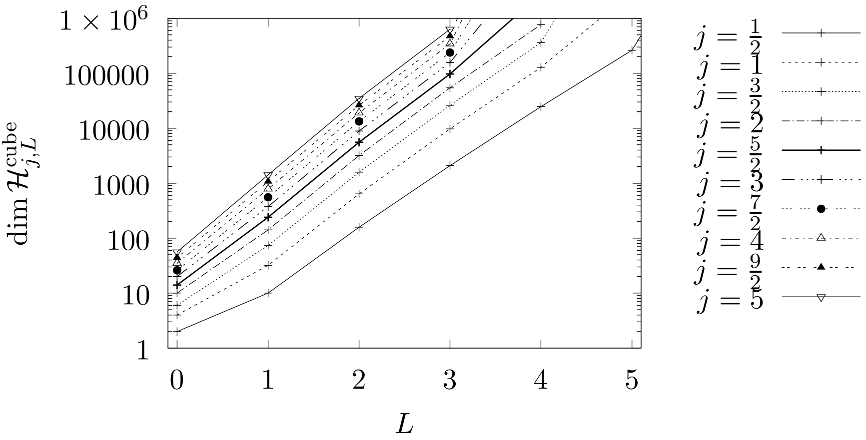
The action of g on an intertwiner ι is:

$$R(g)\iota = (-1)^{2l\,F(g,\mathfrak{l})} \tilde{\sigma}_g \cdot \iota.$$

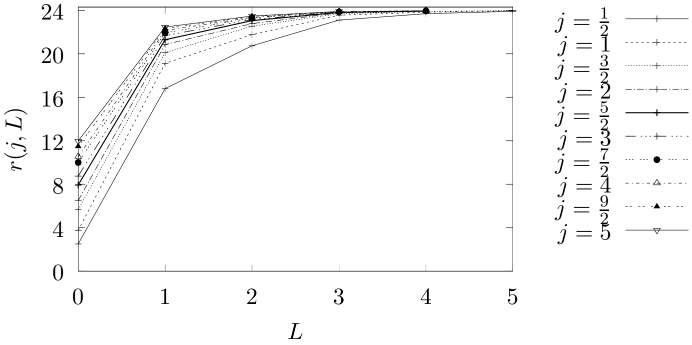
Knowing the action of O_{cube} on $\mathcal{H}^{\mathrm{loop}}_{\Gamma,j,n_0}$ we can write the action of the projection operator P^{cube}_j on a vector $(\mathfrak{l}, \iota) \in \mathcal{H}^{\mathrm{loop}}_{\Gamma,j,n_0}$:

$$P^{\mathrm{cube}}_j(\mathfrak{l}, \iota) = \frac{1}{24} \sum_{g \in O_{\mathrm{cube}}} e^{-i\Phi_j(g)} (g \cdot \mathfrak{l}, R(g)\iota).$$

Dimension of $\mathcal{H}^{\text{cube}}_{j,L}$



The rate of symmetry reduction



The rate of symmetry reduction $r(j, L)$ will be defined to be the ratio of the dimensions of the spaces $\mathcal{H}^{j,L}_{\text{loops}}$ and $\mathcal{H}^{j,L}_{\text{cube}}$.

$$r(j, L) = \frac{\dim \mathcal{H}^{\text{loop}}_{\Gamma, j, n_0, L}}{\dim \mathcal{H}^{\text{cube}}_{j,L}}.$$

Section 6

Summary

Summary & Discussion

- The restriction to homogeneous-isotropic sector of Loop Quantum Gravity that we proposed leads to substantial reduction of the degrees of freedom. From the plethora of spin-network states defined on cubical lattice Γ, it restricts us to lattices with links labelled with the same spin. Moreover, it allows us to restrict the problem of diagonalization of operators $\hat{C}_n$, $n \in \mathrm{Nodes}(\Gamma)$ to single operator $\hat{C}_{n_0}$ at a fixed node n_0.
- The symmetry group acts non-trivially in the intertwiner spaces, leading, after averaging, to further reduction of the degrees of freedom. We have noticed that the averaging should include non-trivial phase factor to accommodate Livine-Speziale coherent intertwiners. We found, that after a truncation of the relevant Hilbert space to spin networks with not more than L loops at each node, the symmetry reduction leads to almost 24 times smaller subspaces of intertwiners.
- Most diagonalization algorithms have complexity close to $\mathcal{O}(n^3)$, where n is the rank of the matrix. This means around 10^4 speedup compared to naive approach. The computing centre were our numerical calculation were done has $3 \cdot 10^4$ cores. This roughly speaking means that instead of running a diagonalization program using all resources of our computing centre, we could just run the program on our laptop.

Outlook

- We hope that this speedup will allow us to find eigenstates of the operators $\hat{C}_x$. We realize that the truncation needs further study. Our result from [Kisielowski M, Class.Quant.Grav. 38 (2021) 19, 195023] suggests that it has an interpretation as a truncation in the volume. We plan to look for its justification using numerical calculations: by varying the truncation and investigating if some of the eigenvalues converge when the number of loops (or volume cut-off) increases.
- Knowing some eigenstates we could construct coherent states analogous to coherent states in LQC and investigate their stability under quantum dynamics generated by the physical Hamiltonian.
- Let us also note that although the analysis is focused on finding eigenstates of $\hat{C}_x$ in the homogeneous-isotropic sector, the results have important impact on the spin-foam amplitudes we proposed in [Kisielowski M and Lewandowski J 2019 Class. Quant. Grav. 36 075006]. Our analysis in particular implies that a history of a state in $\mathcal{H}^j_{\mathrm{cube}}$ is described by states in $\mathcal{H}^j_{\mathrm{cube}}$. This means that in order M of the expansion, the complexity of the problem gets roughly 24^{M-1} times smaller due to symmetry reduction.

Thank you for your attention!