Self-Dual Gravity

or

On a class of (modified) gravity theories
motivated by Plebanski's self-dual formulation of GR

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Terminological Remark

Used to refer to this class of theories as "non-metric gravity".

Turned out to be perfectly metric, albeit unusual.

Hence more correct to use the term (modified) "self-dual gravity" (SDGR).

Not to be confused with SDGR –
The Salisbury Diocesan Guild of Ringers [http://www.sdgr.org.uk/]
that was founded on 14th September 1882 under the patronage of
the Bishop and the presidency of the Right Honorable the Earl Nelson.

Questions you may be asking

Why should I be interested in modified gravity?

What's wrong with the usual one?

We don't understand even the usual GR as well as we would like.

Now people make life even more confusing with all these modifications: Bekenstein (TeVeS), Moffat (SVeTe), brane worlds, Banados (Born-Infeld), etc.

What's wrong with GR?

What's wrong with GR?

Many things!

- Einstein's general relativity
  is non-renormalizable when quantized.
  Was never designed to become a quantum theory -
  as "classical" as it gets.

- Unless you add dark matter and dark energy,
GR is not consistent with astrophysical observations.
Einstein challenged by "dark" forces.

Yes, but...

May be we can quantize GR non-perturbatively?
or
Loop Quantum Gravity: We CAN quantize GR non-perturbatively.

Non-renormalizability is not about non-applicability of perturbative quantization, it is a physical property of the theory: new physics is around the corner. Should be visible in any quantization scheme, when sufficiently developed.

Non-renormalizability does tell us that the UV physics of gravity is not described by GR, not even by quantum GR. It is some other (quantum) theory.

Yes, but...

How can a new theory designed to be the UV completion of GR have anything to do with astrophysics? Gravity on large scales is certainly not quantum!

The classical theory whose quantum version describes gravity in the deep UV regime does not have to coincide with GR on large scales.

Thus, the class of theories of this talk may or may not be UV complete (unknown), but there is certainly enough room for them be different from GR on large scales.

SDGR

Describe geometry of a four-manifold M by something else than the metric.

Theorem: [Atiyah-Hitchin-Singer'78] A metric g is Einstein iff the restriction of the Levi-Civita connection $\Gamma(g)$ to the bundle Λ^+ of self-dual two-forms is self-dual.

Plebanski'78: Einstein's general relativity admits a reformulation in which the main dynamical object is a triple of self-dual two-forms, and the metric appears as a derived concept.

Bengtsson, Peldan'91: A large class of (modified) gravity theories is envisaged in the Ashtekar's Hamiltonian formulation.

KK'06 hep-th/0611182: The Lagrangian formulation of this new class of gravity theories is found.

Hodge operator (of $g_{\mu\nu}$)

Λ^k — k-forms on a 4-manifold M.

$$* : \Lambda^2 \to \Lambda^2$$
$$*B_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma} B_{\rho\sigma}.$$

Here $\epsilon_{\mu\nu\rho\sigma} \in \Lambda^4$ — volume form of $g_{\mu\nu}$.

Properties

•

$$**=\begin{cases}+1 & \text{Riemannian}\\ -1 & \text{Lorentzian}\end{cases}$$

• $g_{\mu\nu} \to \Omega g_{\mu\nu}$ conformal transformation

$$\epsilon_{\mu\nu}{}^{\rho\sigma} \qquad \text{invariant}$$

(continuation)

Theorem: Let $g, \hat{g}$ define $*, \hat{*} : * = \hat{*}$. Then $\exists \Omega : \hat{g} = \Omega g$.
Proof.

$$\hat{\epsilon}_{\mu\nu}{}^{\alpha\beta} = \epsilon_{\mu\nu}{}^{\alpha\beta}.$$

But $\exists \Omega$:

$$\hat{\epsilon}_{\rho\sigma\alpha\beta} = \Omega^2 \epsilon_{\rho\sigma\alpha\beta}.$$

Contracting the two get:

$$\hat{g}_{\mu[\rho}\hat{g}_{\sigma]\nu} = \Omega^2 g_{\mu[\rho}g_{\sigma]\nu}.$$

Work in coordinates x^i in which $\hat{g}_{\mu\nu}$ is diagonal. Prove that $g_{\mu\nu}$ is also diagonal in the same coordinate system. Then

$$\hat{g}_{ii}\hat{g}_{jj} = \Omega^2 g_{ii}g_{jj} \Longrightarrow$$

$$\hat{g}_{ii} = \pm\Omega g_{ii}.$$

(continuation)

Corollary:

Hodge Operator = Metric/Conformal Transformations

(continuation)

Lemma: One knows $*$ if one knows its eigenspace ${}^{+}\Lambda^2$ of self-dual two-forms.
Proof: If one knows ${}^{+}\Lambda^2$, can get ${}^{-}\Lambda^2$ as the complement of ${}^{+}\Lambda^2$ in Λ^2 wrt the (conformal) metric on Λ^2 given by the wedge product. If knows ${}^{+}\Lambda^2$, ${}^{-}\Lambda^2$, then

$$B = B_{sd} + B_{asd},$$

$$*B = B_{sd} - B_{asd} \qquad (*B = i(B_{sd} - B_{asd}))\,.$$

Conformal metrics and self-dual two-forms

Corollary:

Space of conformal metrics at a point $\mathrm{GL}(4,\mathbb{R})/\mathrm{O}(4)$	$=$	Grassmanian of 3-planes ${}^{+}\Lambda^2 \subset \Lambda^2$ $\mathrm{SO}(3,3)/\mathrm{SO}(3)\times\mathrm{SO}(3)$

Plebanski formulation of GR: preliminaries

Let $E \to M$ be a principal $\mathrm{SO}(3)$ bundle over M. Let A be a connection in the associated bundle. Locally $A \in \Lambda^1 \otimes \mathfrak{so}(3)$. Let B be a Lie-algebra valued two-form: $B \in \Lambda^2 \otimes \mathfrak{so}(3)$.

Remark: [spacetime metric] Given B, there is a natural metric g_B defined as follows. Choose a basis τ^i in $\mathfrak{so}(3)$. Decompose $B = B^i \tau^i$. Declare $\mathrm{Span}(B^1, B^2, B^3)$ to be the space ${}^+\Lambda^2$ of self-dual two-forms of some metric g_B. This defines g_B up to conformal factor. Define the later via:

$$(vol)_B = \frac{1}{3}\mathrm{Tr}(B \wedge B).$$

Note that g_B may not be Riemannian and maybe singular.

(continuation)

Remark: [fibre metric] Two-form field B defines a metric h_B in the fibres via

$$(B \wedge_{\otimes} B) = h_B (vol)_B \in \mathfrak{so}(3) \otimes \mathfrak{so}(3) \otimes \Lambda^4.$$

Does not have to coincide with the $\mathrm{SO}(3)$-invariant Killing-Cartan fiber metric. If h_B is positive definite then g_B is a non-singular Riemannian metric.

Remark: [another description] Given a metric g, there is a natural bivector-valued two-form:

$$G^{\rho\sigma}_{\mu\nu} := g^{\rho}_{[\mu} g^{\sigma}_{\nu]}.$$

The space of bivectors $X^{\mu\nu}$ is isomorphic to the $\mathfrak{so}(4)$ Lie-algebra. Thus, the self-dual part of $G^{\rho\sigma}_{\mu\nu}$ is a $\mathfrak{so}(3)$-valued two-form. A very special one, as the fiber metric h_B in this case coincides with the Killing-Cartan one.

Connection

Lemma: Given B, there is a unique connection $A(B) : D_{A(B)}B = 0$. Explicitly:

$$A^i(B) = \frac{1}{\det(B)} * (B^j \wedge *(B^i \wedge *(dB^j))). \tag{1}$$

In general, it is *not* equal to the restriction of the Levi-Civita connection to the bundle of g_B-self-dual two-forms. Only when h_B is the $\mathrm{SO}(3)$-invariant fiber metric.

Remark: To write (1) a metric $g \in [g_B]$ was used. The expression is invariant under $g \to \Omega^2 g$.

Plebanski formulation: main theorem

Theorem: [Plebanski'78] Let $E \to M$ be an $\mathrm{SO}(3)$ bundle (of an appropriate second Chern class), B a Lie-algebra valued two-form such that h_B coincides with the Killing-Cartan $\mathrm{SO}(3)$-invariant metric in $\mathfrak{so}(3)$ and $\exists \Psi \in \mathrm{End}(\mathfrak{so}(3))$:

$$F(A(B)) = \Psi(B), \tag{2}$$

where $F(A)$ is the curvature of A. Then the metric g_B defined by B is Einstein.

(continuation)

Remark: The converse statement is a part of the Atiyah-Hitchin-Singer theorem: for an Einstein metric the curvature of the restriction of the Levi-Civita connection to ${}^{+}\Lambda^2$ is self-dual.

Remark: The condition (2) is as natural in the context of Lie-algebra-valued two-forms as is the Einstein condition in the context of symmetric rank two tensors.

Towards an action functional

Consider the following “natural” action:

$$S_0 = \int B^i \wedge F^i(A). \tag{3}$$

Varying this wrt A^i get:

$$D_A B^i = 0, \tag{4}$$

which has a unique solution for A^i given B^i. But can’t vary wrt B^i, because has to satisfy the constraint $h_B^{ij} = \delta^{ij}$. Need to add a constraint term.

General relativity in Plebanski formulation

The action with the constraint:

$$S' = \int B^i \wedge F^i(A) - \frac{1}{2}\Psi^{ij} B^i \wedge B^j. \tag{5}$$

Here $\Psi^{ij} : \mathrm{Tr}(\Psi) = 0$ are Lagrange multipliers. Variation wrt the Lagrange multipliers gives:

$$B^i \wedge B^j = \delta^{ij} (vol)_B. \tag{6}$$

Einstein equations

Variation wrt B^i gives:

$$F^i(A) = \Psi^{ij} B^j. \tag{7}$$

Not the most general action, for does not allow scalar curvature. The most general one is:

$$S = \int B^i \wedge F^i(A) - \frac{1}{2}\left(\Psi^{ij} - \frac{1}{3}\Lambda\delta^{ij}\right) B^i \wedge B^j, \tag{8}$$

Remark: Ψ^{ij} gets identified with the self-dual part W^+ of the Weyl curvature, and 4Λ with the scalar curvature s.

Physics remark

Interested in this formulation because gravity becomes constrained BF theory (topological). We know how to "quantize" BF, so hope to get some insight into quantization of GR. Spin foam models.

Ashtekar's Hamiltonian formulation of GR is the 3+1 decomposition of the Plebanski action.

Beyond Einstein

We have been keeping $h_B : B^i \wedge B^j = h_B^{ij}(vol)_B$ fixed to the $\mathrm{SO}(3)$-invariant metric, and thus freezing some degrees of freedom described by B^i. Can one allow h_B to become dynamical?

Still want to keep

$$F^i(A) = (\Psi^{ij} - \frac{1}{3}\Lambda\delta^{ij})B^j. \tag{9}$$

These are 18 equations, can no longer keep Ψ^{ij} arbitrary. Set $\Psi^{ij} = \Psi^{ij}(h_B), \Lambda = \Lambda(h_B)$.

Action functional

The corresponding action: [hep-th/0611182]

$$S = \int B^i \wedge F^i(A) - \frac{1}{2}\left(\Psi^{ij} - \frac{1}{3}\delta^{ij}\phi(\Psi)\right) B^i \wedge B^j, \tag{10}$$

with Ψ^{ij} still traceless. Now a variation wrt Ψ^{ij} gives:

$$h_B^{ij} = \delta^{ij} + \frac{\partial \phi}{\partial \Psi^{ij}} \Longrightarrow \Psi^{ij} = \Psi^{ij}((h_B)_{\mathrm{tr-free}}). \tag{11}$$

Remarks

Remark: The essence of the modification is in removing a natural, but too restrictive condition on h_B, and requiring:

$$F(A) = \Psi_B(B), \tag{12}$$

where $\Psi_B \in \mathrm{End}(\mathfrak{so}(3))$ depends on the h_B part of B.

Remark: The modification is parameterized by a function of two arguments $\phi(\Psi) = \phi(\mathrm{Tr}(\Psi)^2, \mathrm{Tr}(\Psi)^3)$. A theory from this class is specified by an infinite number of parameters — coefficients in the Taylor expansion of ϕ.

Remark: The described modification is the most general possible one (compatible with the diffeomorphism and $\mathrm{SO}(3)$ invariance of the action) that leads to EOM with not higher than second derivatives.

Renormalizability motivation

Plebanski action (8) is NOT the most general one compatible with all the symmetries and of the same field content. After G is absorbed into the fields, the mass dimensions are:

$$[A] = 1, \quad [B] = 2, \quad [\Psi] = 0. \tag{13}$$

It is thus obvious that all powers of Ψ can appear.

Thus, in addition to the term $\Psi^{ij} B^i \wedge B^j$ need to add the terms of the form

$$\frac{1}{2}(\Psi^{k_1})^{ij}(\mathrm{Tr}(\Psi^2))^{k_2}\ldots(\mathrm{Tr}(\Psi^n))^{k_n} B^i \wedge B^j \tag{14}$$

The theory does seem to be as non-renormalizable as in the usual perturbative quantum gravity. Usual case: dimensionfull Newton's constant; our case - a field Ψ of mass dimension zero.

Other terms

Clear that all powers of Ψ can get generated also in front of BF and FF terms. The renormalized action with all these counterterms is:

$$\mathcal{L} = \frac{1}{2}\bar{X}(\Psi)^{ij} F^i \wedge F^j + \bar{Y}(\Psi)^{ij} B^i \wedge F^j + \frac{1}{2}\bar{Z}(\Psi)^{ij} B^i \wedge B^j, \tag{15}$$

where $\bar{X}(\Psi), \bar{Y}(\Psi), \bar{Z}(\Psi)$ are all tensors, polynomials in Ψ and its traces. The coefficients of these polynomials are undetermined. Infinite number of them, seemingly no predictive power. Usual non-renormalizability? Not quite.

Field B redefinition

Can redefine the field $B \to B + H(\Psi)F(A)$ to get rid of the $F^i F^j$ term. Can then "rescale" the field B to map the $B^i F^j$ term into its canonical form. After this B field redefinitions one gets

$$\mathcal{L} = \tilde{B}^i \wedge F^j + \frac{1}{2}\tilde{\Psi}(\Psi)^{ij}\tilde{B}^i \wedge \tilde{B}^j, \tag{16}$$

where

$$\tilde{\Psi}(\Psi) = (Y(\Psi)^T Z(\Psi)^{-1} Y(\Psi) - X(\Psi))^{-1}. \tag{17}$$

Field Ψ redefinition

The whole effect of the considered counterterms is to replace the curvature field Ψ^{ij} by a non-trivial, depending on many new parameters (coupling constants) functional $\tilde{\Psi}^{ij}(\Psi)$.

Rewrite:

$$\tilde{\Psi}(\Psi)^{ij} = \Phi^{ij}(\Psi) - \delta^{ij}\frac{1}{3}\phi(\Psi), \tag{18}$$

where $\Phi^{ij}(\Psi)$ is the traceless part of $\tilde{\Psi}$. The field Φ^{ij} just replaces the original field Ψ^{ij} after the renormalization!

Renormalized action

The effect of the considered counterterms is in replacing the bare curvature field Ψ by the renormalized one Φ, and in appearance in the action of a new "trace" term:

$$\int_M B^i \wedge F^j + \frac{1}{2}\left(\Phi^{ij} - \frac{\Lambda}{3}\delta^{ij} - \frac{1}{3}\delta^{ij}\phi(\Phi)\right) B^i \wedge B^j, \tag{19}$$

Still non-renormalizable in the strict sense of the word (as still an infinite number of undetermined constants).

Remark: Note that Λ is just the constant part of $\phi(\Phi)$. The whole effect of the "renormalization" we have considered is to replace the cosmological constant by a "cosmological function". We get the theory (10).

Renormalizable?

If could prove that the counterterms considered are the only ones that appear would prove that the theory (10) is *renormalizable* in the sense that the form of the action is unchanged under the renormalization group flow.

Would be an extremely strong statement. One could then hope that $\phi^*(\Psi) = \lim_{\mu\to\infty} \phi_\mu(\Psi)$ exists. The UV completion of gravity would then be a (quantum) theory from this class with $\phi = \phi^*$.

May be not...

Unfortunately, terms of the form:

$$\Psi \ldots \Psi (D\Psi)^4, \quad \Psi \ldots \Psi (D\Psi)^2 B, \quad \Psi \ldots \Psi (D\Psi)^2 F$$

can appear as well.

In spite of this, conjectured in hep-th/0611182 that this class IS renormalizable.

Motivations

- Most optimistic scenario.
- Counterterms containing $D\Psi$ drastically change the character of the theory - higher derivative EOM (and more DOF).
- The "toy" theory

$$S[A,\Psi] = \int_M (\Psi^{-1})^{ij} F^{+\,i} \wedge F^{+\,j}, \tag{20}$$

where F^+ is the self-dual part of F is renormalizable in the sense described, at least to one loop.

Hamiltonian formulation

The phase space of the new class of theories is very similar to that of GR in the Ashtekar's Hamiltonian formulation: $\{A^i_a, \tilde{\sigma}^{i\,a}\}$.

Constraints are unmodified, except for the Hamiltonian constraint that becomes:

$$\epsilon^{ijk}\tilde{\sigma}^{i\,a}\tilde{\sigma}^{j\,b}F^k_{ab} + \underset{\sim}{\epsilon}_{abc}\tilde{\sigma}^{ai}\tilde{\sigma}^{bj}\tilde{\sigma}^{ck}\phi((\Psi)_{tr-free}) \approx 0, \tag{21}$$

where

$$\Psi^{ij} = \frac{F^i_{ab}\epsilon^{jkl}\tilde{\sigma}^{ak}\tilde{\sigma}^{bl}}{\underset{\sim}{\epsilon}_{abc}\tilde{\sigma}^{ai}\tilde{\sigma}^{bj}\tilde{\sigma}^{ck}}. \tag{22}$$

Constraints compactly:

$$\Psi^{ij} \approx \Psi^{(ij)}, \qquad \mathrm{Tr}(\Psi) + \phi((\Psi)_{tr-free}) \approx 0. \tag{23}$$

Degrees of Freedom

The constraint algebra remains the first class [arXiv:0711.0090]. Theories from this class thus have two propagating DOF, as GR.

This makes them unlike all other modifications of gravity considered in the literature. Not so strongly modified after all!

“Cosmological function”

The “only” modification as compared to Plebanski theory is that

$$\Lambda \rightarrow \phi(\Psi), \tag{24}$$

i.e., the cosmological constant became “curvature” dependent.

The cosmological constant problem may be resolved in this gravity scenario by having $\phi(1/l_p^2) \sim 1/l_p^2$, but then ϕ decreasing and taking a small value $\phi(0) = \Lambda_{obs}$ of relevance in cosmology.

Gravity as a sigma-model

Can rewrite the action in terms of the metric g_B (determined by B^i) and a triple of self-dual (with respect to g_B) two forms B^i.

$$S[B] = \int_M \mathrm{vol}_B \left(G^{ij}(dB^i, dB^j) - \frac{1}{2} V(M_B) \right), \quad (25)$$

where G^{ij} is a certain metric (constructed from B^i) in the space $\Omega^3 \otimes \mathfrak{su}(2)$. Conformally invariant action!

The potential $V(M)$ is the Legendre transform of $\phi(\Psi)$ and

$$M_B^{ij} = h_B^{ij} - \delta^{ij} = \left(\frac{\partial \phi(\Psi)}{\partial \Psi^{ij}} \right)_{tr-free} \quad (26)$$

is the traceless part of the fibre metric h_B.

Summary

- There is an infinitely large class of self-dual gravity theories, parameterized by a function $\phi(\mathrm{Tr}(\Psi)^2, \mathrm{Tr}(\Psi)^3)$ - the "cosmological function". Einstein theory is a member of this class corresponding to $\phi(\Psi) = \Lambda$. All these theories have the same number of DOF as GR.

- This class may be closed under the renormalization group flow (renormalizable).

- Physics that determines the form of $\phi(\Psi)$ is not yet understood, but whatever mechanism takes $\phi(\Psi)$ from Planckian curvatures to zero (small) cosmological curvatures also modifies gravity.

- In addition to the spacetime metric, these theories allow for a non-trivial fibre metric. In quantum theory fluctuates independently of the spacetime metric, leading to a potentially very different behaviour.