Quantization of diffeomorphism invariant theories of connections with a non-compact structure group—an example

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Outline

1. Problem: non-compact gauge groups
2. Solution: projective techniques by Kijowski
3. Application: quantization of a 'toy theory'

Motivation

The main motivation is a wish to quantize GR using *complex* Ashtekar variables whose gauge group is $SL(2, \mathbb{C})$.

GR can be quantized using *real* Ashtekar-Barbero variables (gauge group $SU(2)$) but:

- it lacks Lorentz invariance;
- there is a quantization ambiguity (Immirzi parameter);
- the application of the real variables makes the scalar constraint much more complicated.

$SL(2,\mathbb{C})$ as a gauge group

Canonical quantization of GR expressed in terms of the complex Ashtekar variables faces two obstacles:

1. $SL(2,\mathbb{C})$ is *non-compact*:

   this makes the task of constructing the *space of quantum states* for the theory very difficult;

2. $SL(2,\mathbb{C})$ is *complex*:

   it implies the existence of some complicated constraints called *reality conditions* which have to be included in the structure of the resulting quantum theory.

We are going to focus solely on the *non-compactness* problem.

Space of quantum states built via inductive techniques

Consider a principle bundle $P(\Sigma, G)$ and connections on it.

Given graph γ embedded in Σ, define a Hilbert space

$$\mathcal{H}_\gamma := L^2(\mathcal{A}_\gamma, d\mu_\gamma),$$

where $\mathcal{A}_\gamma \cong G^{\text{number of edges of } \gamma}$ is a space of holonomies along the edges of γ.

For every pair $\gamma' \geq \gamma$ define an embedding

$$p_{\gamma'\gamma} : \mathcal{H}_\gamma \mapsto \mathcal{H}_{\gamma'}$$

such that $\{\mathcal{H}_\gamma, p_{\gamma'\gamma}\}$ is an inductive family. Then

Space of quantum states := inductive limit of $\{\mathcal{H}_\gamma, p_{\gamma'\gamma}\}$.

Inductive techniques fail in the non-compact case

So far we know how to define embeddings $\{p_{\gamma'\gamma}\}$ only when the gauge group G is *compact*.

This construction employs the fact that the constant function I of the value equal to 1 is square-integrable over any G^N. If

$$\gamma' := \gamma \cup \gamma_0 \geq \gamma, \quad \gamma \cap \gamma_0 = \emptyset$$

then

$$\mathcal{H}_\gamma \ni \Psi \mapsto p_{\gamma'\gamma}(\Psi) = \Psi \otimes I \in \mathcal{H}_\gamma \otimes \mathcal{H}_{\gamma_0} = \mathcal{H}_{\gamma'}.$$

In the *non-compact* case this particular construction breaks down and we *do not* know how to use the inductive techniques to build the space of quantum states.

Systems and subsystems

A solution to the problem comes from [Kijowski, 1977]. Given a pair

$$\gamma' \geq \gamma$$

of graphs, Kijowski treats them as a pair

system—subsystem

- with the configuration spaces $\mathcal{A}_{\gamma'}$ and $\mathcal{A}_{\gamma}$, respectively.
- and the Hilbert spaces $\mathcal{H}_{\gamma'}$ and $\mathcal{H}_{\gamma}$, resp.

If so, then it is *not natural* to look for the embedding

$$\mathcal{H}_{\gamma'} \xleftarrow{p_{\gamma'\gamma}} \mathcal{H}_{\gamma}$$

since quantum mechanics does not provide any natural embedding of the Hilbert space of the subsystem into the Hilbert space of the system.

Mixed states and partial trace projection

Thus, given a pair system—subsystem

$$\gamma' \geq \gamma,$$

instead of the embedding

$$\mathcal{H}_{\gamma'} \xleftarrow{p_{\gamma'\gamma}} \mathcal{H}_{\gamma}$$

we should employ a projection

$$\mathcal{D}_{\gamma'} \xrightarrow{\pi_{\gamma\gamma'}} \mathcal{D}_{\gamma}$$

from the space $\mathcal{D}_{\gamma'}$ of *mixed* states of the system γ' onto the space $\mathcal{D}_{\gamma}$ of *mixed* states of its subsystem γ.

The projection $\pi_{\gamma\gamma'}$ is defined by so-called *partial trace*.

Space of quantum states built via projective techniques

Consequently, we obtain a *projective* family

$$\{\mathcal{D}_\gamma, \pi_{\gamma\gamma'}\}$$

and define the *space of quantum states*

$$\mathcal{D} := \text{projective limit of } \{\mathcal{D}_\gamma, \pi_{\gamma\gamma'}\}.$$

Note that $\mathcal{D}$ is *not* a Hilbert space, but is a *convex* set!

Question: does $\mathcal{D}$ correspond to the space of positive linear functionals on a C^*-algebra?

Algebra of quantum observables

Let $\mathcal{B}_\gamma$ be the C^*-algebra of bounded operators on $\mathcal{H}_\gamma$. Then $\mathcal{D}_\gamma$ coincides with the set of (normal) states on $\mathcal{B}_\gamma$.

Consequently, given $\gamma' \geq \gamma$, there is an embedding (dual to the projection $\pi_{\gamma\gamma'}$)

$$\pi^*_{\gamma\gamma'} : \mathcal{B}_\gamma \to \mathcal{B}_{\gamma'}$$

such that $\{\mathcal{B}_\gamma, \pi^*_{\gamma\gamma'}\}$ is an inductive family.

The states in $\mathcal{D}$ can be naturally evaluated on the inductive limit $\mathcal{B}$ of the family, hence we call $\mathcal{B}$ the C^*-algebra of *quantum observables* [Kijowski, 1977].

Thus the resulting quantum model consists of the spaces $\mathcal{D}$ and $\mathcal{B}$ *without* any Hilbert space!

'Toy theory'—Lagrangian formulation

Let $\mathcal{M} \times \mathbb{R}$ be a principle bundle over 4-dimensional 'spacetime' $\mathcal{M}$ with the additive group $\mathbb{R}$ as the structure group.

The 'toy theory' is defined by the following action [Okołów, 2006]:

$$S[A, \sigma, \Psi] := \int_{\mathcal{M}} \sigma \wedge F - \frac{1}{2}\Psi\, \sigma \wedge \sigma,$$

where

- A is a connection on the bundle and $F = dA$ is its curvature form;
- σ is a two-form valued in the Lie algebra of $\mathbb{R}$;
- Ψ is a Lagrange multiplier (a real function on $\mathcal{M}$).

Motivation—Plebański action for GR

Let $\mathcal{M} \times SL(2, \mathbb{C})$ be a principle bundle.

Plebański action is defined as follows:

$$S[A^A{}_B, \Sigma^A{}_B, \Psi_{ABCD}] = \int_{\mathcal{M}} \Sigma^{AB} \wedge F_{AB} - \frac{1}{2} \Psi_{ABCD} \, \Sigma^{AB} \wedge \Sigma^{CD},$$

where

- $A^A{}_B$ is a connection on the bundle and $F^A{}_B$ is its curvature form;
- $\Sigma^A{}_B$ is a two-form valued in the Lie algebra of $SL(2, \mathbb{C})$;
- $\Psi_{ABCD} = \Psi_{(ABCD)}$ is a Lagrange multiplier (a symmetric spinor field on $\mathcal{M}$).

'Toy theory'—Hamiltonian formulation

Assume that $\mathcal{M} = \Sigma \times \mathbb{R}$ and that coordinates (x^i, x^0) are adapted to the decomposition.

The Hamiltonian is of the form:

$$H[\tilde{E}^i, A_i, C, N^i] = -\int_\Sigma d^3x \, (C\partial_i \tilde{E}^i + N^i \tilde{E}^j F_{ij}),$$

where

- $\tilde{E}^i := \frac{1}{2}\tilde{\epsilon}^{ijk}\sigma_{jk}$ is the momentum variable;
- A_i is the configuration variable;
- C, N^i are Lagrange multipliers.

Note that

- the Hamiltonian is a sum of Gauss and vector constraints;
- there is *no* scalar constraint!

Interpretation of the 'toy theory'

Let

- the fields $(\tilde{E}^{iA}{}_B, A_j{}^C{}_D)$ on Σ be the *complex* Ashtekar variables valued in the Lie algebra $sl(2,\mathbb{C})$;
- $\tilde{\mathbb{R}}$ be a subgroup of $SL(2,\mathbb{C})$ isomorphic to $\mathbb{R}$.

We restrict

- the phase space of GR to fields $(\tilde{E}^{iA}{}_B, A_j{}^C{}_D)$ valued in the Lie algebra of $\tilde{\mathbb{R}}$;
- the gauge group $SL(2,\mathbb{C})$ to $\tilde{\mathbb{R}}$.

The restricted theory just defined coincides with the 'toy theory'!

Thus the 'toy theory' describes $1+1$ *degenerate* sector of GR [Jacobson, 1996].

Quantum states of the 'toy theory'

Let $\mathcal{L}$ be a finite set of *analytic* loops embedded in Σ and

$$\mathcal{A}_{\mathcal{L}} \cong \mathbb{R}^{\text{number of loops in } \mathcal{L}}$$

be the set of holonomies along the loops in $\mathcal{L}$. Define

$$\mathcal{H}_{\mathcal{L}} := L^2(\mathcal{A}_{\mathcal{L}}, d\mu_{\mathrm{Lebesque}}).$$

Given $\mathcal{L}' \geq \mathcal{L}$ i.e. $\mathcal{L}' \supset \mathcal{L}$, we have

$$\mathcal{A}_{\mathcal{L}'} = \mathcal{A}_{\mathcal{L}'\setminus\mathcal{L}} \times \mathcal{A}_{\mathcal{L}} \implies \mathcal{H}_{\mathcal{L}'} = \mathcal{H}_{\mathcal{L}'\setminus\mathcal{L}} \otimes \mathcal{H}_{\mathcal{L}}.$$

Hence the projection $\pi_{\mathcal{L}\mathcal{L}'} : \mathcal{D}_{\mathcal{L}'} \to \mathcal{D}_{\mathcal{L}}$ is given by the partial trace with respect to $\mathcal{H}_{\mathcal{L}'\setminus\mathcal{L}}$.

Thus we obtain the projective family $\{D_\gamma, \pi_{\gamma\gamma'}\}$, and the space $\mathcal{D}$ of *Yang-Mills gauge invariant* quantum states (note that each holonomy along a loop is gauge invariant).

Quantum 'toy theory'

There is a C^*-algebra $\mathcal{B}$ of quantum observables associated with $\mathcal{D}$.

The observables in $\mathcal{B}$ are *Yang-Mills gauge invariant* therefore we treat the pair $(\mathcal{D}, \mathcal{B})$ as the quantum 'toy theory' with the Gauss constraint solved.

To solve the vector constraint we need to find *diffeomorphism invariant* states in $\mathcal{D}$. It turned out that the set $\mathcal{D}_{\rm diff} \subset \mathcal{D}$ of such states is quite large.

The resulting quantum 'toy theory' is a pair $(\mathcal{D}_{\rm diff}, \mathcal{B})$.

Remark: there are no non-trivial diff. invariant observables in $\mathcal{B}$, however each expectation value

$$\rho(b), \quad \text{where} \quad \rho \in \mathcal{D}_{\rm diff} \text{ and } b \in \mathcal{B}$$

is diff. invariant.

Summary

- Kijowski's projective techniques may, hopefully, solve the non-compactness problem in GR;
- in particular, they were successfully applied to the 'toy theory' whose gauge group is non-compact.

Warnings!!!

- $SL(2,\mathbb{C})$ unlike $\mathbb{R}$ is *non-Abelian*, hence it generates non-trivial gauge transformations and the non-vanishing scalar constraint!!!
- What about the reality conditions???

References

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