Kerr-NUT-de Sitter spacetimes that admit non-singular horizons

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Plan of the talk

1. Kerr-NUT-(Anti) de Sitter spacetimes
2. Horizons in the K-NUT-(A)dS spacetimes and singularity removability conditions
3. Neighbourhoods of non-singular horizons
4. Summary (& comparison with known results on type D horizons, a puzzle and the solution).

Context

Spacetimes (M, g) satisfying vacuum Einstein Equations with cosmological constant:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}.$$

4-dimensional family of Kerr-NUT-(anti) de Sitter solutions parametrized by

- m - mass of the black hole
- a - Kerr parameter
- Λ - cosmological constant
- l - NUT parameter

Motivation

In the local theory of isolated horizons the Petrov type D horizons were investigated (Lewandowski, Pawłowski, Szereszewski, Dobkowski-Ryłko, Kamiński, Racz). In particular there were defined type D horizons of the topology of the Hopf bundle. Upon imposing the vacuum Einstein Equations with a cosmological constant a 4-dimensional family of type D horizons was derived. On the other hand, there is also a 4-dimensional family of the Kerr-NUT-(Anti) de Sitter spacetimes. The relation seemed to be obvious...

Metric

$$g = -\frac{Q}{\Sigma}(dt - A d\phi)^2 + \frac{\Sigma}{Q} dr^2 + \frac{\Sigma}{P} d\theta^2 + \frac{P}{\Sigma}\sin^2\theta (a dt - \rho d\phi)^2,$$

where

$$\Sigma = r^2 + (l + a\cos\theta)^2,$$
$$A = a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta,$$
$$\rho = r^2 + (l+a)^2 = \Sigma + aA,$$
$$Q = (a^2 - l^2) - 2mr + r^2 - \Lambda\big((a^2 - l^2)l^2 + (\tfrac{1}{3}a^2 + 2l^2)r^2 + \tfrac{1}{3}r^4\big),$$
$$P = 1 + \frac{4}{3}\Lambda a l\cos\theta + \frac{\Lambda}{3}a^2\cos^2\theta.$$

Limits: Reduces to Kerr(-anti) de Sitter for $l = 0$ and to Taub-NUT for $a = 0 = \Lambda$.

Assumption: Lorentzian signature $\implies P > 0$

Killing vector fields: ∂_t, ∂_ϕ.

Conical singularity

The 1-form $Ad\phi$, where

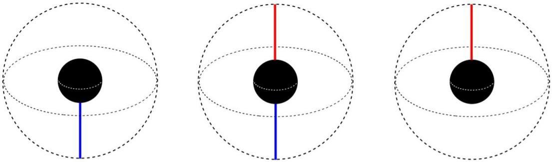
$$A = a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta$$

does not vanish at the pole $\theta = \pi \implies g$ is not continuous at the semi-axis $\theta = \pi$.
New coordinate $t' = t - 4l\phi \implies A' = a\sin^2\theta - 4l\cos^2\frac{1}{2}\theta$ and g is not continuous at the semi-axis $\theta = 0$.
Another coordinate $t'' = t - 2\phi \implies A'' = a\sin^2\theta + 2l\cos\theta$ and g is not continuous at the whole axis.

- Gravitational analogue of the magnetic monopole
- For generic values of parameters singular in at least one pole

Gluing a'la Misner

Consider Taub-NUT limit $a = 0 = \Lambda$.
We glue two non-singular patches at the cost of periodic t.
For Taub-NUT we arrive at the topology $S^3 \times \mathbb{R}$.



Source: [1]

Our result: Not enough for Kerr-NUT-de Sitter, the gluing is differentiable in a neighbourhood of a horizon, iff

$$\frac{3}{\Lambda} = a^2 + 2l^2 + 2r_0^2.$$

[1] R. Durka - "The first law of black hole thermodynamics for Taub-NUT spacetime"

KNdS Horizons

Surfaces $r = r_0$, where $\mathcal{Q}(r_0) = 0$, are Killing horizons.

$$\mathcal{Q} = (a^2 - l^2) - 2mr + r^2 - \Lambda\big((a^2 - l^2)l^2 + (\tfrac{1}{3}a^2 + 2l^2)r^2 + \tfrac{1}{3}r^4\big)$$

- KN(a)dS admits up to 4 Killing horizons
- Developed by the Killing vector $\xi = \partial_t + \Omega_H \partial_\phi$ where $\Omega_H := \frac{a}{\rho_0} = \frac{a}{r_0^2 + (a+l)^2}$

The problem that is not solved by the Misner glueing is non-differentiability of the horizon (degenerate) metric tensor.

Coordinates through horizon

The KNdS metric can be extended through the horizon by introducing:

$$dv = dt + \frac{\rho}{\mathcal{Q}}dr,$$
$$d\tilde{\phi} = d\phi + \frac{a}{\mathcal{Q}}dr.$$

Then the metric takes form

$$ds^2 = -\frac{\mathcal{Q}}{\Sigma}(dv - Ad\tilde{\phi})^2 + 2dr(dv - Ad\tilde{\phi}) + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta(adv - \rho d\tilde{\phi})^2.$$

$$\mathcal{Q}(r_0) = 0 \implies {}^{(2)}q_H = \frac{\Sigma_0}{P}d\theta^2 + \frac{P}{\Sigma_0}\sin^2\theta(adv - \rho_0 d\tilde{\phi})^2$$

$$\Sigma_0 := \Sigma(r_0, \theta) \quad \rho_0 := \rho(r_0)$$

Singular horizon

We introduce coordinates (τ, x^2, x^3) on the horizon, constant along ξ

$$\xi(\tau) = 1, \qquad \xi(x^i) = 0, \qquad i = 2, 3.$$

We choose

$$\begin{bmatrix} \tau \\ x^2 \\ x^3 \end{bmatrix} := \begin{bmatrix} v \\ \theta \\ -\Omega_H v + \tilde{\phi} \end{bmatrix}, \quad \begin{bmatrix} \partial_\tau \\ \partial_2 \\ \partial_3 \end{bmatrix} = \begin{bmatrix} \xi \\ \partial_\theta \\ \partial_{\tilde{\phi}} \end{bmatrix}.$$

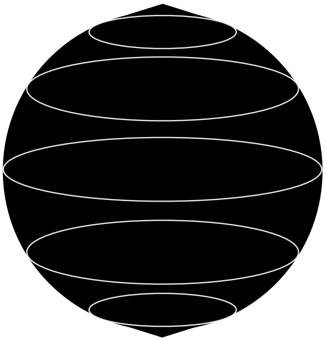
Then the corresponding tangent frame is

$$\begin{bmatrix} \partial_\tau \\ \partial_2 \\ \partial_3 \end{bmatrix} = \begin{bmatrix} \xi \\ \partial_\theta \\ \partial_{\tilde{\phi}} \end{bmatrix}.$$

The degenerate horizon metric becomes

$${}^{(2)}q_H = \frac{\Sigma_0}{P}(dx^2)^2 + \frac{P}{\Sigma_0}\sin^2(x^2)\rho_0^2(dx^3)^2.$$

Conical singularity - illustration



Source: Dobkowski-Ryłko

Singularity removal condition

So far no assumptions that (x^2, x^3) are spherical coordinates:

$$x^2 \in [0, \pi], \quad x^3 \in [0, 2\pi c), \qquad c = \text{ const.}$$

it follows $x^2 = 0$ is a single point, otherwise $x^2 = $ const is a circle. Define the circumference of a circle $x^2 = $ const

$$L(x^2) = \int_0^{2\pi c} \sqrt{q_{33}} dx^3.$$

and its radius for $0 < x^2 \leq \pi/2$

$$R_0(x^2) = \int_0^{x^2} \sqrt{q_{22}} dx^2,$$

and for $\pi/2 < x^2 < \pi$

$$R_\pi(x^2) = \int_{x^2}^{\pi} \sqrt{q_{22}} dx^2.$$

Singularity removal condition

Apart from the poles the horizon metric ${}^{(2)}q_H$ is analytic. Then the horizon would be diffeomorphic to S^2 iff

$$\lim_{x^2\to 0}\frac{L(x^2)}{R_0(x^2)} = 2\pi = \lim_{x^2\to\pi}\frac{L(x^2)}{R_\pi(x^2)}.$$

$$\lim_{x^2\to 0/\pi}\frac{L(x^2)}{R_{0/\pi}(x^2)} = \begin{cases} 2\pi c P(0) & \text{for } x^2 = 0, \\ 2\pi c \dfrac{r_0^2+(l+a)^2}{r_0^2+(l-a)^2}P(\pi) & \text{for } x^2 = \pi. \end{cases}$$

The singularity removal condition becomes

$$P(0) = \frac{r_0^2+(l+a)^2}{r_0^2+(l-a)^2}P(\pi), \quad c = \frac{1}{P(0)}.$$

$$P = 1 + \frac{4}{3}\Lambda a l\cos\left(x^2\right) + \frac{\Lambda}{3}a^2\cos^2(x^2)$$

Known non-singular solutions

$$P(0) = \frac{r_0^2 + (l+a)^2}{r_0^2 + (l-a)^2} P(\pi)$$

The condition recovers known non-singular solution:

- $\left(\Lambda = 0 \quad \Rightarrow \quad P(0) = P(\pi)\right) \implies \left(\underbrace{l = 0}_{\text{Kerr}} \quad \vee \quad \underbrace{a = 0}_{\text{Taub-NUT}}\right)$
- $\left(\underbrace{l = 0 \;\wedge\; \Lambda - \text{free}}_{\text{K(a)dS}} \vee \underbrace{a = 0 \;\wedge\; \Lambda - \text{free}}_{\text{Taub-NUT-(a)dS}}\right) \implies \left(P(0) = P(\pi)\right)$

New non-singular solutions

Our main result:

$$\Lambda a l \neq 0 \implies \Lambda = \frac{3}{a^2 + 2l^2 + 2r_0^2}, \quad c = 1/P(0) = \frac{3}{2\Lambda}$$

Given (a, l, r_0) and the above condition we have

$$m = \frac{a^4 - 2a^2l^2 + l^4 + 2a^2r_0^2 - 6l^2r_0^2 + r_0^4}{2a^2r_0 + 4l^2r_0 + 4r_0^3}.$$

The geometry of the space of null generators of the horizons:

- $(x^2, x^3) \in [0, \pi] \times [0, \frac{2\pi}{P(0)})$
- Area $= 4\pi \left(\frac{3}{2\Lambda}\right)$
- At least twice differentiable at the poles
- Horizon is not (!) $S^2 \times \mathbb{R}$
- Given a Kerr-NUT-dS spacetime defined by (m, a, l, Λ), at most ONE of the horizons may have r_0 that satisfies the singularity removability condition

Space of the orbits - geometric description

If $g(\xi,\xi) \neq 0$ than we can choose coordinates (τ, x^i) such that $\xi = \partial_\tau$ and introduce the metric q on the space of orbits of the Killing vector field ξ:

$$g = g_{\tau\tau} d\tau^2 + 2g_{\tau i} d\tau dx^i + g_{ij} dx^i dx^j = g_{\tau\tau}(d\tau + \omega_i dx^i)^2 + q_{ij} dx^i dx^j$$

where

- $q = q_{ij}(x^k) dx^i dx^j$ is the metric on the space of the orbits
- $d\tau + \omega_i dx^i = d\tau + \frac{g_{\tau i}}{g_{\tau\tau}} dx^i$ is the rotation-connection 1-form
- $g(\xi,\xi) = g_{\tau\tau}$ is the lapse function.

The benefit: The above structure is geometrically defined.

Interpretation: q measures distance between Killing observers.

Neighbourhood of a non-singular horizon

Similarly to the horizon case we choose coordinates

$$\begin{bmatrix} \tau \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} := \begin{bmatrix} t \\ r \\ \theta \\ -\Omega_H t + \phi \end{bmatrix}, \quad \begin{bmatrix} \partial_\tau \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} = \begin{bmatrix} \xi \\ \partial_r \\ \partial_\theta \\ \partial_\phi \end{bmatrix}.$$

and the KNdS metric becomes

$$\begin{aligned} ds^2 = &- \frac{\mathcal{Q}}{\Sigma}\left(d\tau \frac{\Sigma_0}{\rho_0} - A dx^3\right)^2 + \frac{\Sigma}{\mathcal{Q}}(dx^1)^2 \\ &+ \frac{\Sigma}{P}(dx^2)^2 + \frac{P}{\Sigma}\sin^2(x^2)\left(\frac{a}{\rho_0}\left((x^1)^2 - r_0^2\right)d\tau - \rho dx^3\right)^2 \end{aligned}$$

Space of the orbits - coordinate description

Metric q on the space of orbits is defined as

$$q_{ij} := g(\hat{\partial}_i, \hat{\partial}_j)$$

where $\hat{\partial}_i$ are chosen to be orthogonal to ξ

$$\hat{\partial}_i = \partial_i - A_i \xi, \quad \text{and} \quad g(\xi, \hat{\partial}_i) = 0.$$

Finally

$$q = \frac{\Sigma}{\mathcal{Q}}(dx^1)^2 + \frac{\Sigma}{P}(dx^2)^2 + \frac{P\mathcal{Q}\sin^2(x^2)\rho_0^2\Sigma}{\mathcal{Q}\Sigma_0^2 - P\sin^2(x^2)a^2((x^1)^2 - r_0^2)^2}(dx^3)^2.$$

Moduli space metric

$$q = \frac{\Sigma}{Q}(dx^1)^2 + \frac{\Sigma}{P}(dx^2)^2 + \frac{PQ\sin^2(x^2)\rho_0^2\Sigma}{Q\Sigma_0^2 - P\sin^2(x^2)a^2((x^1)^2 - r_0^2)^2}(dx^3)^2.$$

Signature:
- $q_{11} \propto -g(\xi,\xi)$ close to the non-extremal horizon
- $q_{22} > 0$ everywhere
- $q_{33} > 0$ close to the non-extremal horizon

Horizon limit: $q_{33} = \frac{P\sin^2(x^2)\rho_0^2\Sigma}{\Sigma_0^2 - P\sin^2(x^2)a^2(x^1+r_0)^2\frac{(x^1-r_0)^2}{Q}} \implies$ $\lim_{x^1\to r_0}\left(q_{22}(dx^2)^2 + q_{33}(dx^3)^2\right) = {}^{(2)}q_H$ if r_0 is a single root of Q

Singularity: Denominator of $q_{33} \propto g(\xi,\xi)$

Applicability of the moduli metric

The horizon metric ${}^{(2)}q_H$ was defined regardless of the extremality of the horizon.
The moduli metric has good properties if we assume that

- The horizon is non-external.
- The signature is $(+,+,+)$ or $(-,+,+)$ and $g(\xi,\xi) \neq 0$

Thus q is defined either the future or the past of the horizon:

$$r_0 < x^1 < r_0 + \epsilon, \qquad \text{or} \qquad r_0 - \epsilon < x^1 < r_0,$$

such that ϵ satisfied the above conditions.

Non-singular neighbourhoods of the horizon

The moduli metric q is again analytic everywhere apart from the poles $x^2 = 0, \pi$.
We employ the same method of loops around the symmetry axis as for ${}^{(2)}q_H$ for the pullback of the q to $x_1 = \text{const} = r \neq r_0$:

$$
{}^{(2)}q := \frac{\Sigma}{P}(dx^2)^2 + \frac{PQ\sin^2(x^2)\rho_0^2\Sigma}{Q\Sigma_0^2 - P\sin^2(x^2)a^2((x^1)^2 - r_0^2)^2}(dx^3)^2
$$

Surprisingly: The condition for removal of the singularity is independent of r, and coincides with the non-singular horizon condition. Importantly, the rescaling constant c is r independent as well. $P(0) = \frac{r_0^2+(l+a)^2}{r_0^2+(l-a)^2}P(\pi)$

Conclusion: Removing the singularity in a horizon simultaneously removes the singularity from a future/past neighbourhood.

Summary

We have considered above the Kerr-NUT-(A)dS spacetimes parametrized by (m, a, l, Λ).

1. We have examined the geometry of null generators of a horizon and we have found that:
   - Generically it has a non-removable conical singularity.
   - the necessary and sufficient condition for removability of the singularity, is: $\Lambda = \frac{3}{a^2+2l^2+2r_0^2} \vee \Lambda = 0 \vee a = 0 \vee l = 0$
   - The catch is, that given $(m, a, l, \Lambda = \frac{3}{a^2+2l^2+2r_0^2})$ at most one horizon can satisfy that condition.

Summary

2. We have also considered a neighbourhood of a non-singular horizon. - We have found that in a non-extremal case removing the singularity in a horizon removes the singularity in a neighbourhood.
   - The topology of a resulting neighbourhood is $S^3 \times (r - \epsilon, r + \epsilon)$.
   - The topology of each horizon is that of the Hopf bundle $S^3 \to S^2$
3. A puzzle: we known from earlier works, that there is a 4-dimensional family of (non-singular) the Petrov type D vacuum horizons with a cosmological constant that have the Hopf bundle topology. Our very horizons define a 3-dimensional subfamily. Where are the remaining type D horizons?
   Answer: A similar research can be done for accelerated Kerr-NUT-(Anti) de Sitter spacetimes. It leads to a similar singularity removal condition. We have also derived it and solved. The result is a 4-dimensional family of the accelerated Kerr-NUT-(Anti) de Sitter spacetimes that admits non-singular horizons (and neighbourhoods).

The end

Details in an upcoming paper (probably next week).

Thank you for your attention!