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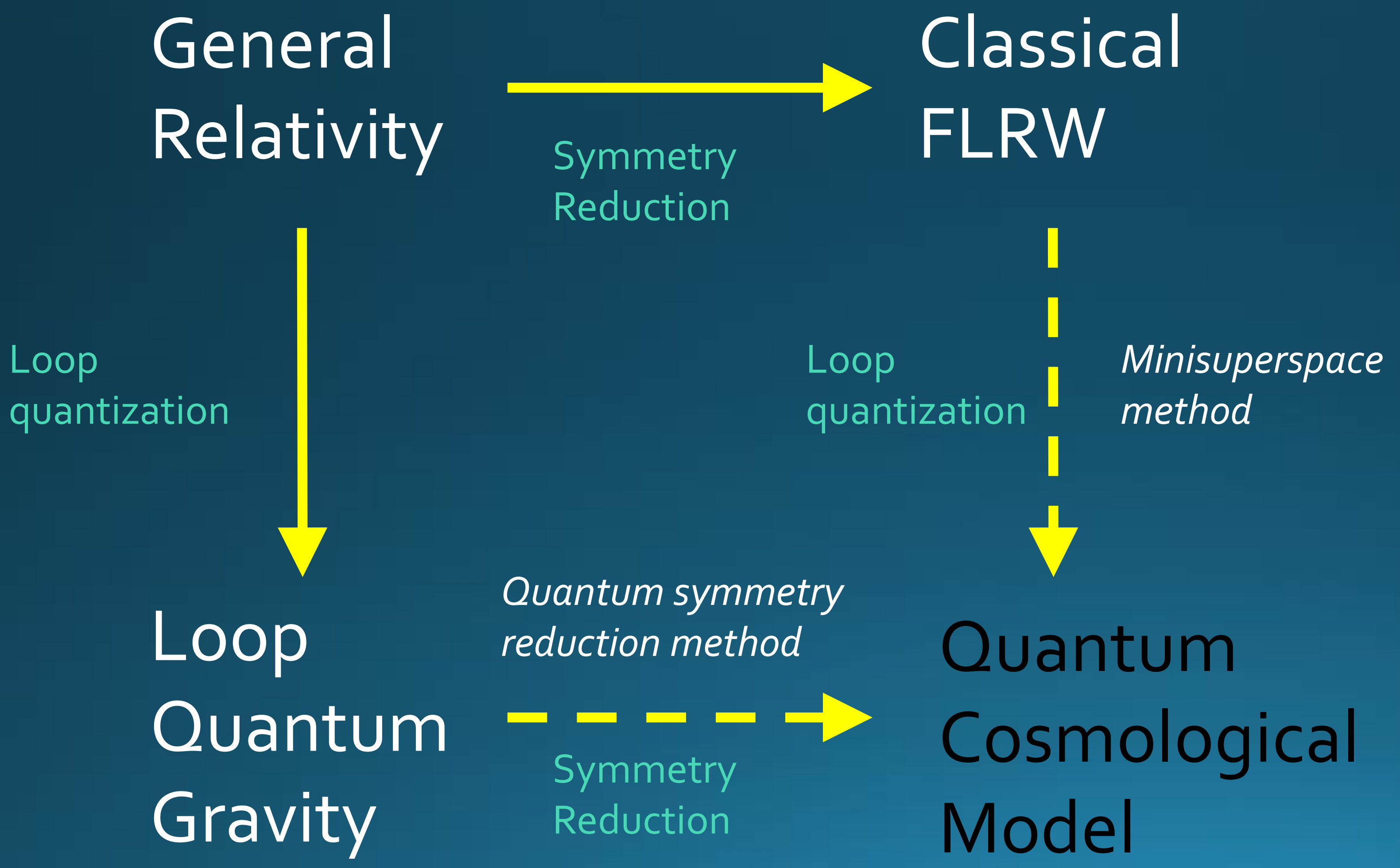
Emanuele Alesci (SISSA)

Panel on LQC and symmetry reduction

International LQG Seminar, May 3rd, 2016

Jonathan Engle (Florida Atlantic University):

Broad picture and background



Neither path is necessarily more reliable.

- Each has its advantages and questions.

Value of considering both paths:

- With agreement, more confidence.
- Allows for feedback to learn about full LQG.

Clarification

- For a while it was commonly believed that the two paths even in principle could not yield the same model in the end. *(See issue related to Heisenberg uncertainty principle in later slide.)*

- But this is absolutely not true! (JE 2006.) Symmetry reduction of a free scalar field theory. Exact match of both paths, as long as quantum symmetry reduction is imposed correctly as the quantization of the corresponding classical condition.

- If there is disagreement, it is either (1.) because of use of different quantization strategies or (2.) due to complications from non-linearity. But at a basic conceptual level, they should agree if done properly.

Quantum symmetry reduction: Broadest conceptual level (JE 2006).

- NOT done by imposing invariance of the state under an action of the symmetry group. Especially in quantum gravity, this is clear: all physical states are already invariant under all diffeos, and hence under all possible symmetry groups, so that this form of symmetry is even vacuous in content.

- Rather, take classical equations expressing symmetry and quantize them like constraints. $\mathcal{L}_\xi \varphi = 0$, $\mathcal{L}_\xi \pi = 0$.
  - Are second class: Cannot impose $\widehat{\mathcal{L}_\xi \varphi}\Psi = 0$ and $\widehat{\mathcal{L}_\xi \pi}\Psi = 0$.
  - Must instead use, e.g., Gupta-Bleuler $\widehat{\mathcal{L}_\xi(\varphi + i\lambda\pi)}\Psi = 0$, Master constraint, or imposition on labels on coherent states,
  - Inherent ambiguity in any of these methods: Choice of annihilation operator, integration measure, or coherent state family.

In simple cases, these choices are equivalent.

In free scalar field case, compatibility with dynamics fixes the ambiguity.

Minisuperspace method:

- Advantage: Simpler. Quantization can be carried out to completion with more control.
- Disadvantage: No direct proof that it represents full LQG faithfully. Even if it does, not clear to what degree and in what way.

Quantum reduction method:

- Advantage: Clearly reflective of full LQG.
- Disadvantage: (1.) Open issues of the full theory. (2.) Second-class nature of symmetry constraints leads to extra subtlety.

Note:

- Canonical vs. Spinfoams has to do with dynamics only. SF cosmology is not covered in this panel. SF cosmology has so far not tried to make contact with LQC. Making contact between spinfoam cosmology and LQC is an open issue.

- In the following I will consider only **homogeneous-isotropic** symmetry groups, and even focus on the **k=o case**. But often what I say also potentially applies to the various homogeneous symmetry groups and other k cases.

Minisuperspace method: Issues fall into two categories:

- (1.) Adaptation of LQG methods to FLRW.
- (2.) Relation of minisuperspace model to full LQG

A quick overview of the method of quantization in LQG.

- Basic continuum variables: Connection A_a^i and conjugate momentum E_i^a. Classical configuration space is $\mathcal{A}$, space of smooth $SU(2)$ connections.

- Basic algebra $\mathfrak{A}$ of configuration variables which become well-defined operators: Algebra generated by parallel transports along analytic paths.

- *Quantum* configuration space $\overline{\mathcal{A}}$ is a certain compactification of $\mathcal{A}$ *determined by* $\mathfrak{A}$: The Gel'fand spectrum of $\mathfrak{A}$. Basically, the manifold on which $\mathfrak{A}$ becomes precisely the space of all continuous functions.

- Measure μ on $\overline{\mathcal{A}}$ is uniquely selected by *diffeomorphism invariance*, yielding the Ashtekar-Lewandowski measure μ_{AL}

- Kinematical space of states is $L^2(\overline{\mathcal{A}}, \mu_{AL})$

- Hamiltonian constraint: Various proposals, but original, best motivated so far is probably first. Curvature is regularized using holonomies around loops. Regulator is removed - each loop shrunk to a point - using diffeomorphism invariance.

Differences in original quantization strategy for *LQC.*

1. Classical configuration space is $\mathcal{A}_S \cong \mathbb{R}$, space of *symmetric* connections $A^i_a = c\mathring{A}^i_a$.
2. Basic configuration algebra $\mathfrak{A}_S$ includes only parallel transports along *straight* edges.
   $\rightsquigarrow$ Leads to $\overline{\mathcal{A}}_S = \mathbb{R}_{\mathrm{Bohr}}$, Bohr compactification of real line.
3. Measure on $\overline{\mathcal{A}}_S$ is selected via mathematical naturalness: The Haar measure on $\mathbb{R}_{\mathrm{Bohr}}$. Not using diffeo invariance.
4. Curvature in Hamiltonian constraint regularized by a square loop not shrunken to a point, but to the size of the area gap. Is the best we can do so far to capture the area gap in the homogeneous context - otherwise area gap is not faithfully represented due to homogeneity.

Notes: Difference 1: We don't want to get rid of this difference. Question is: To what extent can other differences be rid of?
Differences 2 & 3: Can be gotten rid of, and resulting kinematics is still the standard LQC kinematics! (JE and Hanusch 2016, building on Fleischhack 2010).
Difference 4: Difficult to even imagine how it can be gotten rid of. Seems inherent to minisuperspace approach.

Relation of LQC to full theory

Kinematics:

Embedding of quantum configuration spaces: $\bar{A}_S \hookrightarrow \bar{A}$.

Is possible only when difference 2 is addressed (Brunnemann and Fleischhack 2007).

Embedding of *state spaces* $\mathrm{Cyl}^*_S \hookrightarrow \mathrm{Cyl}^*$ (at level of distributional states)

Is possible with original LQC (JE 2013, building on Fleischhack 2010).

Dynamics:

Dynamics of LQC so far only related to versions of *canonical* full theory with gauge-fixing and truncation. (Alesci, Cianfrani, Bodendorfer). Important.

Quantum symmetry reduction

Conceptual issues:

(1.) How to impose symmetry on both configuration and momentum without violating Heisenberg uncertainty?

Non-symmetric components of configuration and momenta are conjugate, so how can you impose that both be zero in quantum theory?

(2.) How to impose symmetry in a diffeomorphism invariant way?

Symmetry is usually imposed via a specific choice of action of the symmetry group: Violates diffeomorphism invariance. With such a method, will never be able to impose the symmetry and solve diffeomorphism constraint in fundamental full theory (which is not gauge fixed).

Successful approaches which side-step issue 2 by starting from gauge-fixed LQG:

Alesci, Cianfrani, and Bodendorfer. Issue 1 solved using coherent states. **See Alesci part of talk for details on his approach.**

Approach attempting to address both issues.

(Beetle, JE, Hogan, Mendonca 2006)

Uses diffeomorphism and gauge-covariant formulation of hom.-isotropy, which can be quantized in full, diff-invariant, non-gauge-fixed LQG.

$$\mathbf{A}_a^i := A_a^i + ite_a^i$$

$$\mathbf{B}[f] := \int f^i{}_j \mathbf{F}^j \wedge e_i, \qquad V[f] := \int f^i{}_i \epsilon$$

$$\boxed{\mathbf{S}[f,g] := \mathbf{B}[g]V[f] - \mathbf{B}[f]V[g] \approx 0}$$

$$\boxed{\widehat{\mathbf{S}}[f,g]\Psi := \left(\widehat{\mathbf{B}}[g]\widehat{V}[f] - \widehat{\mathbf{B}}[f]\widehat{V}[g]\right)\Psi = 0}$$

- Will try to solve the constraint by solving for an embedding of LQC states into it.
- Is an idealistic approach: Should definitely be tried, but even if too hard, it can provide guidance or corroboration for other approaches: E.g., do the states which Emanuel uses satisfy some quantization of the above constraint, at least approximately?
- Has already shed light on projector P_{AW} from Bianchi I LQC to isotropic LQC introduced by Ashtekar and Wilson-Ewing (2009): Adjoint of P_{AW} embeds isotropic LQC into solutions of most natural quantization of the above constraint. Im $P_{AW}{}^\dagger$ satisfies diff-invariant quantum isotropy!

Panel on symmetry reduction: from LQG to LQC

II. Canonical Issues

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ILQGS, May 2016

1 Canonical Quantization

- Given: classical system with first-class constraints

1. Elementary Variables
   - choose separating space $\mathfrak{S}$ of phase space functions
2. Quantization
   - choose "representation" of $\mathfrak{S}$ on some kinematical Hilbert space $\mathcal{H}$, giving self-adjoint constraints
3. Group Averaging
   - choose constraint-invariant dense subset Φ in Hilbert space $\mathcal{H}$
   - solve constraints using Gelfand triple $\Phi \subseteq \mathcal{H} \subseteq \Phi'$

$$\eta(\phi) \ := \ \int_{\mathcal{Z}} \mathrm{d}\mu(Z)\ \overline{Z\phi} \ \in \ \Phi'$$

4. Physical Hilbert Space
   - inner product: $\langle \eta\phi_1, \eta\phi_2 \rangle_{\text{phys}} \ := \ (\eta\phi_1)[\phi_2]$
   - completion of $\eta(\Phi)$ gives physical Hilbert space, self-adjoint dual representation of observable algebra

Ashtekar, Lewandowski, Marolf, Mourão, Thiemann 1995

- Given: classical system

1. Elementary Variables
   - choose separating space $\mathfrak{S}$ of phase space functions

Idea: choose separating algebra $\mathfrak{A} \subseteq C_0(\mathbf{S})$ on configuration space

Problem: no topology on $\mathbf{S} \implies C_0(\mathbf{S})$ inappropriate

Idea: $\mathfrak{A} \subseteq C_b(\mathbf{S})$ often much better

Definition: **Natural Mapping** $\iota : \mathbf{S} \longrightarrow \operatorname{spec}\mathfrak{A}$

$$\begin{aligned} \iota(s) : \quad \mathfrak{A} &\longrightarrow \mathbb{C} \\ a &\longmapsto a(s) \end{aligned}$$

$\mathbf{S}$	$\mathcal{A}$
b	$(h_\gamma)^i_j$
$\mathfrak{A}$	$\overline{\mathrm{Cyl}}$
$\overline{\mathbf{S}}$	$\overline{\mathcal{A}}$

Proposition: 1. $\iota(\mathbf{S})$ dense in $\operatorname{spec}\mathfrak{A}$ — Rendall 1993

2. ι injective $\iff \mathfrak{A}$ separates points in $\mathbf{S}$ — Rendall 1993, CF 2010

Definition: **Quantum Configuration Space** $\overline{\mathbf{S}} := \operatorname{spec}\mathfrak{A}$ — Ashtekar, Isham 1992

2 Symmetries

Given: Embedding σ of classical configuration spaces

Task 1: Extend σ to $\overline{\sigma}$

Task 2: Choose Gelfand triple in LQC

Bojowald 2000–

$$
\begin{tikzcd}
\text{Cosmology} \arrow[d, "Q"'] & \text{Gravity} \arrow[l, "R"'] \arrow[d, "Q"'] \\
\text{Quantum Cosmology} & \text{Quantum Gravity} \arrow[l, "R"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{classical config spaces} & \mathbf{S}_{\mathsf{C}} \arrow[r, hook, "\sigma"] \arrow[d, hook, dotted] & \mathbf{S}_{\mathsf{G}} \arrow[d, hook, dotted] \\
\text{quantum config spaces} & \overline{\mathbf{S}_{\mathsf{C}}} \arrow[r, hook, dotted, "\overline{\sigma}"] & \overline{\mathbf{S}_{\mathsf{G}}} \\
\text{algebras of basic functions} & \mathfrak{B}_{\mathsf{C}} \arrow[d, hook, dotted, "\eta_1"] & \mathfrak{B}_{\mathsf{G}} \arrow[l, "\overline{\sigma}^*"'] \\
\text{quantum states} & \mathfrak{B}_{\mathsf{C}}^* \arrow[r, hook, "\overline{\sigma}^{**}"] & \mathfrak{B}_{\mathsf{G}}^*
\end{tikzcd}
$$

3 Embeddability Criterion

$\mathbb{R}$	$\mathcal{A}$	$\mathbf{S}$
$\mathrm{e}^{il\bullet}$	$(h_\gamma)^i_j$	a
	Cyl	$\mathfrak{B}$
$C_{\mathrm{AP}}(\mathbb{R})$	$\overline{\mathrm{Cyl}}$	$\mathfrak{A}$
$\mathbb{R}_{\mathrm{Bohr}}$	$\overline{\mathcal{A}}$	$\overline{\mathbf{S}}$

Question: Existence, uniqueness, continuity of $\overline{\sigma}$?

$$\begin{tikzcd} \mathbf{S}_1 \arrow[r, "\sigma"] \arrow[d, "\iota_1"] & \mathbf{S}_2 \arrow[d, "\iota_2"'] \\ \overline{\mathbf{S}}_1 \arrow[r, dotted, "\overline{\sigma}"] & \overline{\mathbf{S}}_2 \end{tikzcd}$$

Theorem: continuous $\overline{\sigma}$ exists $\Longleftrightarrow$ $\sigma^*\mathfrak{B}_2 \subseteq \mathfrak{A}_1$ *(unital case)*

Corollary: continuous $\overline{\sigma}$ exists $\Longrightarrow$ $\overline{\sigma}$ unique

Lesson: $\mathbb{R} \hookrightarrow \mathcal{A}$ can**not** be continuously extended to $\mathbb{R}_{\mathrm{Bohr}} \hookrightarrow \overline{\mathcal{A}}$. Brunnemann, CF 2008

Theorem: injective continuous $\overline{\sigma}$ exists $\Longleftrightarrow$ $\mathfrak{A}_1 = C^*(\sigma^*\mathfrak{B}_2)$

Lesson: **Defining**

$$\mathfrak{B}_{\mathrm{LQC}} \quad := \quad \sigma^*\mathfrak{B}_{\mathrm{LQG}}$$

is the **only** way to get a continuous embedding of LQC into LQG.

$\mathcal{A} \hookrightarrow \overline{\mathcal{A}}$ \ $\mathbb{R} \hookrightarrow \overline{\mathbb{R}}$	same as for LQG	piecewise linear	in fixed geodesic	incommensurable	
piecewise analytic	$+$	$-$	$-$	$-$	Ashtekar/Lewandowski
piecewise smooth	$+$	$-$	$-$	$-$	Baez/Sawin, CF
piecewise C^k	$+$	$-$	$-$	$-$	CF
piecewise linear	$+$	$+$	$+$	$-$	Zapata, Engle
in fixed graph	$+$	$(-)$	$(-)$	$(-)$	Giesel/Thiemann
in fixed PL graph	$+$	$(\circ)$	$(\circ)$	$(-)$	Giesel/Thiemann
barycentric subdivision	$+$	$(\circ)$	$(\circ)$	$(-)$	Aastrup/Grimstrup
	CF	Bojowald	Bojowald, Engle	Thiemann, CF	

$+$ . . . cont. inj. $\overline{\sigma}$

$\circ$ . . . cont. non-inj. $\overline{\sigma}$

$-$ . . . no cont. $\overline{\sigma}$

4 Embeddable LQC

CF 2010, 2014 *Fundamental Algebra*

Task: Determine $\mathfrak{B}_{\mathrm{LQC}} := \sigma^* \mathfrak{B}_{\mathrm{LQG}}$ for homogeneous isotropic case

matrix functions of solutions of

$$\begin{aligned} \dot g(t) &= -c\, A_*\big(\dot\gamma(t)\big)\, g(t) \\ g(0) &= \mathbf{1} \end{aligned}$$

Theorem: C^*-algebra underlying homogeneous isotropic LQC:

CF 2010
Hanusch, CF 2013

$$C_0(\mathbb{R}) + C_{\mathrm{AP}}(\mathbb{R})$$

Task: Determine spectrum of $C_0(\mathbb{R}) + C_{\mathrm{AP}}(\mathbb{R})$

Theorem: $\mathfrak{A} \;=\; C_0(X) + \mathfrak{A}_1 \;\subseteq\; C_b(X) \quad\Longrightarrow\quad \operatorname{spec}\mathfrak{A} \;=\; X \sqcup \operatorname{spec}\mathfrak{A}_1$

locally compact

unital C^*

topology generated by

$$\begin{gathered} V \sqcup \varnothing \\ K^c \sqcup \operatorname{spec}\mathfrak{A}_1 \\ f^{-1}(U) \sqcup \widetilde{f}^{-1}(U) \end{gathered}$$

Result: $\overline{\mathbb{R}} \;=\; \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$

Task: Find measures on $\overline{\mathbb{R}} \;=\; \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$

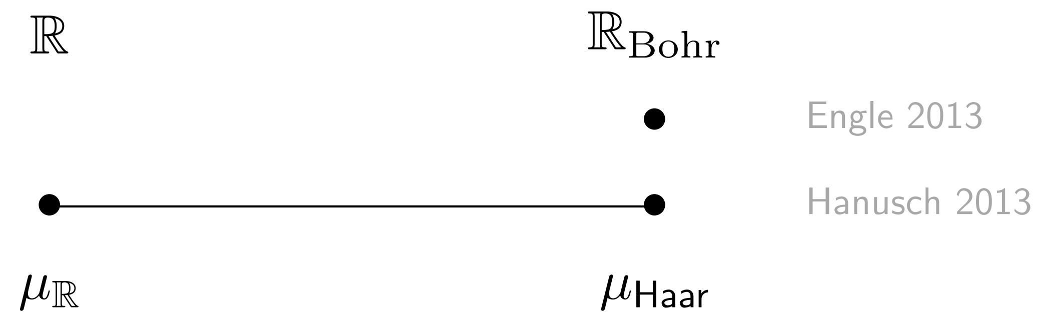
Task: Find measures on $\overline{\mathbb{R}} = \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$ ($\Longrightarrow$ states Engle 2007, 2013)

Problem: No continuous group structure on $\overline{\mathbb{R}}$ $\Longrightarrow$ no Haar measure

Hanusch 2013

Idea:

- Determine measures separately on $\mathbb{R}$ and $\mathbb{R}_{\mathrm{Bohr}}$



- Invariant measures: Extend $\mathbb{R}$-action from $\mathbb{R}$ to $\overline{\mathbb{R}}$

Theorem: All invariant finite Radon measures on $\overline{\mathbb{R}}$:

measure	Dirac	Dirac	Haar	
support	0	0_{Bohr}	$\mathbb{R}_{\mathrm{Bohr}}$	
$\mathbb{R}$ -invariant			•	Hanusch 2014
$\mathbb{R}^*$-invariant	•	•	•	Engle, Hanusch 2016

5 Uniqueness

Homogeneous isotropic cosmology

Engle, Hanusch 2016

- **Theorem:** μ_{Haar} is the **only** normalized dilation invariant Radon measure on $\overline{\mathbb{R}}$ admitting a hermitian momentum operator p on $L^2(\overline{\mathbb{R}}, \mu)$ with

$$[p, \varphi] \quad = \quad -\mathrm{i}\,\dot{\varphi} \qquad \text{for } \varphi \in \operatorname{dom} p$$

- **Remark** True for both $\overline{\mathbb{R}} = \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$ and $\overline{\mathbb{R}} = \mathbb{R}_{\mathrm{Bohr}}$.

Homogeneous cosmology

Ashtekar, Campiglia 2013

- **Theorem:** There is a **unique** cyclic state on the reduced Weyl algebra that is invariant under volume preserving anisotropic dilations.
- **Remark**
  - Maps $W(\boldsymbol{\mu}, \boldsymbol{\eta})$ to $\delta_{\mathbf{0}\boldsymbol{\mu}}$.
  - True for quantum configuration space $\mathbb{R}^3_{\mathrm{Bohr}}$; unknown for $\overline{\mathbb{R}^3}$.

6 Reduction vs Quantization

$$
\begin{tikzcd}
\mathbf{S}_1 \arrow[r, "\sigma"] \arrow[d, "\iota_1"'] \arrow[dr, phantom, "\mathfrak{B}_1 = \sigma^* \mathfrak{B}_2"] & \mathbf{S}_2 \arrow[r, "\tau"] \arrow[d, "\iota_2"'] \arrow[dr, phantom, "\mathfrak{B}_2 = \tau^* \mathfrak{B}_3"] & \mathbf{S}_3 \arrow[d, "\iota_3"'] \\
\overline{\mathbf{S}}_1 \arrow[r, "\overline{\sigma}"'] & \overline{\mathbf{S}}_2 \arrow[r, "\overline{\tau}"'] & \overline{\mathbf{S}}_3
\end{tikzcd}
\qquad \mathfrak{B}_1 = (\tau \circ \sigma)^* \mathfrak{B}_3
$$

Theorem: Functoriality $\qquad \overline{\tau \circ \sigma} \; = \; \overline{\tau} \circ \overline{\sigma}$

6 Reduction vs Quantization

Hanusch 2013

$$\begin{tikzcd}
\mathbf{S} \arrow[r, "\sigma"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\mathfrak{B} = \sigma^*\mathfrak{B}"] & \mathbf{S} \arrow[r, "\tau"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\mathfrak{B} = \tau^*\mathfrak{B}"] & \mathbf{S} \arrow[d, "\iota"'] \\
\overline{\mathbf{S}} \arrow[r, "\overline{\sigma}"'] & \overline{\mathbf{S}} \arrow[r, "\overline{\tau}"'] & \overline{\mathbf{S}}
\end{tikzcd}$$

Theorem: Functoriality $\quad \overline{\tau \circ \sigma} = \overline{\tau} \circ \overline{\sigma}$ CF 2014

Application: Lifting of symmetries to quantum level

6 Reduction vs Quantization

Hanusch 2013

$$\begin{tikzcd} \mathbf{S}_{\mathsf{R}} \arrow[r, "\sigma"] \arrow[d, "\iota_{\mathsf{R}}"'] & \mathbf{S} \arrow[r, "\tau"] \arrow[d, "\iota"'] & \mathbf{S} \arrow[d, "\iota"'] \\ \overline{\mathbf{S}_{\mathsf{R}}} \arrow[r, "\overline{\sigma}"'] & \overline{\mathbf{S}} \arrow[r, "\overline{\tau}"'] & \overline{\mathbf{S}} \end{tikzcd}$$

Theorem: Functoriality $\quad \overline{\tau \circ \sigma} = \overline{\tau} \circ \overline{\sigma}$ CF 2014

Application: Lifting of symmetries to quantum level

- Embeddability criterion $\equiv$ Invariance of $\mathfrak{B}$ under symmetry group T
- Functoriality $\Longrightarrow$ Lifting of symmetry group action to quantum level
- Invariant quantum configuration space $\quad \overline{\mathbf{S}}_{\mathsf{R}} := \{\overline{s} \mid \overline{\tau}(\overline{s}) = \overline{s} \quad \forall \overline{\tau} \in \overline{T}\} \subseteq \overline{\mathbf{S}}$
- Invariant classical configuration space $\quad \mathbf{S}_{\mathsf{R}} := \{s \mid \tau(s) = s \quad \forall \tau \in T\} \subseteq \mathbf{S}$

Theorem: Always $\quad \overline{\mathbf{S}_{\mathsf{R}}} \cong \overline{\sigma}(\overline{\mathbf{S}_{\mathsf{R}}}) \subseteq \overline{\mathbf{S}}_{\mathsf{R}}$

But in general $\quad \overline{\sigma}(\overline{\mathbf{S}_{\mathsf{R}}}) \subset \overline{\mathbf{S}}_{\mathsf{R}}$

Lesson: Quantization and reduction do **not** commute.

7 Lessons

- Relation between config spaces: established
- Lifting of config space symmetries: established

- Examples: mostly homog iso
- Quantization vs reduction: not (always) given
- Uniqueness theorems: first results
- Relation between phase spaces: first hints
- States: only via measures

- Dynamics: unknown

Panel on symmetry reduction: From LQG to LQC

III. Quantum Reduced Loop Gravity

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3th May 2016

Quantum Reduced LG

GOAL:

Implement on the SU(2) Kinematical Hilbert space of LQG the classical reduction:

1) $\boxed{A_a^i = c^i(t,x)\delta_a^i}$ 2) $\boxed{E_i^a = p_i(t,x)\delta_i^a}$

$$\{p^i(x,t), c_j(y,t)\} = 8\pi G\gamma\delta_j^i\delta^3(x-y)$$

<u>Restrict the holonomies to curves along edges</u> e_i parallel to fiducial δ_a^i vectors

The SU(2) classical holonomies associated to the reduced variables are

$${}^R h_{e_i} = P(e^{\int_{e_i} c^i \delta_a^i dx^a(s)\tau_i})$$

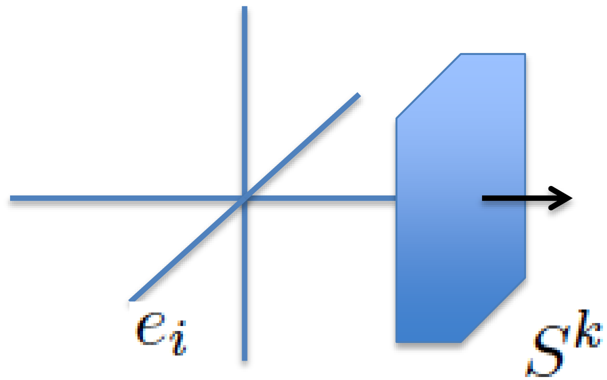
NO sum over i

$${}^R h_{e_i}^j = \exp\left(i\alpha^i\tau_i\right)$$

Holonomy belong to the U(1) subgroup generated by τ_i

Three different U(1) subgroups, one per direction

Consider fluxes across surfaces $x^a(u,v)$ with normal vectors parallel to the fiducial ones



The classical reduction implies

$$E_i(S^k) = \int E_i^a \delta_a^i \, du \, dv = \delta_i^k \int p_i \, du \, dv$$

For consistency only the diagonal part of the matrix $E_i(S^j)$ is non vanishing

$$\chi_i = \sum_{l,k} \epsilon_{il}{}^{k} E_k(S^l) = 0$$

Second class with the Gauss constraint

How to implement the reduction on the holonomies and consistently impose $\chi_i=0$?

Strategy: Mimic the spinfoam procedure

2) Impose the second class constraint weakly to find a "Physical Hilbert space"

Engle, Pereira, Rovelli,Livine '07- '08

1) Embed U(1) cylindrical functions in SU(2) ones:
   Projected spinnetworks (Alexandrov, Livine '02) with the Dupuis-Livine map (Dupuis Livine '10)

Use a Projector P_χ on Physical reduced states and project the constraints

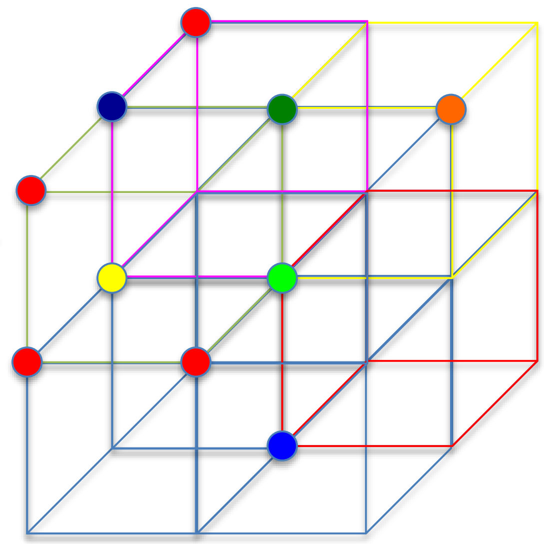
$\hat{G}_i(A,E) \Rightarrow P_\chi^\dagger \hat{G}_i P_\chi$ — reduced intertwiners

$\hat{V}_a(A,E) \Rightarrow P_\chi^\dagger \hat{V}_a P_\chi$ — s-knot states

Ashtekar, Lewandowski, Marolf, Mourao, Thiemann

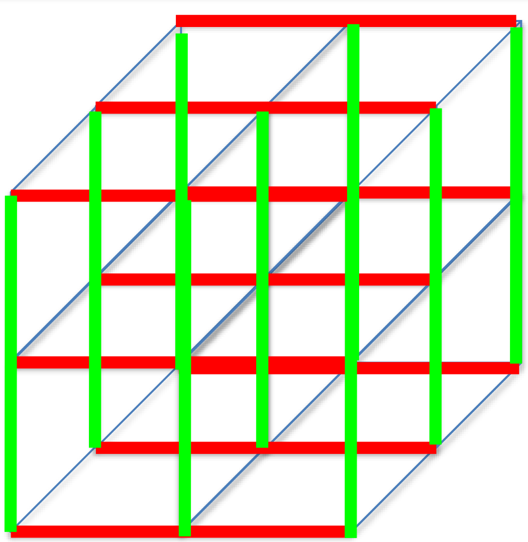
The Inhomogenous sector

Different
Reduced SU(2)
intertwiners:
inhomogeneities

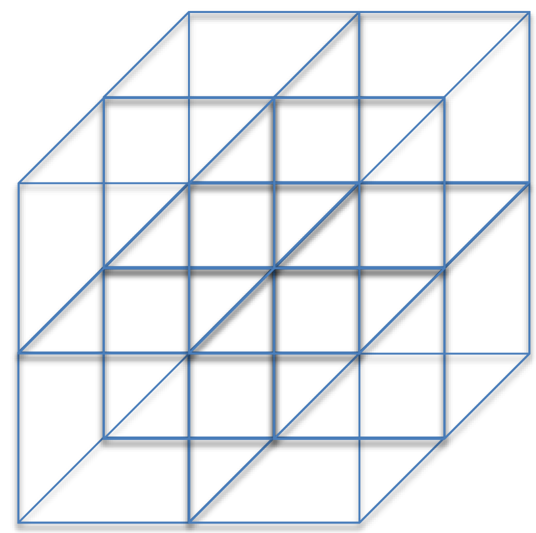


Different
Spin labels:
Anisotropies

Homogeneous and anisotropic sector



Homogeneous and Isotropic sector



How general is this framework ?

Observation: 3-metrics can be taken to diagonal form by a 3d diffeomorphism, with a residual gauge freedom (reduced diffeomorphisms)

Restriction to cubic lattice can be seen as a gauge fixing at the quantum level of the diffeomorphisms on the 3-metric *Alesci, Cianfrani, Rovelli*

$$\eta_x^{km} = \delta^{ij} E_i(S_x^k) E_j(S_x^m) = 0, \qquad k \neq m, \quad \forall x \in \Sigma$$

$$\langle \psi | \eta_x^{km} | \phi \rangle = 0, \qquad k \neq m, \quad \forall x \in \Sigma$$

Weak solution: SU(2) spinnetworks, restricted to reduced graphs with Livine-Speziale intertwiners, synchronized with the frame that diagonalize the metric.

Loop Quantum Gravity in diagonal triad gauge ?

Yes but non trivial Hamiltonian (the evolution may not preserve the gauge; in the BKL hypotesis it does). Same kind of construction as in the radial gauge case.

Bodendorfer, Lewandowski, Świeżewski

Study the Hamiltonian à la Thiemann ('96-'98) using operators defined in the reduced Hilbert space on coherent states

Hall, Thiemann, Winkler, Sahlmann, Bahr

Large distance asymptotic behaviour Bianchi Magliaro Perini

$$\Psi_{H_l}(h_l) \simeq \sum_{j_l, i_n} \prod_l e^{-\frac{(j_l - j_l^0)^2}{2\sigma_l^2}} e^{-i\xi_l j_l} \left(\prod_n \Phi_{i_n}\right) \Psi_{j_l, i_n}(h_l)$$

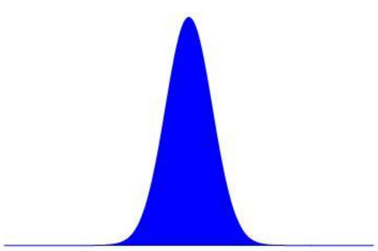
Codes the intrinsic geometry

Codes the extrinsic curvature

Livine-Speziale Intertwiners

$$j_0 = \frac{|E|}{8\pi G\hbar\gamma}$$

$$\xi \sim K = c$$



j_0, c

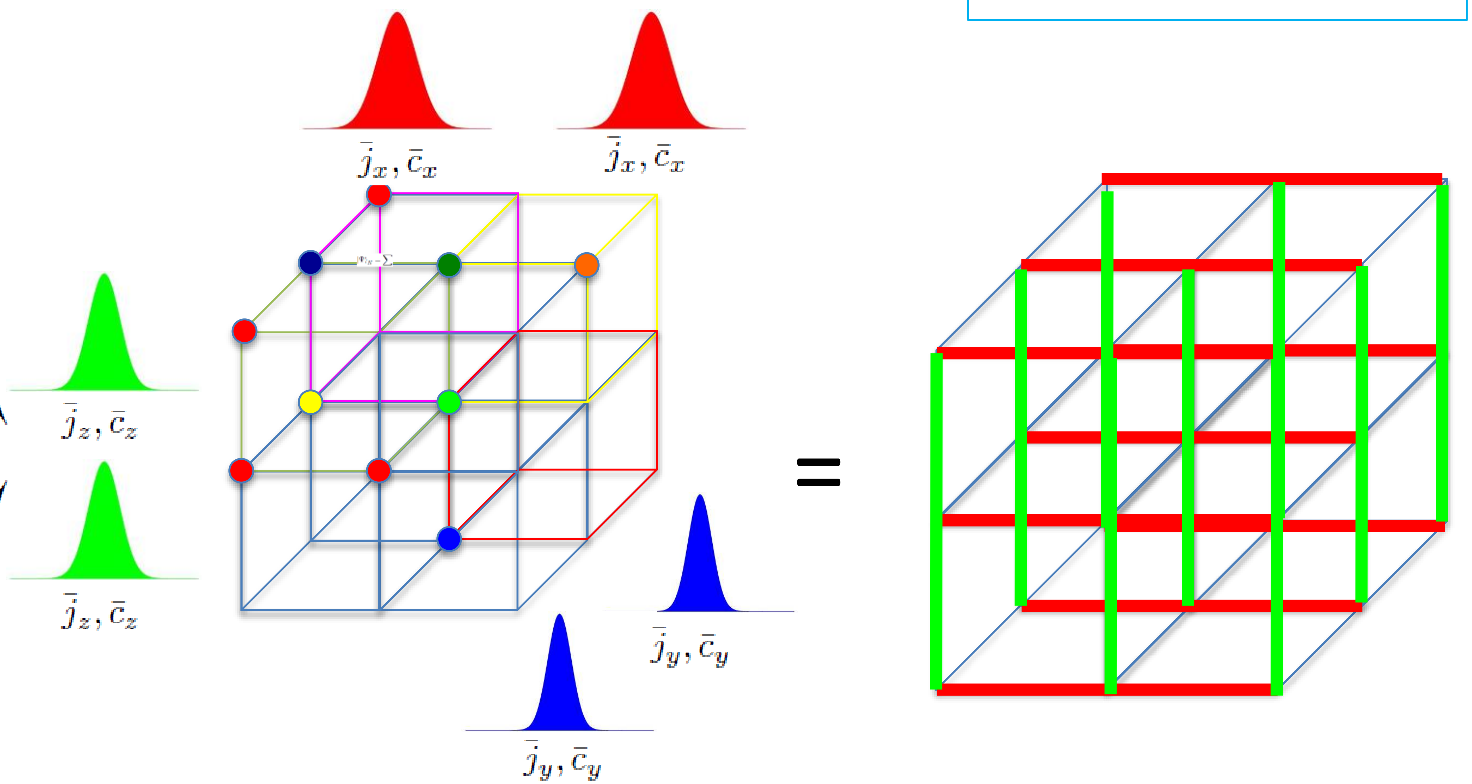
Project the coherent states in the reduced Hilbert space

Collective Coherent state

$$N = N_x N_y N_z$$

Total number of nodes

$$|\Psi\rangle_N = \sum$$



$$P_x = N_y N_z p_x \qquad P_y = N_z N_x p_y \qquad P_z = N_x N_y p_z$$

$$C_x = N_x c_x \qquad C_y = N_y c_y \qquad C_z = N_z c_z$$

Collective Variables

Comparison with LQC

$$H=\frac{2}{\gamma^2}\mathcal{N}\left(\sqrt{\frac{p^xp^y}{p^z}}\frac{\sin\mu_xc_x}{\mu_x}\frac{\sin\mu_yc_y}{\mu_y}+\sqrt{\frac{p^yp^z}{p^x}}\frac{\sin\mu_yc_y}{\mu_y}\frac{\sin\mu_zc_z}{\mu_z}+\sqrt{\frac{p^zp^x}{p^y}}\frac{\sin\mu_zc_z}{\mu_z}\frac{\sin\mu_xc_x}{\mu_x}\right)$$

$$\langle {}^R\hat{H}^{1/2}\rangle_N\approx\frac{2}{\gamma^2}\mathcal{N}\left(N_xN_y\sqrt{\frac{P^x\,P^y}{P^z}}\;\sin\frac{C_x}{N_x}\sin\frac{C_y}{N_y}+\right.$$
$$+N_yN_z\sqrt{\frac{P^y\,P^z}{P^x}}\;\sin\frac{C_y}{N_y}\sin\frac{C_z}{N_z}+$$
$$\left.+N_zN_x\sqrt{\frac{P^z\,P^x}{P^y}}\;\sin\frac{C_z}{N_z}\sin\frac{C_x}{N_x}\right)$$

$$\bar{\mu}_x\bar{\mu}_y=\frac{\Delta l_P^2}{p^z},\quad\bar{\mu}_y\bar{\mu}_z=\frac{\Delta l_P^2}{p^x},\quad\bar{\mu}_z\bar{\mu}_x=\frac{\Delta l_P^2}{p^y}$$

$$\frac{1}{N_x}\frac{1}{N_y}=\frac{8\pi\gamma l_P^2}{P^z}j_z\,,\quad\frac{1}{N_y}\frac{1}{N_z}=\frac{8\pi\gamma l_P^2}{P^x}j_x\,,\quad\frac{1}{N_z}\frac{1}{N_x}=\frac{8\pi\gamma l_P^2}{P^y}j_y$$

Ashtekar, Wilson-Ewing

$$\mu_i=\frac{1}{N_i}$$

Volume corrections

$$\frac{1}{\sqrt{\bar{p}_i}}\to\frac{1}{\sqrt{\bar{p}_i}}\left[1+\left(\frac{\pi\gamma l_P^2}{\bar{p}_i}\right)^2\right]$$

$$\frac{1}{\sqrt{\bar{P}_i}}\to\frac{1}{\sqrt{\bar{P}_i}}\left[1+\frac{N^2}{N_i^2}\left(\frac{\pi\gamma l_P^2}{\bar{P}_i}\right)^2\right]$$

Improved LQC from QRLG

Homogeneous case

$$p = 8\pi\gamma\ell_P N^{2/3} j \qquad c = N^{1/3}\theta$$

Collective variables

$$\langle N, j, \theta | \hat{H}^{grav} | N, j, \theta \rangle = \frac{3}{8\pi G\gamma^2}\sqrt{p} N^{2/3} \sin^2(N^{-1/3} c)$$

Collective coherent states

FIXED N

$$\mu_0 = N^{-1/3}$$

Is there a way to reproduce the improved scheme ?

(without a graph changing H)

Statistical approach: density matrix

Count microstates (N,j,θ) compatible with macroscopic configuration (c,p)

Few big or many small ? Both !

$$\rho_{p,c} = \sum_N c_N \, |N, j(p,N), \theta(c,N)\rangle\langle N, j(p,N), \theta(c,N)|$$

Key Observation: for fixed area, j has a minimum → N has a maximum

$$\boxed{p = 8\pi\gamma\ell_P^2 N_{max}^{2/3} j_0}$$

$$c_N = \frac{1}{2^{N_{max}}}\binom{N_{max}}{N} \qquad Tr(\rho_{p,c}H) = \frac{3}{8\pi G\gamma^2}\sqrt{p}\, f(c, N_{max})$$

Approximating the binomial with a Gaussian

$$f(c, N_{max}) \sim \frac{\sin^2(\tilde{\mu}c)}{\tilde{\mu}^2} \qquad \tilde{\mu} = (N_{max}/2)^{-1/3}$$

$$p\tilde{\mu}^2 = \tilde{\Delta} \qquad \tilde{\Delta} = (2)^{2/3} 8\pi\gamma\ell_{pl}^2 j_0$$

Improved scheme from QRLG

Perspectives

- QFT on quantum spacetime ? Phenomenology (scalar field included)
  EA Cianfrani Bilski
- Study the Physical Hilbert space, Graph Changing Hamiltonian
- Link to LQC phenomenology *Ashtekar, Agullo, Barrau, Bojowald, Cailletau, Campiglia, Corichi, Giesel, Hofmann, Grain, Henderson, Kaminski, Lewandowski, Mena Marugan, Nelson, Pawlowski, Pullin, Singh, Sloan, Taveras, Thiemann, Winkler, Wilson-Ewing*
- Link to Spinfoam Cosmology *Bianchi Krajewski Martin-Benito Rennert Rovelli Sloan Vidotto Wilson-Ewing*
- Link to GFT Cosmology *Calcagni Gielen Oriti Sindoni Wilson-Ewing*
- Full theory in a gauge? (Perturbations) *Bodendorfer, Lewandowski, Świeżewski*
- Test the new Hamiltonian: *Alesci Assanioussi Makinen Lewandowski*
- <u>Arena for the canonical theory</u>:

AQG, Master constraint, deparametrized theories.. Computable!