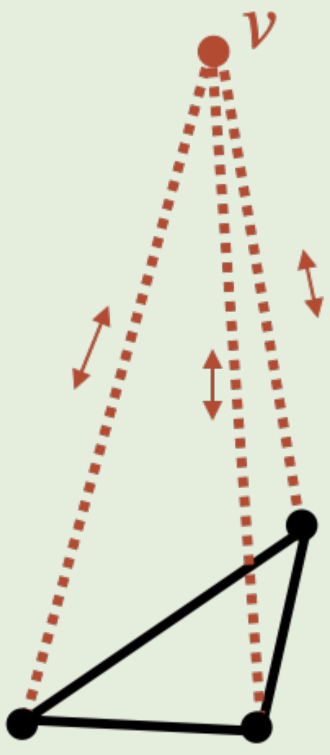


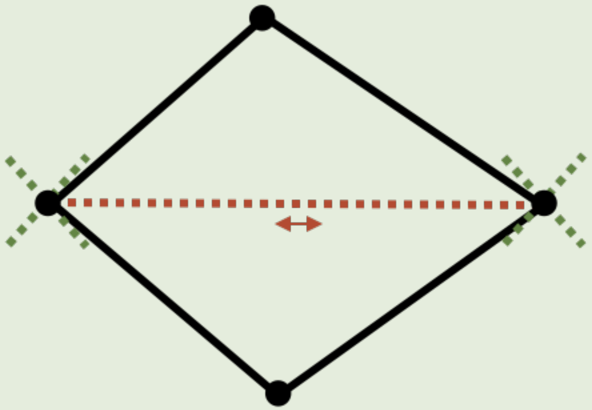

# Spikes and spines in 3D and 4D Lorentzian simplicial quantum gravity


Dongxue Qu

Collaboration with: Johanna Borissova, Bianca Dittrich, Marc Schiffer



arXiv: 2406.19169, 2407.13601

*Oct. 22nd @ ILQGS*


PERIMETER INSTITUTE

# Outline

- **Motivation**
- **LORENTZIAN GEOMETRY OF SIMPLICES**
  - Complex Regge action
  - Light-cone irregularity
  - Generalized triangle inequalities
  - Asymptotic behavior of Regge action
- **3D REGGE ACTION ASYMPTOTICS**
- **4D REGGE ACTION ASYMPTOTICS**

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- **Challenges in Quantum Gravity**: Ensuring **finiteness** of the path integral and avoiding **singularities**.

[J. Ambjorn, J. Borissova, B. Dittrich, J. L. Nielsen, J. Rolf , T. Regge, DQ, G. K. Savvidy, M. Schiffer, R. Sorkin, K. Tate, M. Visser]

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- **Challenges in Quantum Gravity**: Ensuring **finiteness** of the path integral and avoiding **singularities**.
- **Simplicial Quantum Gravity Approaches**: Quantum Regge Calculus, Spinfoams etc

  Triangulations as Regulators: $\infty$-dim path integral $\Longrightarrow$ integral over finitely many edge lengths

[J. Ambjorn, J. Borissova, B. Dittrich, J. L. Nielsen, J. Rolf , T. Regge, DQ, G. K. Savvidy, M. Schiffer, R. Sorkin, K. Tate, M. Visser]

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- **No guarantee** of the **finiteness** of the path integral: configurations with edge lengths can become arbitrarily large.
  - Euclidean Quantum Regge Calculus, e.g., 2D Euclidean Regge [J. Ambjorn, J. L. Nielsen, J. Rolf, and G. K. Savvidy] $\Longrightarrow$ $\infty$ partition function and length expectation values.

[J. Ambjorn, J. Borissova, B. Dittrich, J. L. Nielsen, J. Rolf , T. Regge, DQ, G. K. Savvidy, M. Schiffer, R. Sorkin, K. Tate, M. Visser]

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[J. Ambjorn, J. Borissova, B. Dittrich, J. L. Nielsen, J. Rolf , T. Regge, DQ, G. K. Savvidy, M. Schiffer, R. Sorkin, K. Tate, M. Visser]

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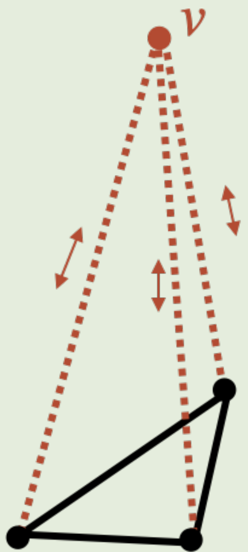


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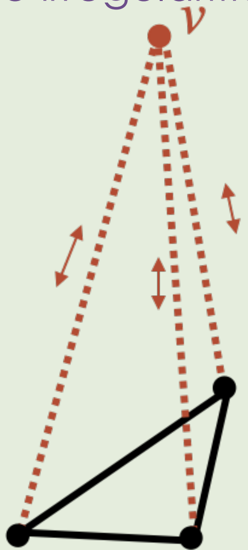
- **Significance of Spikes and Spines:**
  - **Spikes:** ☑ Lorentzian Quantum Regge Calculus $\Longrightarrow$ finite partition function and length expectation values

    ☑ Light-cone irregularities can appear $\Longrightarrow$ how to handle light-cone irregular configurations

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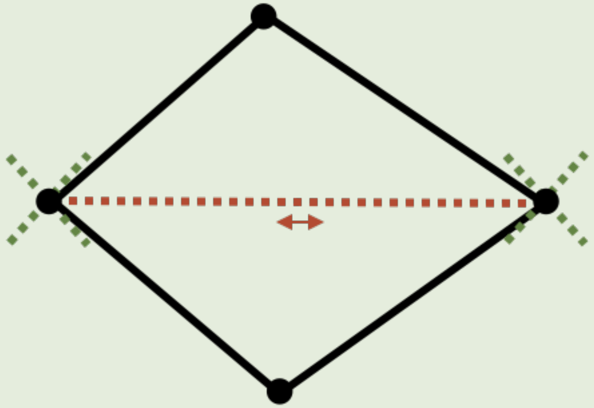

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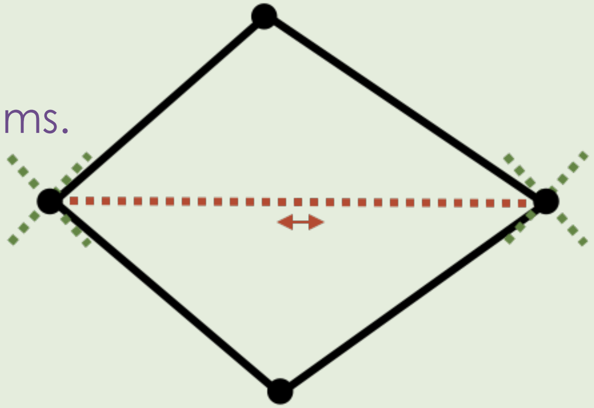
- **Significance of Spikes and Spines:**
  - **Spikes:**
  - **Spines:**
    - ☑ Finite partition function and length expectation values in Lorentzian Quantum Regge Calculus
    - ☑ Light-cone irregular configurations also appear.
    - ☑ can only appear in Lorentzian triangulations.
    - ☑ These have not been discussed so far in approaches e.g., Spinfoams.



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- **Significance of Spikes and Spines:**
  - **Spikes:**
  - **Spines:**
- **3D vs. 4D Lorentzian Settings:** A comparison of how spikes and spines behave in different dimensions.
- **Research Goals:**
  - Address issues of divergence in the Lorentzian path integral
  - Investigate the behavior of spike and spine configurations to ensure finite results.
  - Understand the implications of these configurations for quantum gravity.

[J. Ambjorn, J. Borissova, B. Dittrich, J. L. Nielsen, J. Rolf , T. Regge, DQ, G. K. Savvidy, M. Schiffer, R. Sorkin, K. Tate, M. Visser]

# LORENTZIAN GEOMETRY OF SIMPLICES

**Signature (Metric)**: $(-,+,+,\cdots)$

# Complex Regge action

- **Regge Path Integral:** $Z \sim \int \mathscr{D}s_e \, e^{\imath S}$

[S. Asante, B. Dittrich, D. Jia, J. Padua-Arguelles, R. D. Sorkin]

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$d$-dim simplex $\sigma^d = (012\cdots d)$, $s_{ij}$ is the squared edge length

Caley-Menger determinant

$$\mathbb{V}_{\sigma^d} = \frac{(-1)^{d+1}}{2^d(d!)^2}\det\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & s_{01} & s_{02} & \cdots & s_{0d} \\ 1 & s_{01} & 0 & s_{12} & \cdots & s_{1d} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_{0d} & s_{1d} & s_{2d} & \cdots & 0 \end{pmatrix} = \begin{cases} >0, & \text{spacelike} \\ =0, & \text{null} \\ <0, & \text{timelike} \end{cases}$$

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Deficit angle: $\begin{cases} \epsilon_h^{(\text{bulk})} = 2\pi + \sum_{\sigma\supset h}\theta_{\sigma,h} \\ \epsilon_h^{(\text{bdry})} = \pi + \sum_{\sigma\supset h}\theta_{\sigma,h} \end{cases}$

$$\vec{a}\cdot\vec{b} = \frac{d^2}{\mathbb{V}_h}\frac{\partial\mathbb{V}_\sigma}{\partial s_{\bar{h}}}, \quad \vec{a}\cdot\vec{a} = \frac{\mathbb{V}_{\rho_a}}{\mathbb{V}_h}, \quad \vec{b}\cdot\vec{b} = \frac{\mathbb{V}_{\rho_b}}{\mathbb{V}_h}.$$

$$\log(-r) = \log r - \imath\pi$$

$$\sqrt{-r} = \imath\sqrt{r}, \quad r \in \mathbb{R}_+$$

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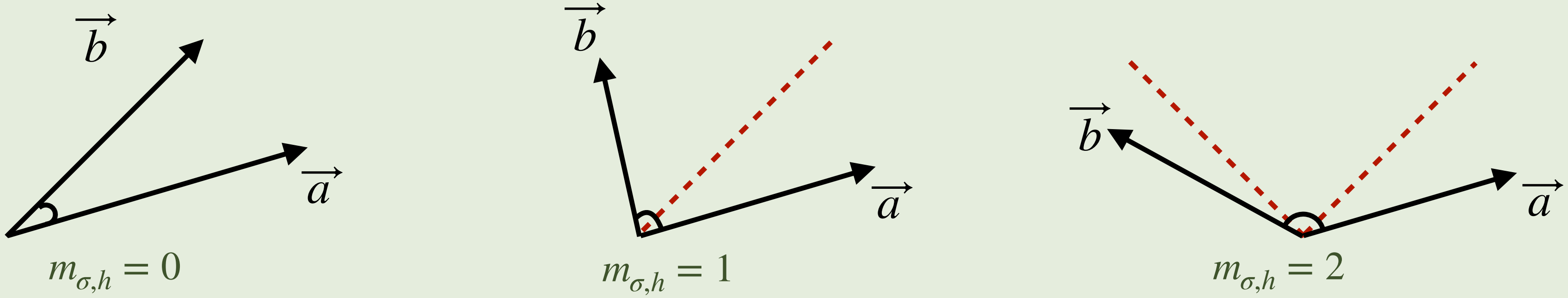


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$$\mathcal{N}_h = \sum_{\sigma\supset h} m_{\sigma,h} = \begin{cases} 4, \\ \neq 4 \end{cases}$$

- *Light-cone regular config.*
- *Light-cone irregular config.*

$$\Longrightarrow S \supset S_h \begin{cases} \text{Real} \\ \text{includes + or − imaginary part} \end{cases}$$

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# Light-cone irregularity

There is a **branch cut** in the action along the light-cone irregular configuration.

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$\Uparrow$

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[S. Asante, B. Dittrich, T. Jacobson, J. Padua-Arguelles, R. D. Sorkin]

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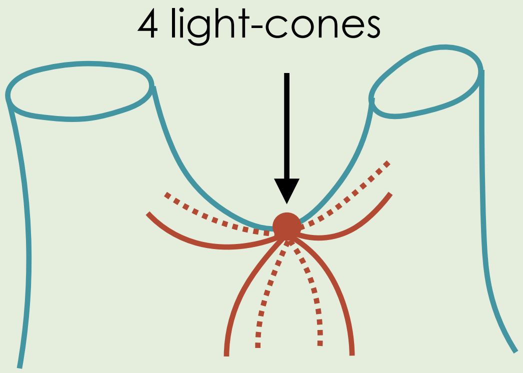
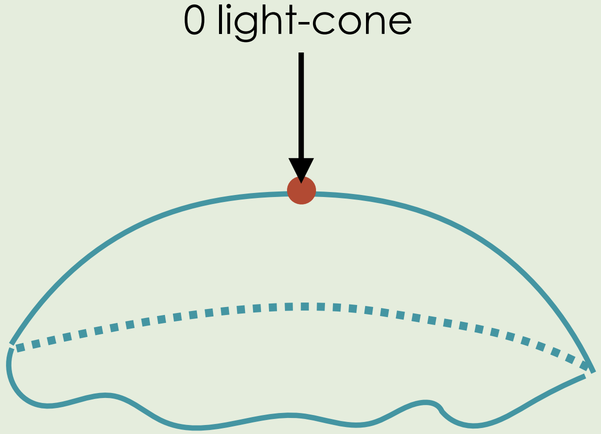
- Examples of Light-Cone Irregularities:
  - A topological change that plays a key role in deriving entropy from the Lorentzian path integral.




Trouser config.




yarmulke config.

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  - These configurations introduce imaginary terms into the complex Regge action with a "+" or "-"sign, depending on how the squared edge length is complexified: $s_e \to s_e \pm \imath\varepsilon$ and $\varepsilon \to 0$.

[S. Asante, B. Dittrich, T. Jacobson, J. Padua-Arguelles, R. D. Sorkin]

# Light-cone irregularity

There is a **branch cut** in the action along the light-cone irregular configuration.

$$\epsilon^L_{\sigma,h} = 2\pi - \frac{\pi}{2}\left(\sum_{\sigma\supset h} m_{\sigma,h}\right) - \imath \sum_{\sigma\supset h} \beta_{\sigma,h}, \qquad \mathcal{N}_h = \sum_{\sigma\supset h} m_{\sigma,h} \neq 4$$

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  - 2. Select the side that **suppresses** contributions from light-cone irregular configurations when the branch cuts are infinitely long.

[S. Asante, B. Dittrich, T. Jacobson, J. Padua-Arguelles, R. D. Sorkin]

# Generalized triangle inequalities

$$\mathbb{V}_{\sigma^d} = \frac{(-1)^{d+1}}{2^d (d!)^2} \det \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & s_{01} & s_{02} & \cdots & s_{0d} \\ 1 & s_{01} & 0 & s_{12} & \cdots & s_{1d} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_{0d} & s_{1d} & s_{2d} & \cdots & 0 \end{pmatrix} = \begin{cases} > 0, & \text{spacelike} \\ = 0, & \text{null} \\ < 0, & \text{timelike} \end{cases}$$

- A Euclidean or spacelike (non-degenerate) simplex $\sigma$:

$$\mathbb{V}_\sigma > 0, \quad \mathbb{V}_\rho > 0$$

For the simplex $\sigma$ itself and for all subsimplices $\rho \subset \sigma$

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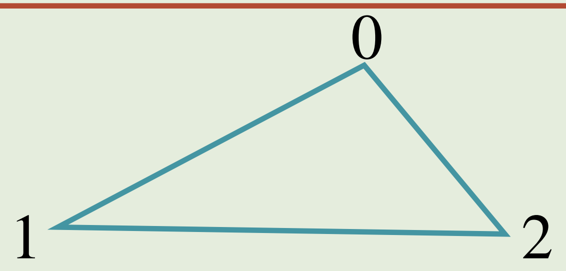


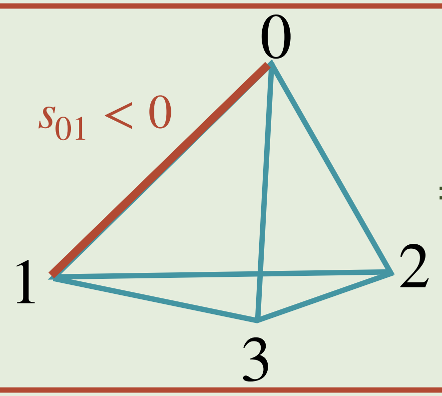

$$\mathbb{V}_{(012)} > 0, \quad s_{01} > 0 \quad s_{02} > 0, \quad s_{12} > 0$$

[K. Tate and M. Visser]

# Generalized triangle inequalities

- A Lorentzian simplex $\sigma$, whose subsimplices $\rho$ can be timelike, spacelike, or null:

$$\rho \subset \sigma, \mathbb{V}_\rho \leq 0 \quad \Rightarrow \quad \forall \rho' \supset \rho : \; \mathbb{V}_{\rho'} \leq 0$$

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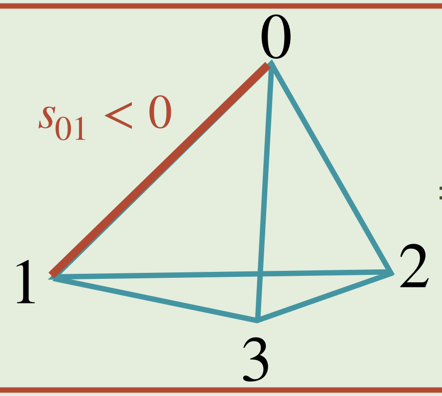


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With one spacelike edge and two timelike edges, or one timelike edge and two spacelike edges, the Lorentzian triangle inequality is always satisfied

$$\mathbb{V}_{(012)} = -\frac{1}{16}(s_{01}^2 + (s_{02} - s_{12})^2 - 2s_{01}s_{02} - 2s_{01}s_{12})$$

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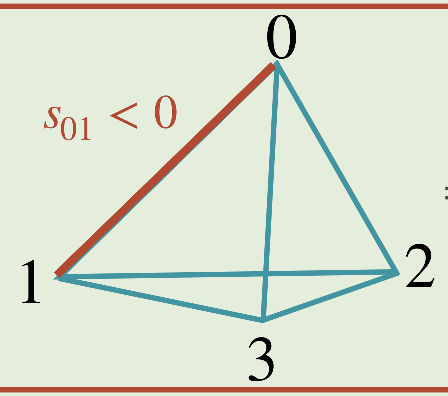


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In the EPRL spinfoam, 5-1 move with spacelike boundary tetrahedra and large spacelike bulk edges is **forbidden**.

[K. Tate and M. Visser]

# Asymptotic behavior of Regge action

- Asymptotic behavior of the Regge action $S = -\imath \sum_h \sqrt{\mathbb{V}_h}\epsilon_h$
  - Volumes in the limit of single/multiple large edges: e.x., $\mathbb{V}_{(0123)} = \pm \frac{1}{9}\mathbb{V}_{(123)}\,\lambda + \mathcal{O}(\lambda^0) \Longrightarrow$ **Dimensional reduction**

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$$\theta_{(0123),(02)} = -\imath \log\left(\frac{bx^2 - \imath\sqrt{-x^2 s_{02}\left(b^2 - s_{02} + x^2\right)}}{\sqrt{s_{02} - b^2}\sqrt{x^2 s_{02} - x^4}}\right) \xrightarrow{x\to\infty} -\imath \log\left(-\imath\frac{b + \sqrt{s_{02}}}{\sqrt{s_{02} - b^2}}\right) = \mathcal{O}\left(s_{01}^0\right) ,$$

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Volumes, complex dihedral angles, and deficit angles can all be computed explicitly, leading to quite **simple results** in the asymptotic regime.

- Asymptotic behavior of Regge action is simple (dimensional reduction),

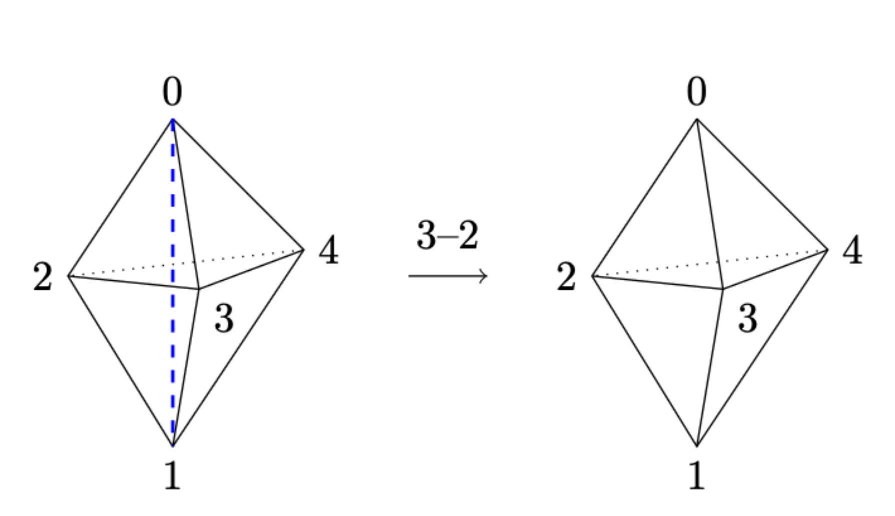
  - e.g., in the 3-2 move: $S^{3-2} = -\imath\, 2\pi\sqrt{s_{01}} + \ \mathcal{O}(\log s_{01})$

  - In 2D, the Regge action is invariant, we don't observe a boundary data dependence for the leading coefficient.

# 3D REGGE ACTION ASYMPTOTICS

# Spines in 3D

3 - 2 Pachner move $\Longleftrightarrow$ Spine configuration in 3D




Regge Path Integral: $Z \sim \int \mathscr{D}s_e \, e^{\imath S}$

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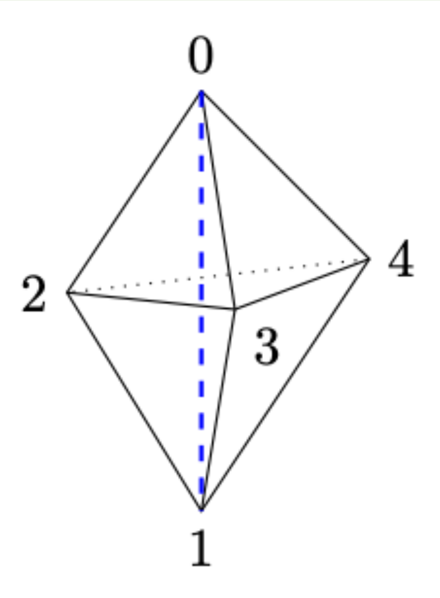


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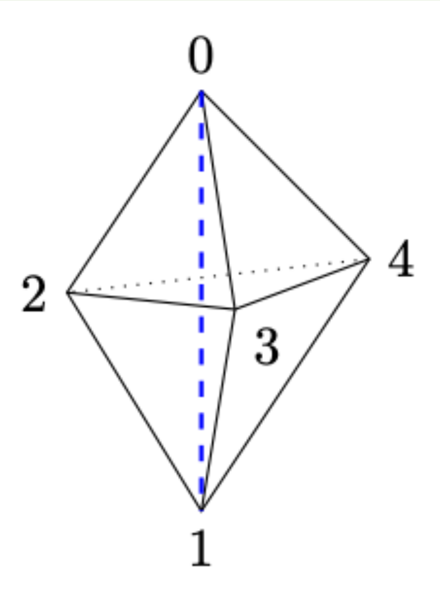


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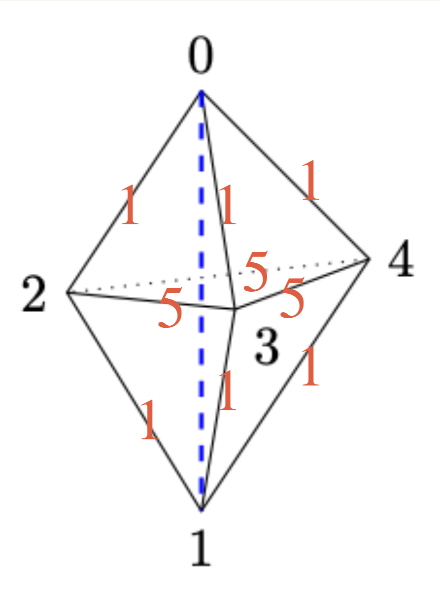


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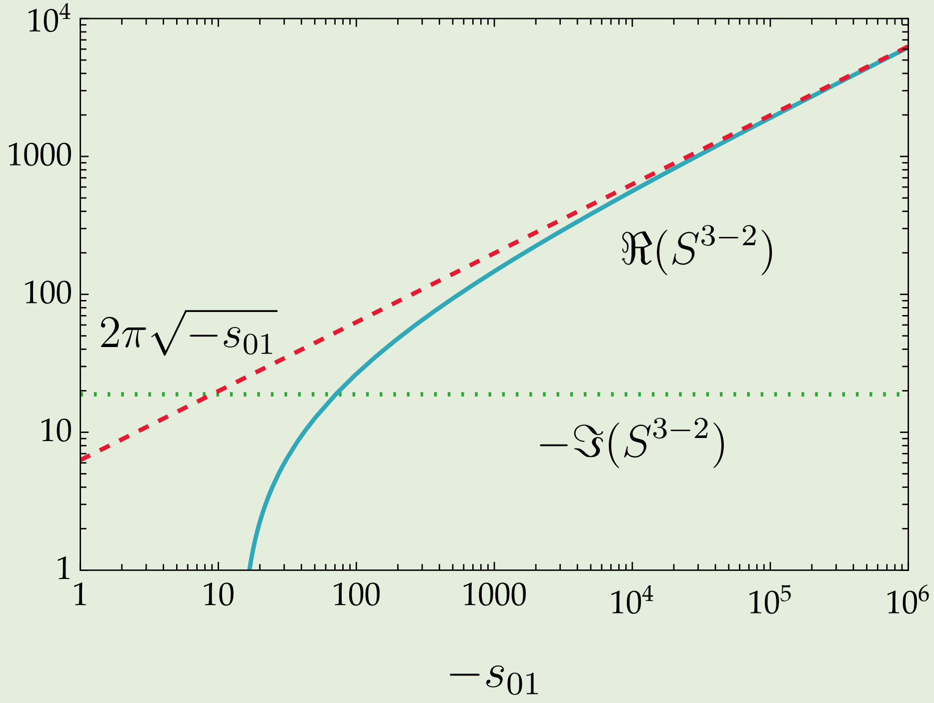


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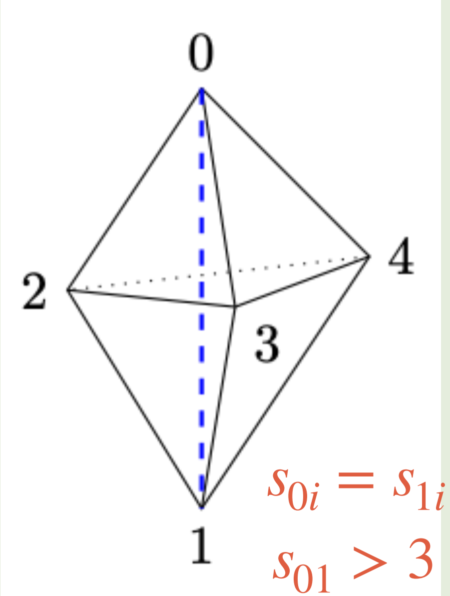


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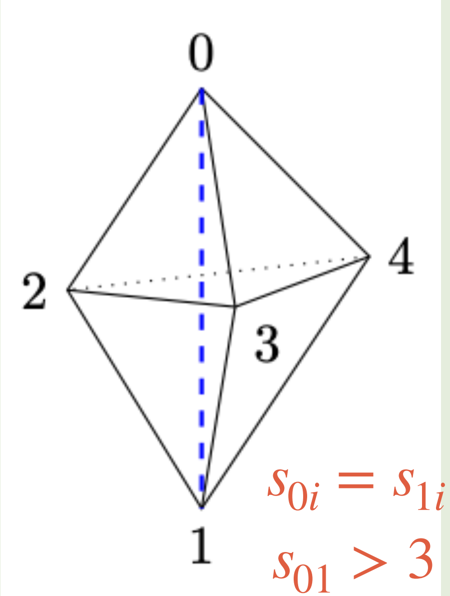


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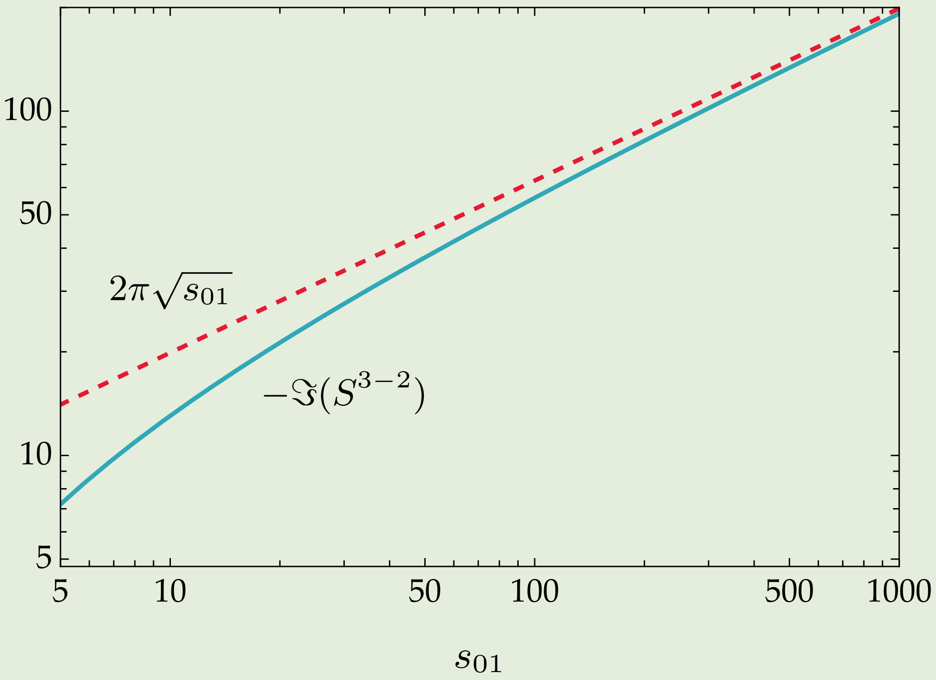


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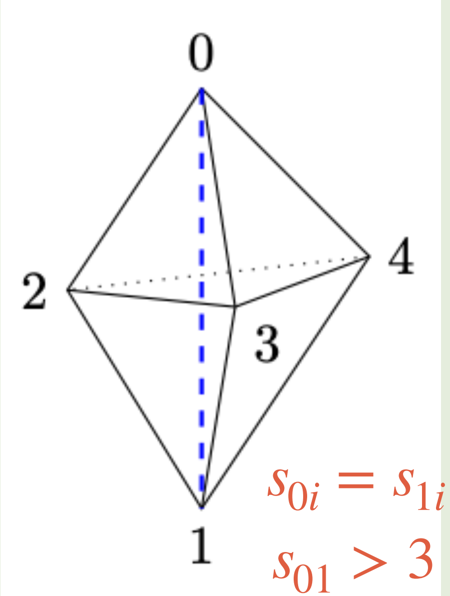


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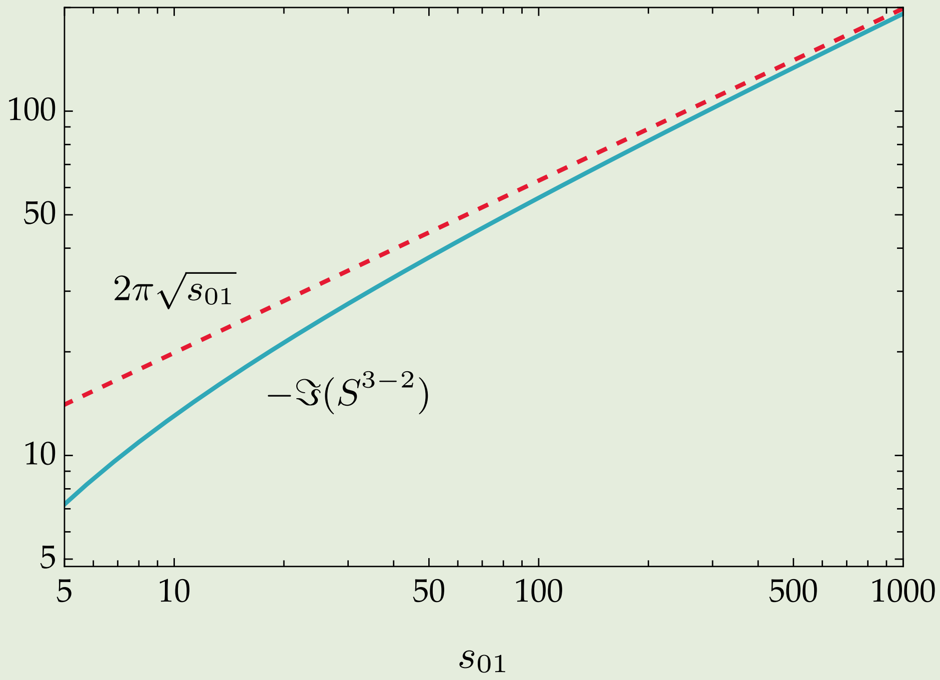


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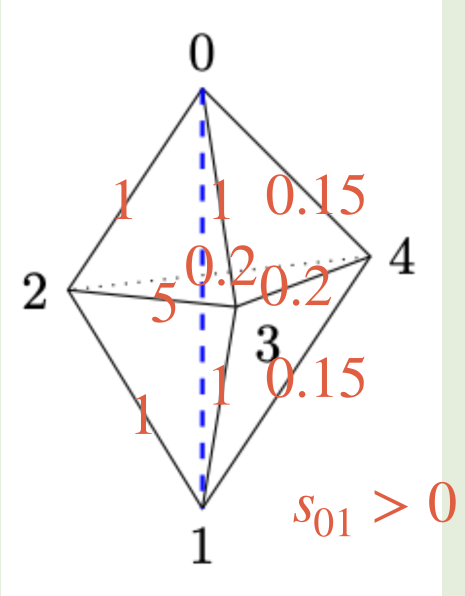


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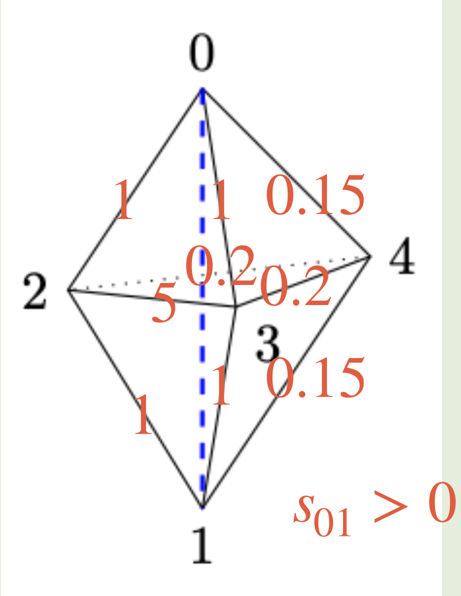


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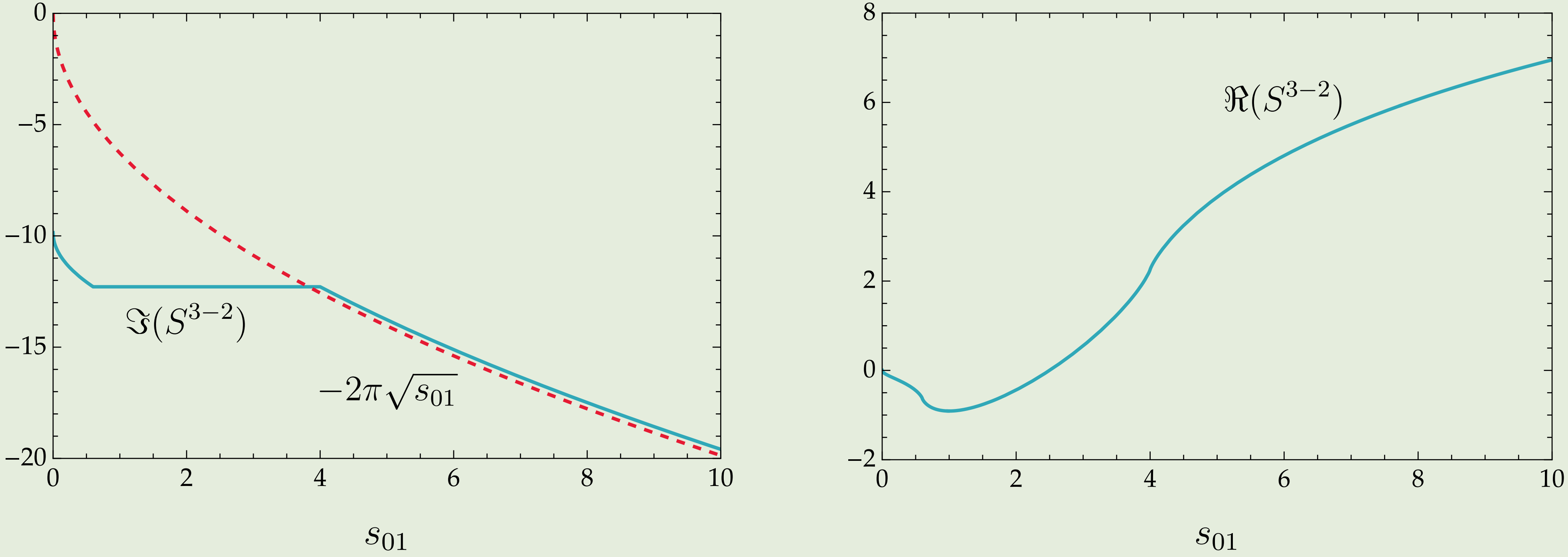

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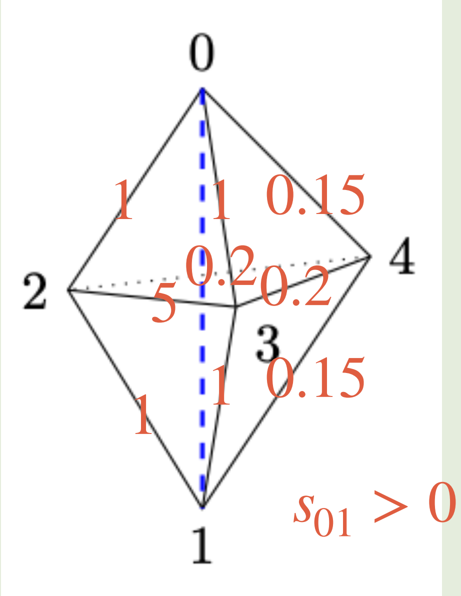


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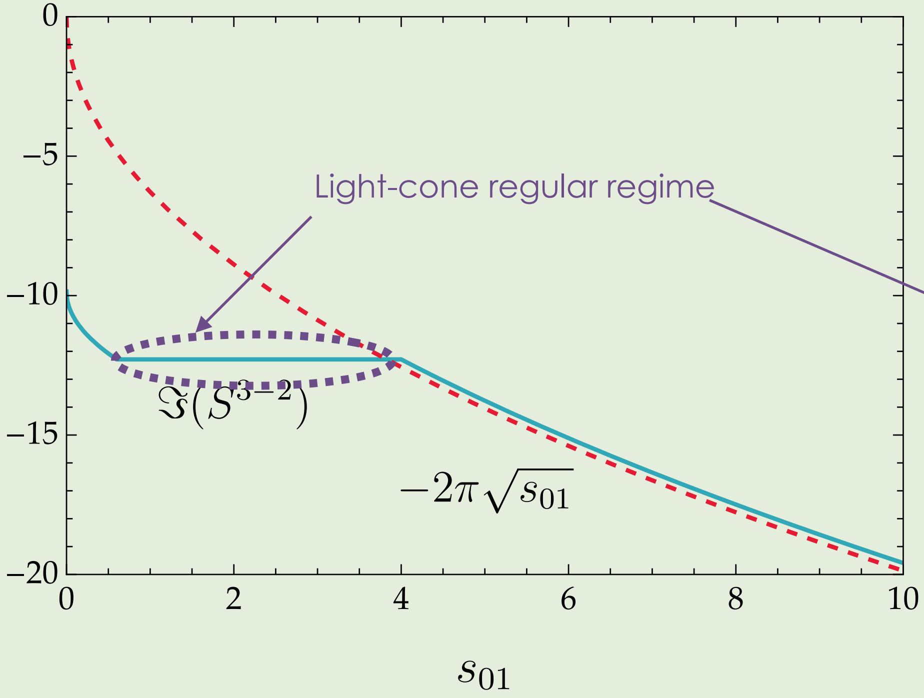


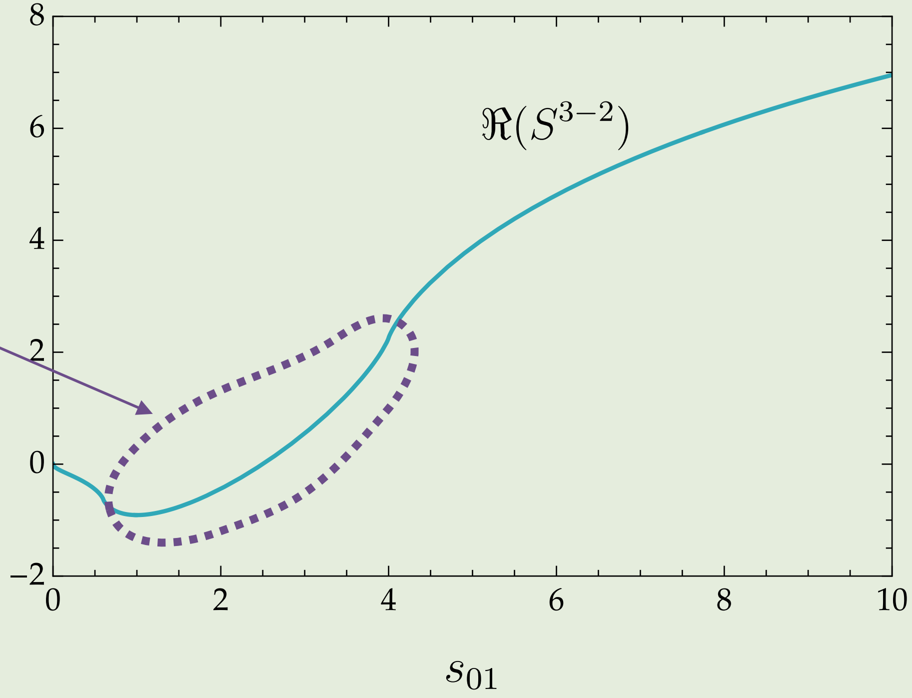

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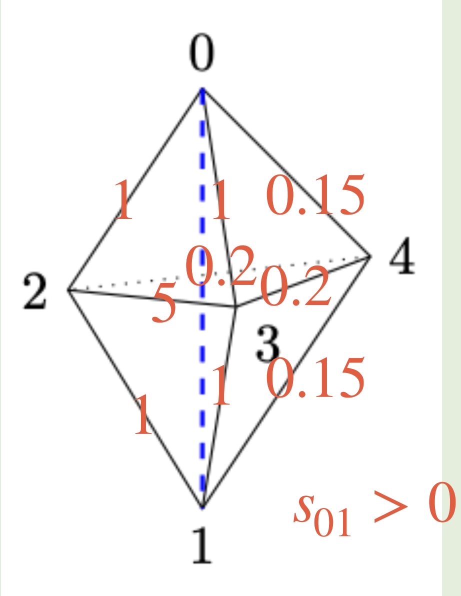


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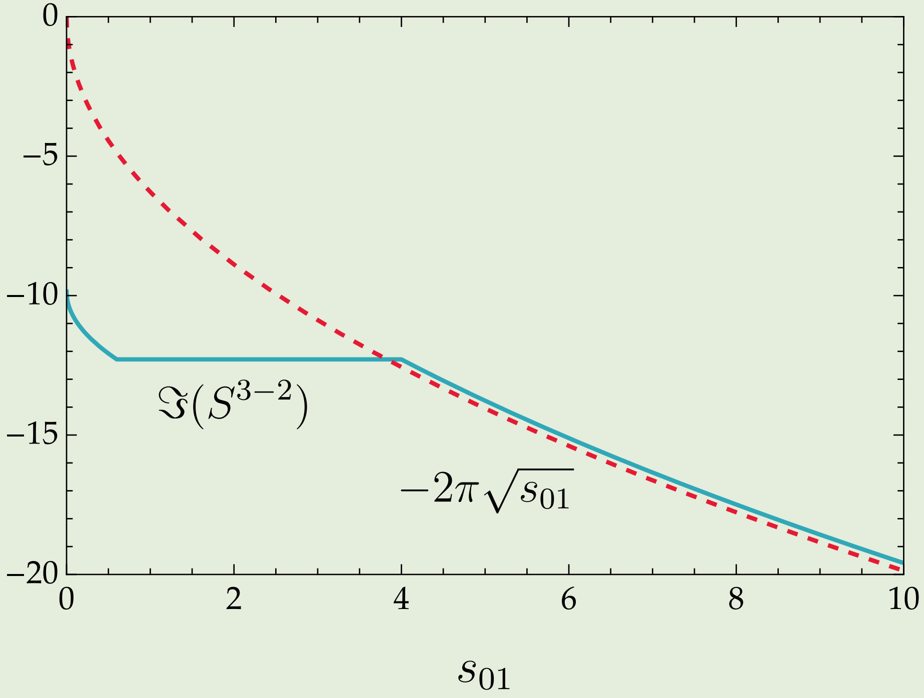
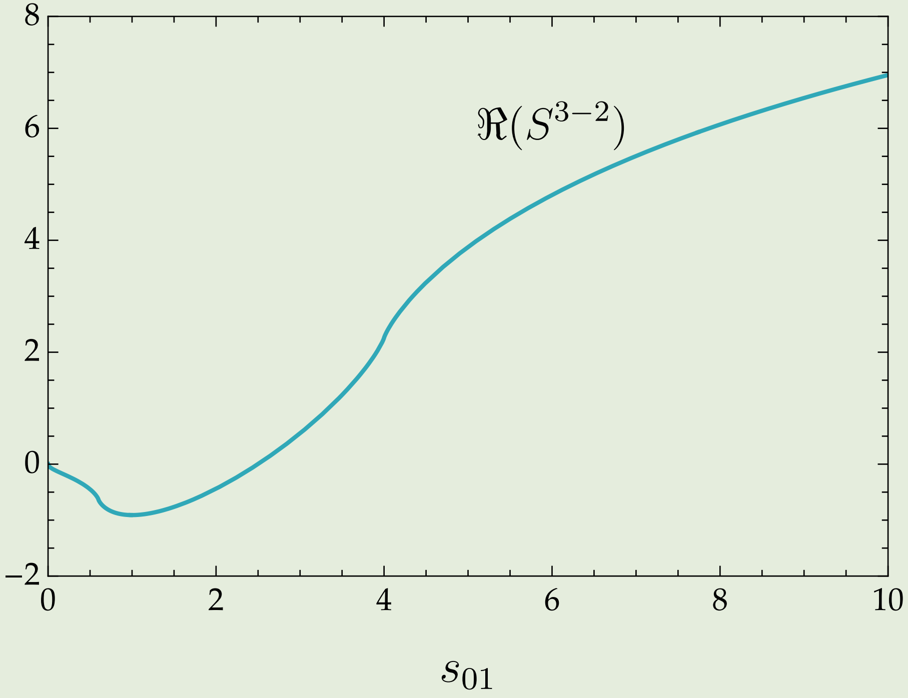
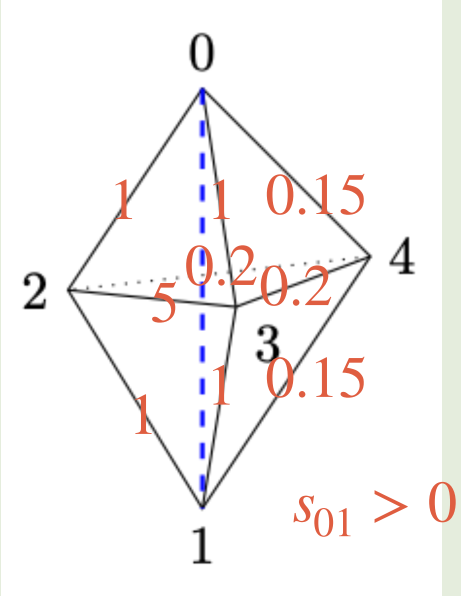


# Spines in 3D

3 - 2 Pachner move $\Longleftrightarrow$ Spine configuration in 3D




Initial congif.: tetrahedra $(0123), (0124), (0134)$ share one edge $(01)$

$$s_{(01)} \to \pm \infty$$

Asymptotic behavior of Regge action: $S^{3-2} = -\imath\, 2\pi\sqrt{s_{01}} + \mathcal{O}(\log s_{01})$

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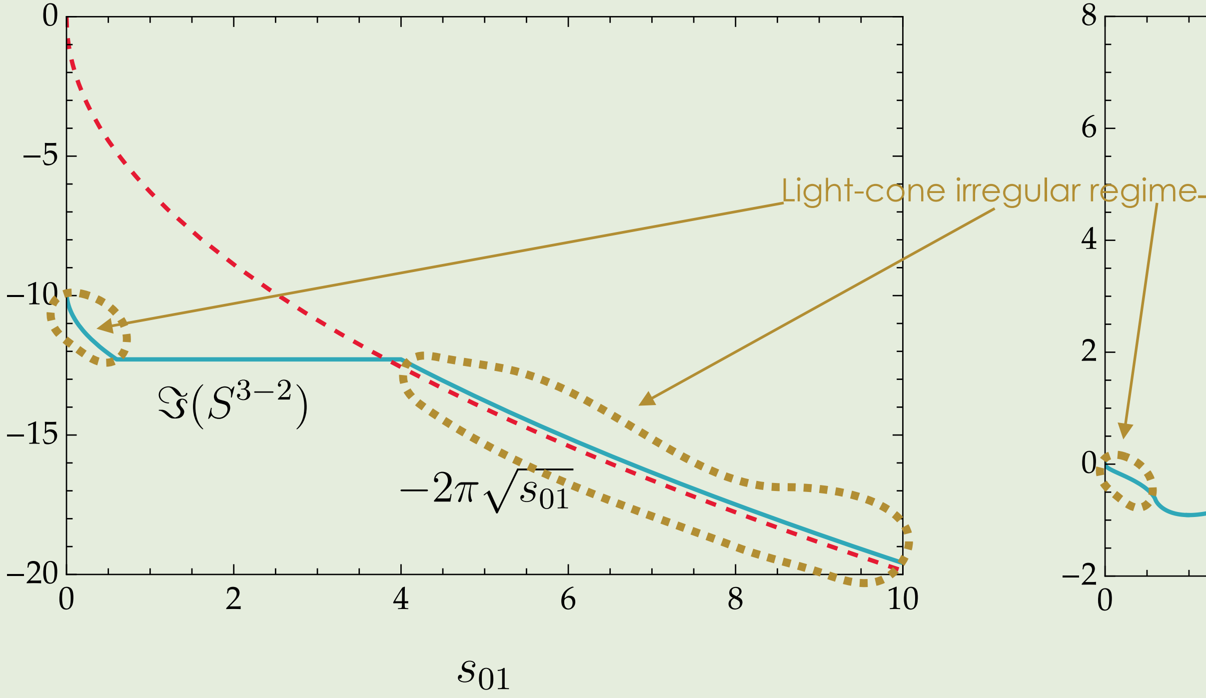


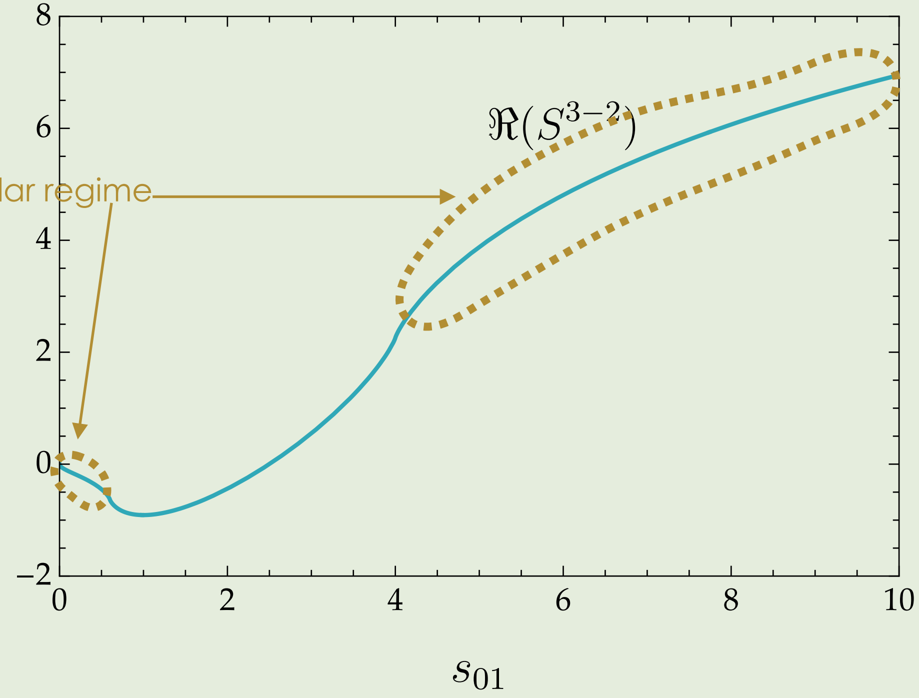

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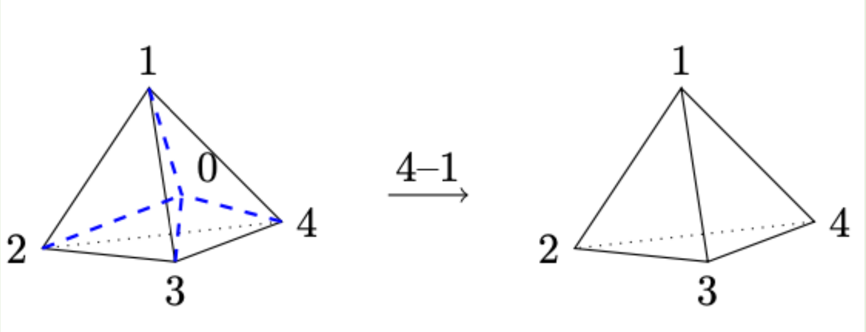
  - For a spacelike edge → the boundary deficit angle is non-compact → Length operator has a continuous spectrum.
  - In the **Lorentzian Ponzano-Regge model,** the spectrum for the timelike length behaves as:

$$T \sim j \in \mathbb{N} \gg 1 \to \; Z \sim \exp(\imath 2\pi j) = 1$$

where we ignore measure terms that might suppress the amplitude for large edge lengths.

# Spikes in 3D

4 - 1 Pachner move ⟺ Spike configuration in 3D



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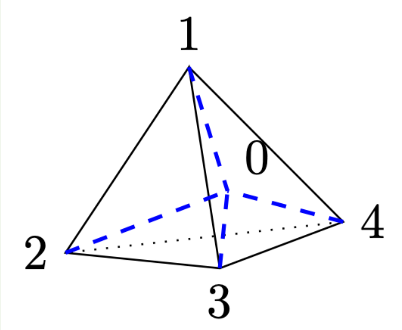


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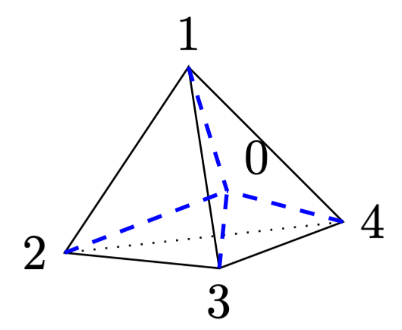
Initial congif.: tetrahedra $(0123), (0124), (0134), (0234)$ share one vertex $(0)$ with 4 bulk edges $s_{0i}$, $i = 1,\cdots,4$.

4 bulk variables with a constant Regge action condition $\Longrightarrow$ 3-dim gauge orbits $\Longrightarrow$ gauge fixing: $|s_{0i}| = \lambda$

(additive scaling)

# Spikes in 3D

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| 4-1 move | Asymptotic behavior of the Regge Action | Properties |
|---|---|---|
| **Homogenous case** $s_{0i} = \pm\lambda, \quad i = 1,\cdots,4$ | $S^{4-1} = -4\pi\imath\sqrt{\pm\lambda} + \mathcal{O}(\lambda^0)$ | 1. **Timelike** bulk edges are always **light-cone regular**.<br>2. The asymptotic regime is **light-cone irregular** for **spacelike** bulk edges. |
| **Inhomogenous case** $s_{01} = \pm\lambda, \quad s_{0j} = \mp\lambda$ $j = 2,3,4$ | $S^{4-1} = -2\pi\imath(1+\imath)\sqrt{\lambda} + \mathcal{O}(\log\lambda)$ | 1. Edge (01) is **light-cone irregular** if (01) is **spacelike**.<br>2. If three edges are **spacelike**, then at least one of them is **irregular**. |
| **Inhomogenous case** $s_{01} = s_{02} = \pm\lambda$ $s_{03} = s_{04} = \mp\lambda$ | $S^{4-1} = -2\pi\imath\,(1+\imath)\sqrt{\lambda} + \mathcal{O}(\log\lambda)$ | 1. **Light-cone irregular** configuration<br>2. All **spacelike** bulk edges are **light-cone irregular**. |

# Spines and Spikes in 3D

Timelike Bulk Edges:

- The Regge action remains **real**, which ensures **light-cone regularity**.
- This results in the **exponential suppression** of amplitudes, leading to the **convergence** of the path integral.

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- Integration along the side where **Im(S) < 0** causes **divergence** in the path integral.

# Finite Expectation values

3-2 Pachner move (spine configuration in 3D)

- Expectation Values of arbitrary power of length(Wynn's epsilon algorithm): $\mathscr{E}_{3-2}(m,c) = \int_{\sqrt{c}}^{\infty} \mathrm{d}\lambda\, \lambda^m\, e^{\imath S^{3-2}(-\lambda)}$

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- Asymptotic approximations to investigate the convergence behavior of the expectation values:
  - Asymptotic behavior of Regge Action for Timelike Bulk Edge: $S^{3-2} = 2\pi\sqrt{\lambda} + O(\log \lambda)$
  - Measure in Asymptotic Regime: $\mu^{3-2} \propto \lambda^{-3/2}$

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Finite for $m \in \mathbb{R}$

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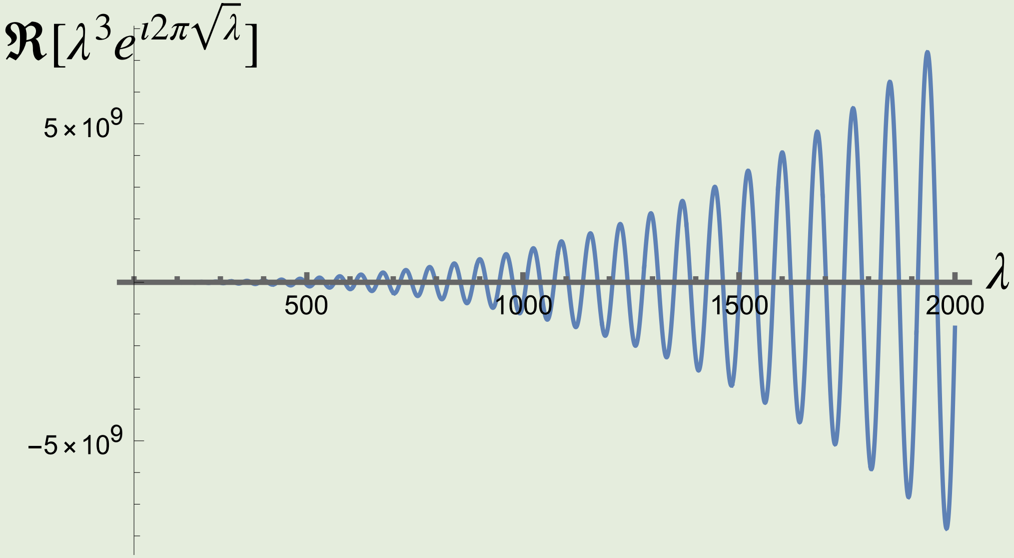
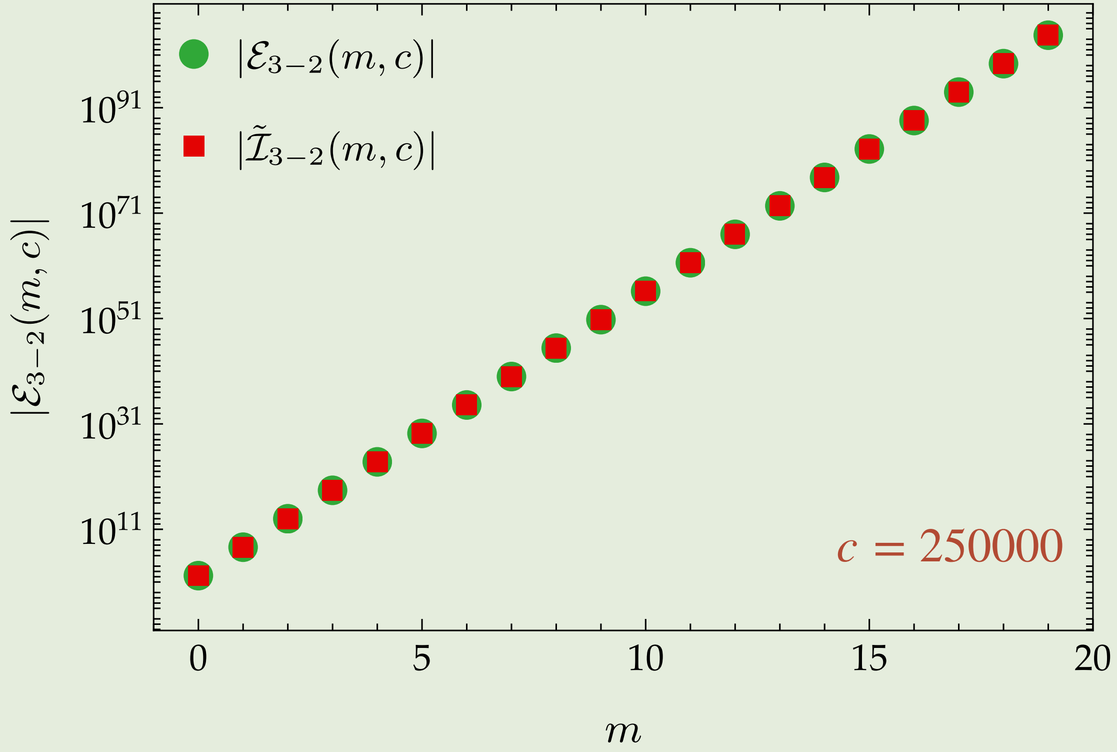



Oscillatory nature of the Lorentzian path integral ensures finite results

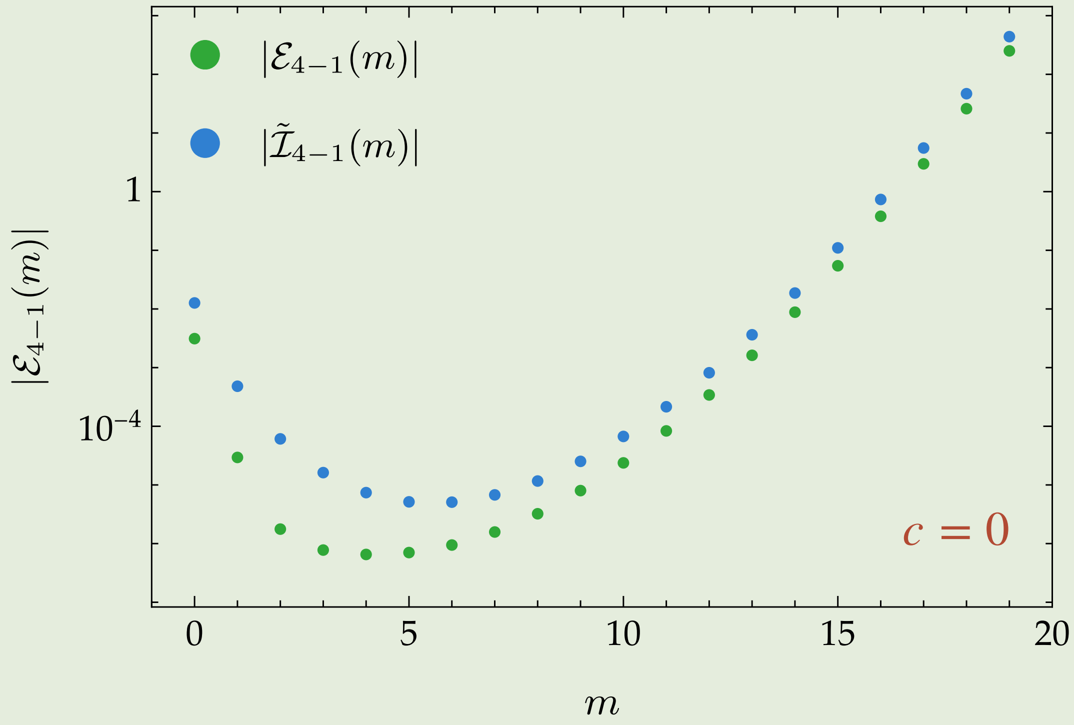
# Finite Expectation values

Expectation Values of arbitrary power of length in the 3-2 move and 4-1 move

The exact numerical result uses the Wynn-Epsilon algorithm method




The approximated result becomes more accurate for larger $m$.



# Discussion for 3D cases

**Key Results**:

1. Discovered many cases with **light-cone irregularities**, which introduce complex behavior in the path integral.
2. The **asymptotic behavior** of the action is **linear** in the length of large bulk edges with coefficients being simple constants like **2π** and **4π** for configurations sharing edges.
3. A simple method was found to identify **allowed asymptotic regimes** based on triangle inequalities, highlighting differences between Lorentzian and Euclidean signature configurations.
4. Found that certain configurations, such as those with **large spacelike bulk edges**, are **forbidden** by Lorentzian triangle inequalities.

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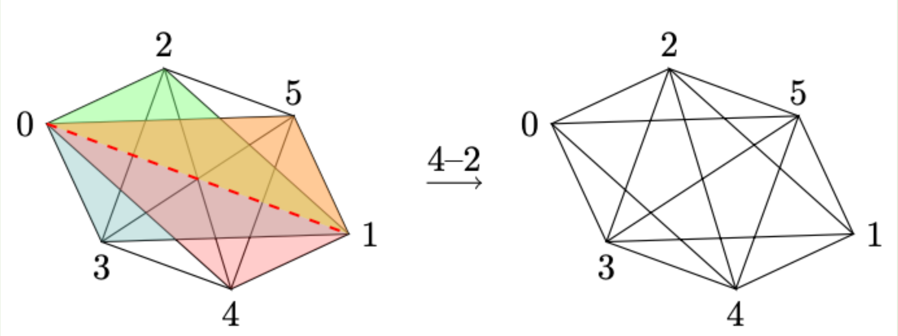
**Finiteness of Path Integral**:

- Either exclude irregular configurations or to choose the suppressing side of the branch cut for the integration contour.
- These finiteness results hold using both **asymptotic and numerical methods** (e.g., **Wynn-Epsilon algorithm**).
- The oscillatory nature of the Lorentzian path integral ensures finite results.

# 4D REGGE ACTION ASYMPTOTICS

# Spines in 4D




4 - 2 Pachner move $\Longleftrightarrow$ Spine configuration in 4D

Initial configuration: four 4-simplices $(01234), (01235), (01245), (01345)$ share one edge $(01)$

Regge Path Integral: $Z \sim \int \mathscr{D}s_e \, e^{\imath S}$

Complex Regge action: $S = -\imath \sum_h \sqrt{\mathbb{V}_h} \epsilon_h$

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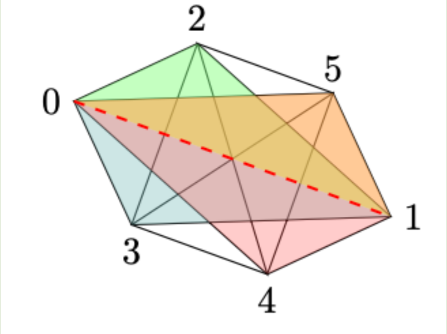


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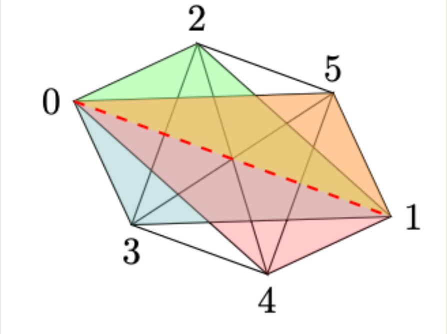


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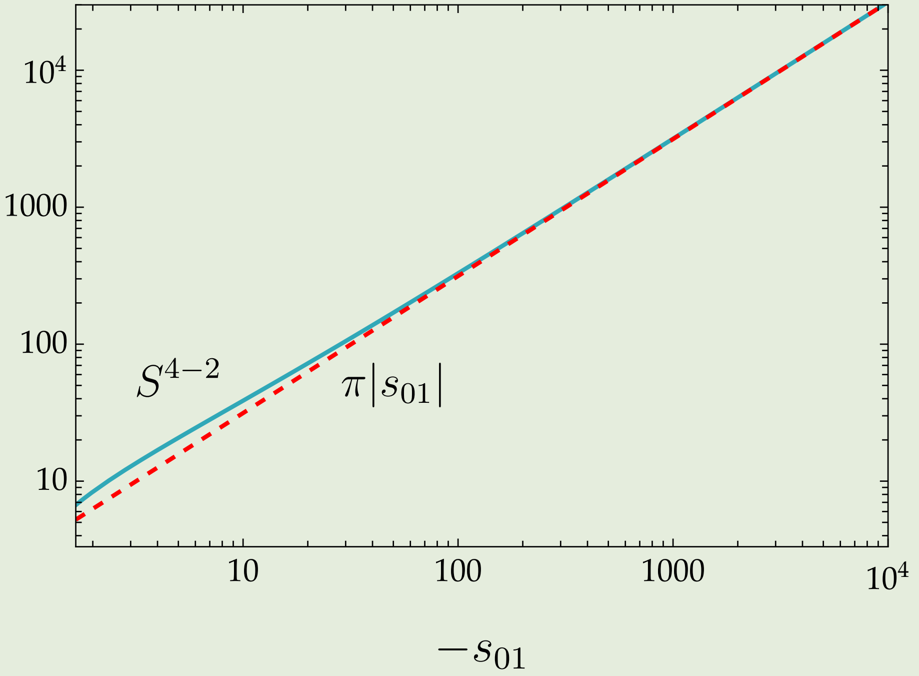


$$S^{4-2} = \pi|s_{01}| + \mathcal{O}(\log s_{01})$$

- There is no asymptotic regime with light-cone irregular bulk triangles.
- Bulk triangles are necessarily timelike in the asymptotic regime.

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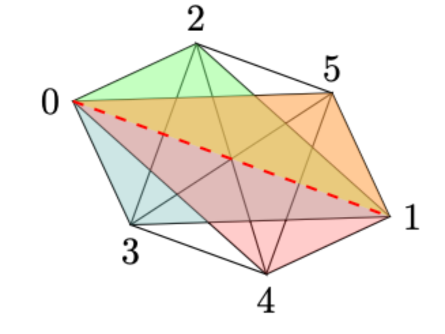


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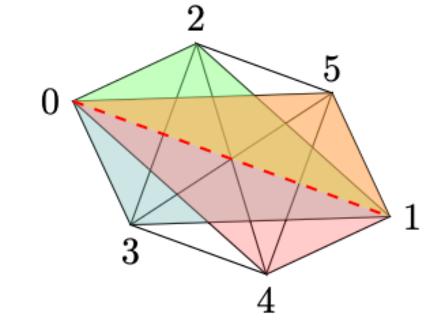


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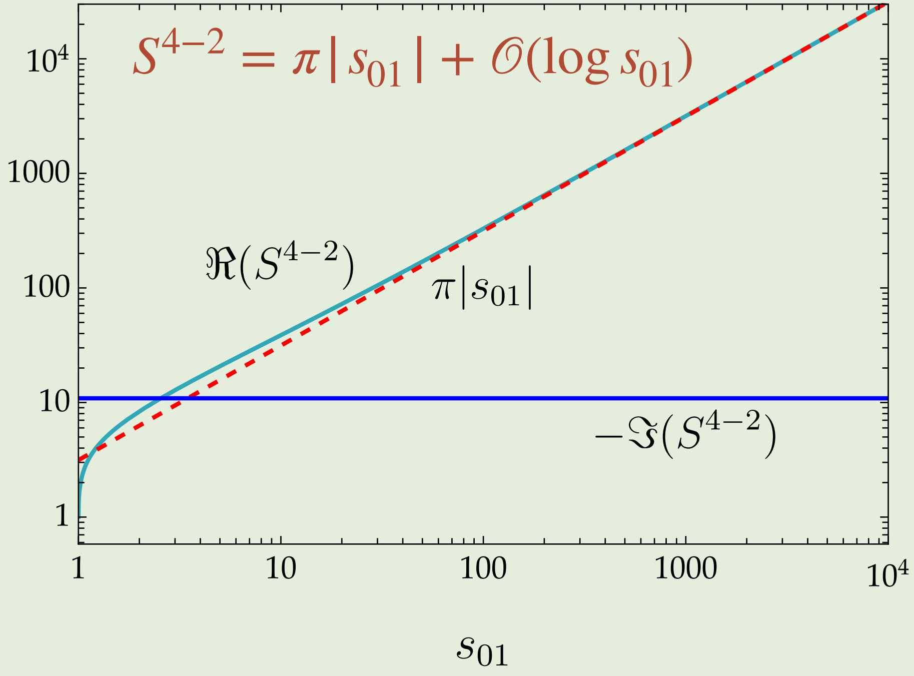


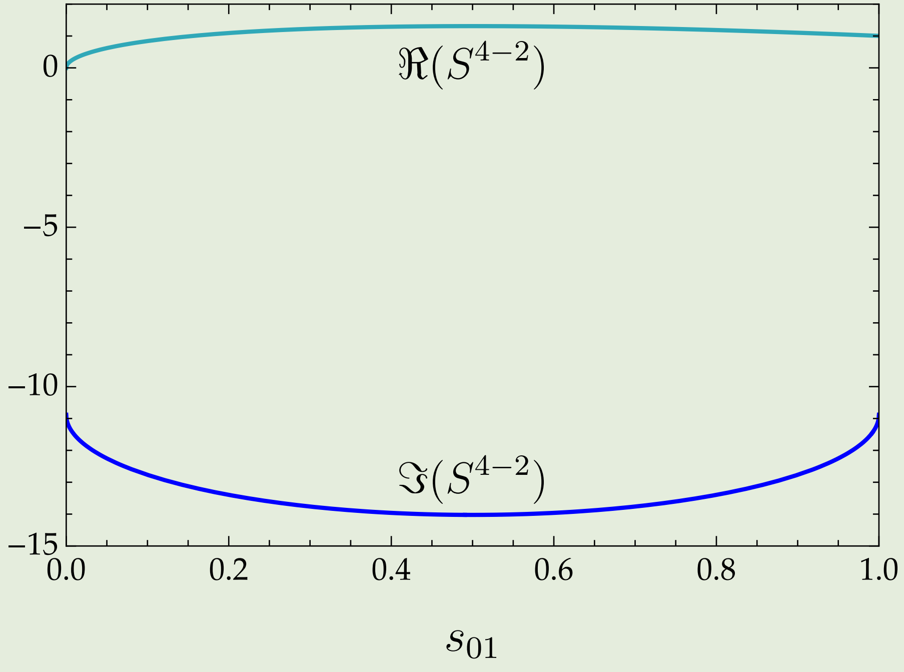


There is no asymptotic regime involving light-cone irregular bulk triangles.

# N 4-simplices sharing an edge

Generalization to $N$ 4-simplices sharing an edge $(01)$ with $T$ bulk triangles [J. Padua-Arguelles]:

$$S = \pi |s_{01}| \frac{(2T - N)}{4} + \mathcal{O}(\log s_{01}), \quad s_{(01)} \to \pm \infty$$

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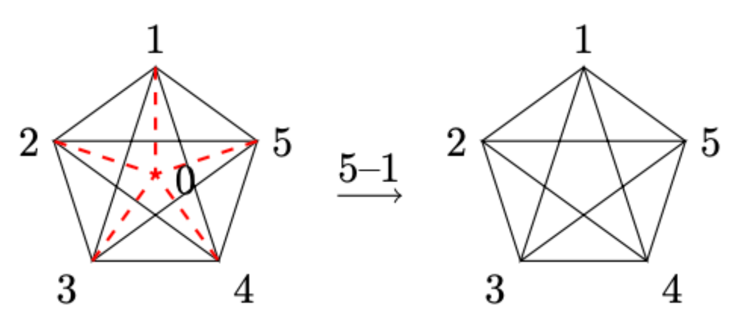
- **Asymptotic Formula for Amplitude:**

$$A \sim \frac{|s_{01}|}{4} \to S \sim A\pi\,(2T - N) \xrightarrow{4-2\text{ move: }2T-N=4} Z \sim 1,$$

where we ignore measure terms that may suppress the amplitude for large values of the area.

# Spikes in 4D




5 - 1 Pachner move $\Longleftrightarrow$ Spike configuration in 4D

Initial configuration:

- Five 4-simplices $(01234), (01235), (01245), (01345), (02345)$
- One vertex $(0)$
- Five bulk edges (to be integrated out)
- Ten bulk triangles

Five bulk variables with a constant Regge action condition $\Longrightarrow$ 4-dim gauge orbits $\Longrightarrow$ gauge fixing: $|s_{0i}| = \lambda$

(additive scaling)

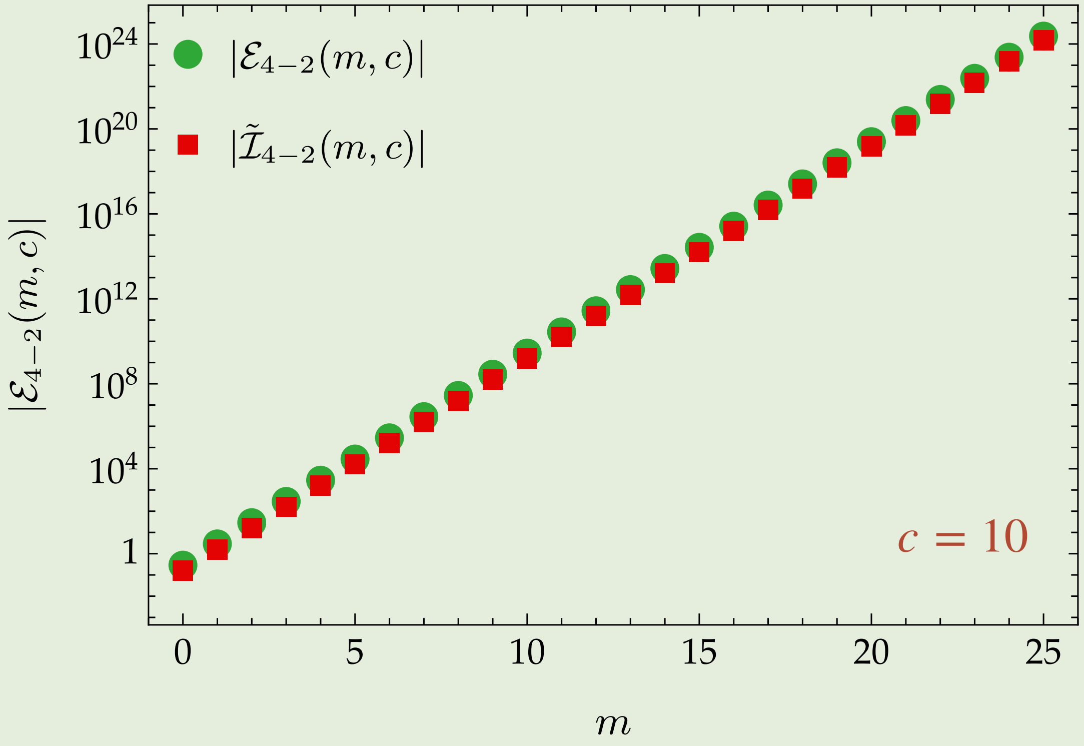
# Spikes in 4D

| 5-1 move | Asymptotic behavior of the Regge Action | Properties |
|---|---|---|
| **Homogenous case** $s_{0i} = -\lambda$ $i = 1,\cdots,5$ | $S^{5-1} = -\frac{\sqrt{\lambda}}{2} S^{E,3D} + \mathcal{O}(\lambda^0), \quad S^{E,3D} = -\sum_{1\le i<j\le 5} \sqrt{s_{ij}}\epsilon_{(ij)}$ | 1. Bdry tetra are spacelike. 2. Light-cone regular bulk triangles |
| **Homogenous case** $s_{0i} = \lambda$ $i = 1,\cdots,5$ | $S^{5-1} = \frac{\sqrt{\lambda}}{2} S^{L,3D} + \mathcal{O}(\lambda^0), \quad S^{L,3D} = -\imath \sum_{1\le i<j\le 5} \sqrt{s_{ij}}\epsilon_{(ij)}$ | Light-cone irregularities depend on the 3D dihedral angles at edge (ij) in the bdry tetra. |
| **Inhomogenous case** $s_{01} = -\lambda, \quad s_{0j} = \lambda$ $j = 2,\cdots,5$ | $S^{5-1} = 2\pi\lambda + \frac{\sqrt{\lambda}}{2} S^{L,3D}_{(2345)} + \mathcal{O}(\lambda^0)$ | |
| **Inhomogenous case** $s_{01} = \lambda, \quad s_{0j} = -\lambda$ $j = 2,\cdots,5$ | $S^{5-1} = 2\pi\lambda - \frac{\sqrt{\lambda}}{2} S^{E,3D}_{(2345)} + \mathcal{O}(\lambda^0)$ | 1. All bulk triangles are timelike 2. Light-cone regular bulk triangles |
| **Inhomogenous case** $s_{01} = s_{02} = \lambda, \quad s_{0j} = -\lambda$ $j = 3,4,5$ | $S^{5-1} = 2\pi\lambda - \imath\lambda^{1/2}\left(\pi\sqrt{\lvert s_{12}\rvert} + \frac{\imath\pi}{2}\sum_{3\le k<l\le 5}\sqrt{\lvert s_{kl}\rvert}\right) + \mathcal{O}(\log\lambda)$ | Spacelike bulk triangles are light-cone irregular and of yarmulke type. |
| **Inhomogenous case** $s_{01} = s_{02} = -\lambda, \quad s_{0j} = +\lambda$ $j = 3,4,5$ | $S^{5-1} = 2\pi\lambda - \imath\lambda^{1/2}\left(\imath\pi\sqrt{\lvert s_{12}\rvert} + \frac{\pi}{2}\sum_{3\le k<l\le 5}\sqrt{\lvert s_{kl}\rvert}\right) + \mathcal{O}(\log\lambda)$ | |

# Finite Expectation values in 4D

4-2 Pachner move (spine configuration in 4D)

The asymptotic expression for the Regge action is real: $S^{4-2} = \pi\lambda + \mathcal{O}(\log\lambda)$ .

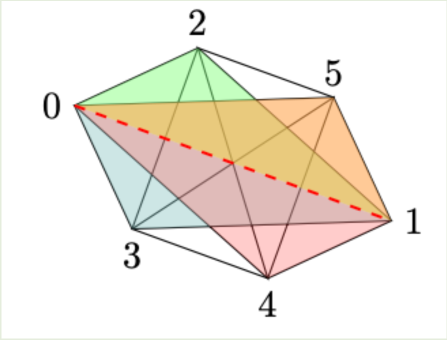



Analytical approximation of the expectation values

$$\tilde{I}_{4-2}(m,c) = \int_c^\infty \mathrm{d}\lambda\,\lambda^m e^{\imath\pi\lambda} = c^{m+1} E_{-m}(-\pi\imath c)$$

Exact numerical result of the expectation values (Wynn-Epsilon algorithm)

$$\mathscr{E}_{4-2}(m,c) = \int_c^\infty \mathrm{d}\lambda\,\lambda^m\, e^{\imath S^{4-2}(\lambda)}$$




$s_{ij} = 2$

$s_{0i} = s_{1i} = 1/4$

$s_{01} > 0$

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- **Finite Path Integrals**: The Lorentzian Regge path integral for 5-1 and 4-2 moves is finite, even for arbitrary powers of length expectation values and a wide range of measures (handled with the Wynn algorithm).

# Thank you for your attention!